Supplementary Material for On the Electrostatic Interaction between Point Charges due to Dielectrical Shielding

I. CHECKING THE BOUNDARY CONDITIONS

We only have to check the boundary conditions at one of the interfaces, because for the other it is similar. Let us pick the interface between Region 1 and Region 0, and choose a cylindrical coordinate system in which the central axis passes through both point-charges q_1 and q_2, r is the radial distance away from that axis (see Fig. 1).

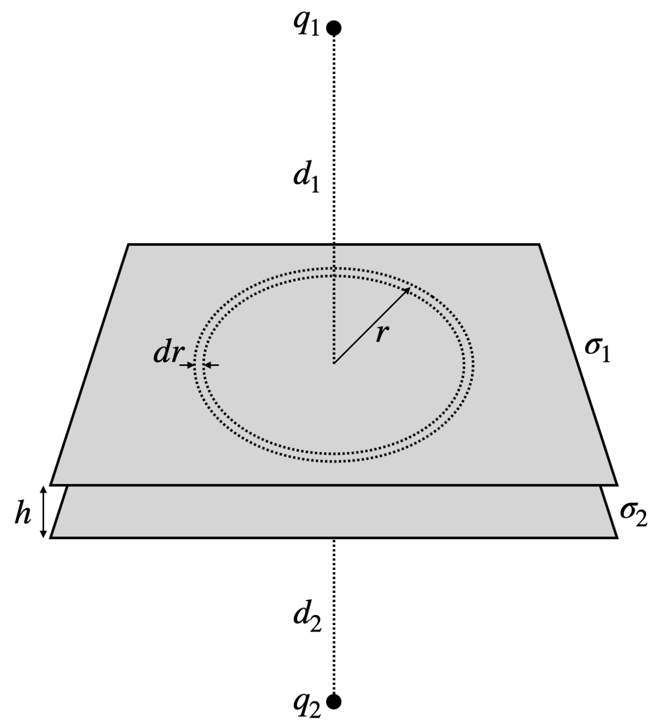

FIG. 1: The cylindrical coordinate system we use in Section 1 and Section 2.

A. Tangential Condition: $E_{\parallel 1} = E_{\parallel 0}$

- On the interface, inside Region 1:

$$
\begin{aligned}
E_{\parallel 1} = \frac{1}{4\pi\epsilon_0}\Bigg\{ & \frac{q_1 r}{[d_1^2 + r^2]^{3/2}} + \frac{q_1' r}{[d_1^2 + r^2]^{3/2}} + \sum_{k=0}^{\infty} \frac{q_1^{(1,k)} r}{[(d_1 + 2h + 2kh)^2 + r^2]^{3/2}} \\
& + \sum_{k=0}^{\infty} \frac{q_2^{(1,k)} r}{[(d_2 + h + 2kh)^2 + r^2]^{3/2}} \Bigg\} \\
= \frac{1}{4\pi\epsilon_0}\Bigg\{ & \frac{2q_1}{(1+\epsilon)} \frac{r}{[d_1^2 + r^2]^{3/2}} + \sum_{k=0}^{\infty} \frac{q_1^{(1,k)} r}{[(d_1 + 2h + 2kh)^2 + r^2]^{3/2}} \\
& + \sum_{k=0}^{\infty} \frac{q_2^{(1,k)} r}{[(d_2 + h + 2kh)^2 + r^2]^{3/2}} \Bigg\}.
\end{aligned} \tag{1}
$$

- On the interface, inside Region 0:

$$\begin{aligned}
E_{\parallel 0} &= \frac{q_1^{(0,0)} r}{4\pi\epsilon_0 [d_1^2+r^2]^{3/2}} + \frac{1}{4\pi\epsilon_0}\sum_{k=0}^{\infty}\left\{\frac{q_1^{(0,2k+1)} r}{[(d_1+2h+2kh)^2+r^2]^{3/2}} + \frac{q_1^{(0,2k+2)} r}{[(d_1+2h+2kh)^2+r^2]^{3/2}}\right\} \\
&\quad + \frac{1}{4\pi\epsilon_0}\sum_{k=0}^{\infty}\left\{\frac{q_2^{(0,2k)} r}{[(d_2+h+2kh)^2+r^2]^{3/2}} + \frac{q_2^{(0,2k+1)} r}{[(d_2+h+2kh)^2+r^2]^{3/2}}\right\} \\
&= \frac{1}{4\pi\epsilon_0}\left\{ q_1^{(0,0)} \frac{r}{[d_1^2+r^2]^{3/2}} + \sum_{k=0}^{\infty}\frac{\left[q_1^{(0,2k+1)} + q_1^{(0,2k+2)}\right] r}{[(d_1+2h+2kh)^2+r^2]^{3/2}} \right. \\
&\quad \left. + \sum_{k=0}^{\infty}\frac{\left[q_2^{(0,2k)} + q_2^{(0,2k+1)}\right] r}{[(d_2+h+2kh)^2+r^2]^{3/2}} \right\}.
\end{aligned} \tag{2}$$

Since $\frac{2q_1}{1+\epsilon} = q_1^{(0,0)}$, $q_1^{(1,k)} = q_1^{(0,2k+1)} + q_1^{(0,2k+2)}$ and $q_2^{(1,k)} = q_2^{(0,2k)} + q_2^{(0,2k+1)}$, we get $E_{\parallel 1} = E_{\parallel 0}$.

B. Normal Condition: $E_{\perp 1} = \epsilon E_{\perp 0}$

- On the interface, inside Region 1:

$$\begin{aligned}
E_{\perp 1} &= \frac{1}{4\pi\epsilon_0}\left\{ -\frac{q_1 d_1}{[d_1^2+r^2]^{3/2}} + \frac{q_1' d_1}{[d_1^2+r^2]^{3/2}} + \sum_{k=0}^{\infty}\frac{q_1^{(1,k)}(d_1+2h+2kh)}{[(d_1+2h+2kh)^2+r^2]^{3/2}} \right. \\
&\quad \left. + \sum_{k=0}^{\infty}\frac{q_2^{(1,k)}(d_2+h+2kh)}{[(d_2+h+2kh)^2+r^2]^{3/2}} \right\} \\
&= \frac{1}{4\pi\epsilon_0}\left\{ -\frac{2\epsilon}{1+\epsilon}\frac{q_1 d_1}{[d_1^2+r^2]^{3/2}} + \sum_{k=0}^{\infty}\frac{q_1^{(1,k)}(d_1+2h+2kh)}{[(d_1+2h+2kh)^2+r^2]^{3/2}} \right. \\
&\quad \left. + \sum_{k=0}^{\infty}\frac{q_2^{(1,k)}(d_2+h+2kh)}{[(d_2+h+2kh)^2+r^2]^{3/2}} \right\}.
\end{aligned} \tag{3}$$

- On the interface, inside Region 0:

$$\begin{aligned}
E_{\perp 0} &= -\frac{q_1^{(0,0)} d_1}{4\pi\epsilon_0 [d_1^2+r^2]^{3/2}} + \frac{1}{4\pi\epsilon_0}\sum_{k=0}^{\infty}\left\{\frac{q_1^{(0,2k+1)}(d_1+2h+2kh)}{[(d_1+2h+2kh)^2+r^2]^{3/2}} - \frac{q_1^{(0,2k+2)}(d_1+2h+2kh)}{[(d_1+2h+2kh)^2+r^2]^{3/2}}\right\} \\
&\quad + \frac{1}{4\pi\epsilon_0}\sum_{k=0}^{\infty}\left\{\frac{q_2^{(0,2k)}(d_2+h+2kh)}{[(d_2+h+2kh)^2+r^2]^{3/2}} - \frac{q_2^{(0,2k+1)}(d_2+h+2kh)}{[(d_2+h+2kh)^2+r^2]^{3/2}}\right\} \\
&= \frac{1}{4\pi\epsilon_0}\left\{ -q_1^{(0,0)}\frac{d_1}{[d_1^2+r^2]^{3/2}} + \sum_{k=0}^{\infty}\frac{\left[q_1^{(0,2k+1)} - q_1^{(0,2k+2)}\right](d_1+2h+2kh)}{[(d_1+2h+2kh)^2+r^2]^{3/2}} \right. \\
&\quad \left. + \sum_{k=0}^{\infty}\frac{\left[q_2^{(0,2k)} - q_2^{(0,2k+1)}\right](d_2+h+2kh)}{[(d_2+h+2kh)^2+r^2]^{3/2}} \right\}.
\end{aligned} \tag{4}$$

Since $\frac{2\epsilon q_1}{1+\epsilon} = \epsilon q_1^{(0,0)}$, $q_1^{(1,k)} = \epsilon\left[q_1^{(0,2k+1)} - q_1^{(0,2k+2)}\right]$ and $q_2^{(1,k)} = \epsilon\left[q_2^{(0,2k)} - q_2^{(0,2k+1)}\right]$, we get $E_{\perp 1} = \epsilon E_{\perp 0}$.

II. CONFIRMATION OF NEWTON'S THIRD LAW

We need to take into account the total forces acting on the dielectric slab, which is equal to $f_1' - f_2'$ where f_1' is the electrostatic force of point-charge q_1 exerts on the slab pulling toward and f_2' is the electrostatic force of point-charge

q_2' exerts on the slab pulling toward. From Newton's third law, all internal forces need to cancel out, which means we need to prove that:

$$f_1 + (f_1' - f_2') - f_2 = 0\ . \tag{5}$$

The surface charge density σ_1 on the interface between Region 1 and Region 0 are given by:

$$\sigma_1 = -\epsilon\left(E_{\perp 1} - E_{\perp 0}\right) = -\epsilon_0\left(1 - \frac{1}{\epsilon}\right)E_{\perp 1}\ , \tag{6}$$

in which $E_{\perp 1}$ can be calculated as described in Eq. (3). Similarly, we can obtain σ_2.

The force exerted by q_1 pulling the dielectric slab (acting on the surface charges on both interfaces) toward it is given by:

$$\begin{aligned} f_1' =& -\int_0^\infty \frac{q_1}{4\pi\epsilon_0}\cdot\frac{2\pi\sigma_1 r dr}{d_1^2 + r^2}\cdot\frac{d_1}{\sqrt{d_1^2 + r^2}} - \int_0^\infty \frac{q_1}{4\pi\epsilon_0}\cdot\frac{2\pi\sigma_2 r dr}{(d_1+h)^2 + r^2}\cdot\frac{d_1 + h}{\sqrt{(d_1+h)^2 + r^2}} \\ =& \frac{q_1}{4\pi\epsilon_0}\int_0^\infty \frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)\Bigg\{ -\frac{2\epsilon}{1+\epsilon}\frac{q_1 d_1}{[d_1^2 + r^2]^{3/2}} + \sum_{k=0}^{\infty}\frac{q_1^{(1,k)}(d_1 + 2h + 2kh)}{[(d_1 + 2h + 2kh)^2 + r^2]^{3/2}} \\ &+ \sum_{k=0}^{\infty}\frac{q_2^{(1,k)}(d_2 + h + 2kh)}{[(d_2 + h + 2kh)^2 + r^2]^{3/2}}\Bigg\}\frac{d_1 r dr}{(d_1^2 + r^2)^{3/2}} + \frac{q_1}{4\pi\epsilon_0}\int_0^\infty \frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)\Bigg\{ -\frac{2\epsilon}{1+\epsilon}\frac{q_2 d_1}{[d_2^2 + r^2]^{3/2}} \\ &+ \sum_{k=0}^{\infty}\frac{q_2^{(2,k)}(d_2 + 2h + 2kh)}{[(d_2 + 2h + 2kh)^2 + r^2]^{3/2}} + \sum_{k=0}^{\infty}\frac{q_1^{(2,k)}(d_1 + h + 2kh)}{[(d_1 + h + 2kh)^2 + r^2]^{3/2}}\Bigg\}\frac{(d_1 + h) r dr}{[(d_1+h)^2 + r^2]^{3/2}}. \end{aligned} \tag{7}$$

Here, we note an useful integration:

$$\int_0^\infty \frac{abxdx}{(a^2 + x^2)^{3/2}(b^2 + x^2)^{3/2}} = \frac{1}{(a+b)^2}\ , \tag{8}$$

where a and b are positive real numbers. Hence, with that, we obtain:

$$\begin{aligned} f_1' =& \frac{q_1}{4\pi\epsilon_0}\frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)\Bigg\{ -\frac{2\epsilon}{1+\epsilon}\frac{q_1}{(2d_1)^2} + \sum_{k=0}^{\infty}\frac{q_1^{(1,k)}}{(2d_1 + 2h + 2kh)^2} + \sum_{k=0}^{\infty}\frac{q_2^{(1,k)}}{(d_1 + d_2 + h + 2kh)^2}\Bigg\} \\ &+ \frac{q_1}{4\pi\epsilon_0}\frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)\Bigg\{ -\frac{2\epsilon}{1+\epsilon}\frac{q_2}{(d_1 + d_2 + h)^2} + \sum_{k=0}^{\infty}\frac{q_2^{(2,k)}}{(d_1 + d_2 + h + 2h + 2kh)^2} + \sum_{k=0}^{\infty}\frac{q_1^{(2,k)}}{(2d_1 + 2h + 2kh)^2}\Bigg\} \\ =& \frac{q_1}{4\pi\epsilon_0}\Bigg\{ -\frac{\epsilon - 1}{\epsilon + 1}\frac{q_1}{(2d_1)^2} + \sum_{k=0}^{\infty}\frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)\frac{q_1^{(1,k)} + q_1^{(2,k)}}{(2d_1 + 2h + 2kh)^2} + \sum_{k=0}^{\infty}\frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)\frac{q_2^{(1,k+1)} + q_2^{(2,k)}}{(d_1 + d_2 + h + 2h + 2kh)^2} \\ &+ \left(-\frac{\epsilon - 1}{\epsilon + 1}q_2 + \frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)q_2^{(1,0)}\right)\frac{1}{(d_1 + d_2 + h)^2}\Bigg\}. \end{aligned} \tag{9}$$

Since $\frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)\left(q_1^{(1,k)} + q_1^{(2,k)}\right) = q_1^{(1,k)}$, $\frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)\left(q_2^{(1,k+1)} + q_1^{(2,k)}\right) = q_2^{(1,k+1)}$ and $-\frac{\epsilon-1}{\epsilon+1}q_2 + \frac{1}{2}\left(1 - \frac{1}{\epsilon}\right)q_2^{(1,0)} = q_2^{(1,0)} - q_2$, from Eq. (7) in the main manuscript we get:

$$f_1 - f_1' = \frac{q_1 q_2}{4\pi\epsilon_0(d_1 + h + d_2)^2}\ . \tag{10}$$

Similarly we can obtain $f_2 - f_2'$, which is the negative of the above expression, thus verify Eq. (5).

III. SOLVING THE POISSON'S EQUATION

Consider the simple case where the setting is symmetric, $q_1 = q_2 = q$ and $d_1 = d_2 = d$. Choose a Cartesian $Oxyz$ coordinate system, in which the origin is inside the dielectric slab at the middle of two charges, the xy-plane is parallel to the slab thus the z-axis passes through both charges. Decompose the potential into Fourier-modes in the xy-plane:

$$V(x,y,z) = \int \frac{dk_x dk_y}{(2\pi)^2} e^{ik_x x + ik_y y}\tilde{V}_k(z)\ , \tag{11}$$

where $k = \sqrt{k_x^2 + k_y^2}$ is a sufficient index due to rotational symmetry in the xy-plane. Excluding the two interfaces and the positions of two charges we get the Laplace's equation $\nabla^2 V = 0$, hence the solution will be of the form (due to the setting is symmetric and far-away potential should go to 0):

$$\begin{aligned}
z > d + \frac{h}{2} \ &: \ \tilde{V}_k(z) = A e^{-kz} \ , \\
d + \frac{h}{2} > z > \frac{h}{2} \ &: \ \tilde{V}_k(z) = B e^{-kz} + C e^{+kz} \ , \\
\frac{h}{2} > |z| \ &: \ \tilde{V}_k(z) = D \left(e^{-kz} + e^{+kz} \right) \ , \\
d + \frac{h}{2} > -z > \frac{h}{2} \ &: \ \tilde{V}_k(z) = C e^{-kz} + B e^{+kz} \ , \\
-z > d + \frac{h}{2} \ &: \ \tilde{V}_k(z) = A e^{+kz} + D e^{+kz} \ .
\end{aligned} \tag{12}$$

The continuity conditions of $\tilde{V}_k(z)$ at the charges' positions and the interfaces give:

$$\begin{aligned}
&A e^{-k(d+h/2)} = B e^{-k(d+h/2)} + C e^{-k(d+h)/2} \ , \\
&B e^{-kh/2} + C e^{+kh/2} = D \left(e^{+kh/2} + e^{+kh/2} \right) \ .
\end{aligned} \tag{13}$$

The jumping conditions of $\partial_z \tilde{V}_k(z)$ at the charges' positions give:

$$-\frac{q}{\epsilon_0} = \left(-A k e^{-k(d+h/2)} \right) - \left(-B k e^{-k(d+h/2)} + C k e^{+k(d+h/2)} \right) \ . \tag{14}$$

The continuity conditions of $\partial_z \tilde{V}_k(z)$ at the interfaces give:

$$0 = \left(-B k e^{-kh/2} + C k e^{+kh/2} \right) - \epsilon D \left(-k e^{-kh/2} + k e^{+kh/2} \right) \ . \tag{15}$$

Four unknowns A, B, C, D can be solved with four equations (13), (14), (15):

$$\begin{aligned}
A &= \frac{2q e^{kh} \left(\cosh(kd) \cosh(\frac{kh}{2}) + \epsilon \sinh(kd) \sinh(\frac{kh}{2}) \right)}{\epsilon_0 k \left((1-\epsilon) + (1+\epsilon) e^{kh} \right)} \ , \\
B &= \frac{q e^{-k(d-h/2)} \left((1+\epsilon) + (1-\epsilon) e^{kh} \right)}{2\epsilon_0 k \left((1-\epsilon) + (1+\epsilon) e^{kh} \right)} \ , \\
C &= \frac{q e^{-k(d+h/2)}}{2\epsilon_0 k} \ , \\
D &= \frac{q e^{-k(d-h/2)}}{\epsilon_0 k \left((1-\epsilon) + (1+\epsilon) e^{kh} \right)} \ .
\end{aligned} \tag{16}$$

Now let's look at the charge's position $z = d + h/2$ and the self-contribution:

$$\begin{aligned}
z > d + \frac{h}{2} \ &: \ \tilde{V}_k^{(s)}(z) = A_+^{(s)} e^{-kz} \ , \\
d + \frac{h}{2} > z \ &: \ \tilde{V}_k^{(s)}(z) = A_-^{(s)} e^{+kz} \ .
\end{aligned} \tag{17}$$

The continuity condition of $\tilde{V}_k^{(s)}(z)$:

$$A_+^{(s)} e^{-k(d+h/2)} = A_-^{(s)} e^{+k(d+h/2)} \ . \tag{18}$$

The jumping conditions of $\tilde{V}_k^{(s)}(z)$:

$$-\frac{q}{\epsilon_0} = \left(-A_+^{(s)} k e^{-k(d+h/2)} \right) - \left(A_-^{(s)} k e^{+k(d+h/2)} \right) \ . \tag{19}$$

From (18) and (19), we get:

$$A_{\pm}^{(s)} = \frac{q e^{\pm k(d+h/2)}}{2\epsilon_0 k} \tag{20}$$

The regularized $\tilde{V}_k^{(r)}(z) = \tilde{V}_k(z) - \tilde{V}_k^{(s)}(z)$ is continuous and smooth at that charge's position, and can be used to determine the gradient right there:

$$\begin{aligned} \partial_z \tilde{V}_k^{(r)}(z)\Big|_{z=d+\frac{h}{2}} &= \left(A - A_{+}^{(s)}\right) \partial_z e^{-kz}\Big|_{z=d+\frac{h}{2}} \\ &= -\frac{q e^{-2kd}}{2\epsilon_0} \frac{1-\epsilon \tanh(\frac{kh}{2})}{1+\epsilon \tanh(\frac{kh}{2})} \ . \end{aligned} \tag{21}$$

Thus the force acting on the charge can be calculated with:

$$\begin{aligned} f &= -q \partial_z V^{(r)}(0,0,z)\Big|_{z=d+\frac{h}{2}} \\ &= -q \int_0^\infty \frac{2\pi k dk}{(2\pi)^2} \partial_z \tilde{V}_k^{(r)}(z)\Big|_{z=d+\frac{h}{2}} \\ &= \frac{q^2}{4\pi\epsilon_0} \int_0^\infty dk k e^{-2kd} \frac{1-\epsilon \tanh(\frac{kh}{2})}{1+\epsilon \tanh(\frac{kh}{2})} \ . \end{aligned} \tag{22}$$

Define $\chi = kh/2$ then:

$$f = \frac{q^2}{4\pi\epsilon_0 h^2} \times 4 \int_0^\infty d\chi \chi e^{-4\chi d/h} \frac{1-\epsilon \tanh\chi}{1+\epsilon \tanh\chi} \ . \tag{23}$$

When $d = h$, we get back the result Eq. (13) in the main manuscript.