Progress In Electromagnetics Research, PIER 32, 271–299, 2001

HIGHER ORDER WHITNEY FORMS

R. Hiptmair

Sonderforschungsbereich 382
Universität Tübingen
72076 Tuebingen, Germany

Abstract—The calculus of differential forms can be used to devise a unified description of discrete differential forms of any order and polynomial degree on simplicial meshes in any spatial dimension. A general formula for suitable degrees of freedom is also available. Fundamental properties of nodal interpolation can be established easily. It turns out that higher order spaces, including variants with locally varying polynomial order, emerge from the usual Whitney-forms by local augmentation. This paves the way for an adaptive p-version approach to discrete differential forms.

1. INTRODUCTION

It is generally acknowledged that the calculus of differential forms is the right device for mathematical modeling in many fields of physics, in particular in electromagnetism [8, 3, 14]. For instance, it turns out that the electric field should be regarded as a 1-form, and that the differential operators div and **curl** are just different manifestations of the exterior derivative d of differential forms.

As differential forms capture the nature of continuous fields, it is natural to resort to discrete differential forms when approximations of these fields are desired. Here, “discrete” means that the form is completely determined by only a finite number of degrees of freedom. By definition, the basic relationships of the calculus of differential forms must hold for the discrete differential forms. Thus, essential properties of the continuous physical models like conservation principles are preserved in the discrete setting.

Special discrete differential forms that are associated with triangulations of a computational domain give rise to finite elements that offer appropriate spaces for the corresponding physical quantities. For instance, they supply the hugely popular edge elements of computational electromagnetism. In conjunction with variational principles, discrete differential forms thus become a powerful tool for the approximate numerical solution of boundary value problems.

With hindsight, it is startling that after the discovery of discrete differential forms by Whitney [27] it took a long time before their significance as computational tool was realized [6, 7]. Without referring to differential geometry, several authors [22, 23, 25] had devised vector valued finite elements that can be regarded as special cases of discrete differential forms. Their constructions are formidably intricate and require much technical effort.

A substantial simplification can be achieved as was demonstrated in [18]: One should exploit the facilities provided by differential geometry for a completely coordinate-free treatment of discrete differential forms. Once we have shed the cumbersome vector calculus, everything can be constructed and studied with unmatched generality and elegance. In particular, all orders of forms and all degrees of polynomial approximation can be dealt with in the same framework. This can be done for simplicial meshes in arbitrary dimension.

The purpose of this paper is twofold. First the results in [18] should be made accessible to a wider audience beyond theoretical numerical analysts. Secondly, special attention is paid to discrete differential forms of higher polynomial degree. The rationale is the arrival of so-called p-adaptive finite element schemes. Their principle is to adjust the polynomial degree of the approximating finite elements to local features of the solution in order to gain accuracy with minimum costs. When this idea is to be applied to discrete differential forms it becomes vital to know suitable local basis functions for various polynomial degrees. Higher order discrete differential forms also form the foundation of hierarchical a-posteriori error estimators [4].

This paper complements and caps several earlier investigations into the subject of higher order edge elements [1, 13, 17, 24, 16, 20, 28, 26]. However, it does not discuss special schemes, but emphasizes governing principles. Once these are understood, they offer guidance on how to tailor higher order discrete differential forms to particular needs.

I should stress that the entire presentation is set in a purely affine context and, consequently, never resorts to vectorfields or functions representing differential forms. This is in stark contrast with the usual treatment of higher order discrete differential forms in literature. Thus my exposure might strike many a reader as peculiar. Yet, metric structures are alien to discrete differential forms. Of course, ultimately orthogonal coordinates come handily for actual computations, but introducing them already for the construction of discrete differential forms distracts from generic properties.

2. EXTERIOR CALCULUS

This section summarizes a few fundamental concepts of exterior calculus. It does not attempt to give an introduction to the subject. For a comprehensive discussion of the theory of differential forms see the monograph [11] or consult your favorite textbook on advanced differential calculus.

In the most general sense, Ω may be a (piecewise) smooth oriented and bounded n-dimensional Riemannian manifold, $n \in \mathbb{N}$, with a piecewise smooth boundary. A differential form ω of order l, $0 \leq l \leq n$, is a mapping from Ω into the $\binom{n}{l}$-dimensional space $\triangle^l(T_\Omega(\mathbf{x}))$ of alternating l–multilinear forms on the n-dimensional tangent space $T_\Omega(\mathbf{x})$ at Ω in $\mathbf{x} \in \Omega$ [11, Sect. 2.1]. In the sequel, $D^l_k(\Omega)$ stands for the space of l-forms on Ω of class C^k.

A fundamental concept in the theory of differential forms is the *integral* of a p-form over a piecewise smooth p-dimensional oriented manifold. Through integration a differential form assigns a value to each suitable manifold, modeling the physical procedure of measuring a field. We write $D^l(\Omega)$ for the vector space of l-forms on Ω, whose Riemann-integrals exist for any compact piecewise smooth l-dimensional submanifold of Ω.

From alternating l-multilinear forms differential l-forms inherit the exterior product $\wedge : D^l_0(\Omega) \times D^k_0(\Omega) \mapsto D^{l+k}_0(\Omega)$, $0 \leq l, k, l+k \leq n$, defined in a pointwise sense. Moreover, remember that the trace $\mathbf{t}_\Sigma\omega$ of an l-form $\omega \in D^l(\Omega)$, $0 \leq l < n$, onto some piecewise smooth $n-1$-dimensional submanifold $\Sigma \subset \bar{\Omega}$ yields an l-form on Σ [19, Sect. 1.10]. It can be introduced by restricting $\omega(\mathbf{x}) \in \triangle^l(T_\Omega(\mathbf{x}))$, $\mathbf{x} \in \Sigma$, to the tangent space of Σ in $\mathbf{x}$. The trace commutes with the exterior product and exterior differentiation, i.e. $d\,\mathbf{t}_\Sigma\omega = \mathbf{t}_\Sigma d\omega$ for $\omega \in D^l_1(\Omega)$.

Another crucial device is the exterior derivative d, a linear mapping from differentiable l-forms into $l+1$-forms. Moreover, Stokes' theorem makes it possible to define the exterior derivative $d\omega \in D^{l+1}(\Omega)$ of $\omega \in D^l(\Omega)$. A fundamental fact about exterior differentiation is that $d(d\,\omega) = 0$ for any sufficiently smooth differential form ω. Under some restrictions on the topology of Ω the converse is also true:

THEOREM 2.1(Poincaré's lemma) *For a contractible domain $\Omega \subset \mathbb{R}^n$ every $\omega \in D^l_1(\Omega)$, $l \geq 1$, with $d\omega = 0$ is the exterior derivative of an $l-1$–form over Ω.*

This result is sometimes referred to as the *exact sequence property.* A second main result about the exterior derivative is the integration by parts formula [19, Sect. 3.2]

$$\int_\Omega d\omega \wedge \eta + (-1)^l \int_\Omega \omega \wedge d\eta = \int_{\partial\Omega} \omega \wedge \eta \tag{1}$$

for $\omega \in D^l(\Omega)$, $\eta \in D^k(\Omega)$, $0 \leq l, k < n-1$, $l + k = n-1$. Here, the boundary $\partial\Omega$ is endowed with the induced orientation.

Finally, we recall the pullback $\omega \mapsto \Phi^*\omega$ under a change of variables described by a diffeomorphism Φ. This transformation commutes with both the exterior product and the exterior derivative, and it leaves the integral invariant.

3. LOCAL SPACES

Following the classical approach in the theory of finite elements [12], we first try to establish suitable spaces $X^l_k(T)$ of discrete l-forms, $0 \leq l \leq n$, on individual simplices T_i of a triangulation $\mathcal{T}_h := \{T_i\}$ of $\Omega \subset \mathbb{R}^n$. Here, $k \in \mathbb{N}_0$ stands for the "polynomial degree" of the forms, a notion that will be explained in a moment.

The transformation of differential forms immediately suggests that we should opt for a construction based on affine equivalence. This means that the local spaces $X^l_k(\widehat{T})$ of discrete differential forms are first specified on some reference simplex $\widehat{T}$. An arbitrary simplex T of the mesh $\mathcal{T}_h$ can be mapped onto $\widehat{T}$ by an affine transformation $\mathbf{\Phi}_T : T \mapsto \widehat{T}$. Then the corresponding pullback converts the discrete forms on $\widehat{T}$ into those on T according to

$$X^l_k(T) := \mathbf{\Phi}^*_T X^l_k(\widehat{T}) \; . \tag{2}$$

A tenet of affine equivalence is that the choice of $\widehat{T}$ must not matter in the end. The reader can always imagine $\widehat{T}$ to be the simplex spanned by the canonical basis vectors of $\mathbb{R}^n$. At least, without loss of generality, assume $0 \in \widehat{T}$.

In the choice of the spaces $X_k^l(\widehat{T})$ we are guided by a few basic requirements stipulated by prerequisites for sensible finite elements, natural properties of differential forms, and the goal of a unified treatment. Firstly, we legitimately expect the exterior derivative of a discrete l-form to be a discrete $l+1$-form of the same degree, $0 \leq l \leq n-1$, i.e.

$$dX_k^l(\widehat{T}) \subset X_k^{l+1}(\widehat{T}) \ . \tag{3}$$

If the bid to save the exact sequence property for discrete forms is to succeed, we must demand even more, namely

$$\{\omega \in X_k^l(\widehat{T}),\, d\omega = 0\} = dX_k^{l-1}(\widehat{T}) \,, \quad 1 \leq l \leq n \ . \tag{4}$$

Secondly, if S is some sub-simplex of $\widehat{T}$, i.e. the closed convex hull of some of its vertices, then $\mathbf{t}_S\omega \in D_k^l(S)$, if $\omega \in D_k^l(\widehat{T})$. This property should be inherited by the discrete forms, i.e.

$$\mathbf{t}_S X_k^l(\widehat{T}) = X_k^l(S) \ . \tag{5}$$

It goes without saying that $X_k^l(S)$ is generated according to (2) as well. In particular, (5) means that the restriction of a discrete differential form to some face of $\widehat{T}$ yields a valid discrete differential form in dimension $n-1$.

Finally, the discrete differential forms have to possess satisfactory approximation properties. They are guaranteed through Bramble-Hilbert arguments as soon as piecewise polynomial forms of a prescribed degree are contained in the spaces (cf. [9, Ch. 4]). But, what is a polynomial differential form? Remember that given a basis $dx_1, \dots, dx_n$ of the dual space of $T_\Omega(\mathbf{x})$ the set

$$\{dx_{i_1} \wedge \ldots \wedge dx_{i_l}\,;\, i_j \in \{1, \dots, n\}, 1 \leq j \leq l, i_1 < i_2 < \ldots < i_l\} \tag{6}$$

furnishes a basis for the space of alternating l-multilinear forms on $T_\Omega(\mathbf{x})$. Thus any $\omega \in D^l(\Omega)$ has a representation

$$\omega = \sum_{(i_1, \dots, i_l)} \varphi_{i_1, \dots, i_l}\, dx_{i_1} \wedge \ldots \wedge dx_{i_l} \ , \tag{7}$$

where the indices run through all combinations admissible according to (6) and the $\varphi_{i_1, \dots, i_l} : \Omega \mapsto \mathbb{R}$ are coefficient functions.

Therefore, we call a differential form polynomial of degree k, $k \in \mathbb{N}_0$, if all its coefficient functions in (7) are polynomials of that degree. For convenience, we adopt special notations for spaces of "polynomial type" over an m–dimensional simplex S (A tilde will invariably tag spaces of homogeneous polynomials): $P_k(S) :=$ Space of m–variate polynomials of degree $\leq k$ on S, $k \in \mathbb{Z}$; $DP^l_k(S) := \{\omega \in D^l_\infty(S);\ \varphi_{i_1,..,i_l} \in P_k(S) \text{ in rep. (7)}\}, l \in \mathbb{Z}$; $\tilde{P}_k(S) :=$ Space of m–variate homogeneous polynomials of degree $k \in \mathbb{Z}$ on S; $\widetilde{DP}^l_k(S) := \left\{\omega \in D^l_\infty(S);\ \varphi_{i_1,\dots,i_l} \in \tilde{P}_k(S) \text{ in rep. (7)}\right\}$, $l \in \mathbb{Z}$. We follow the convention that for $l < 0$, $l > m$ and $k < 0$ these spaces are to be trivial. With polynomial spaces at our disposal we can state the third requirement as

$$DP^l_k(\widehat{T}) \subset X^l_k(\widehat{T}) \subset DP^l_{k+1}(\widehat{T}) \ . \tag{8}$$

I point out that the above requirements are affine invariant, i.e. they remain true for any $X^l_k(T)$ given by (2). The properties of the pullback guarantee this for (4) and (5). On top of that, be aware that the definition of the spaces $DP^l_k(S)$ and $\widetilde{DP}^l_k(S)$ is utterly independent of the choice of the basis (6). Hence, affine pullbacks leave the spaces invariant and (8) is seen to be affine invariant, too. In other words, if $X^l_k(\widehat{T})$ meets the requirements, they carry over to any $X^l_k(T)$, $T \in T_h$.

Surprisingly, the three requirements already fix the dimension of the local spaces $X^l_k(\widehat{T})$: We start with the formal direct sum

$$X^l_k(\widehat{T}) = DP^l_k(\widehat{T}) \oplus Y^l_k(\widehat{T}) \quad , \quad Y^l_k(\widehat{T}) \subset \widetilde{DP}^l_{k+1}(\widehat{T}) \ ,$$

which means

$$\dim X^l_k(\widehat{T}) = \dim DP^l_k(\widehat{T}) + \dim Y^l_k(\widehat{T}) \ . \tag{9}$$

Once we know $X^l_k(\widehat{T})$, the space $Y^l_k(\widehat{T})$ is uniquely defined, because homogeneous polynomials of different degree are linearly independent. We claim that

$$\widetilde{DP}^l_{k+1}(\widehat{T}) = Y^l_k(\widehat{T}) \oplus \widetilde{HN}^l_{k+1}(\widehat{T}) \tag{10}$$

is a direct sum. To see this, we make use of the elementary fact

$$\omega \in \widetilde{DP}^l_{k+1}(\widehat{T}) \quad \Rightarrow \quad d\omega \in \widetilde{DP}^{l+1}_k(\widehat{T}) \ , \quad 0 \leq l \leq n-1,\ k \geq 1 \ . \tag{11}$$

Then, pick $\omega \in Y^l_k(\widehat{T}) \cap \widetilde{HN}^l_{k+1}(\widehat{T})$, that is $d\omega = 0$. From (4) we conclude $\omega \in dX^{l-1}_k(\widehat{T})$ and, by (11), $\omega \in DP^l_k(\widehat{T})$. Hence

$\omega \in \widetilde{DP}^l_{k+1}(\widehat{T}) \cap DP^l_k(\widehat{T}) = \{0\}$. On the other hand, for each $\eta \in \widetilde{DP}^l_{k+1}(\widehat{T})$ we have by (11) that $d\eta \in \widetilde{DP}^{l+1}_k(\widehat{T}) \subset X^{l+1}_k(\widehat{T})$. By (4) there is a $\phi \in X^l_k(\widehat{T})$ such that $d\phi = d\eta$. Imagine that ϕ is split into homogeneous polynomials of different degree. This decomposition is unique and non-vanishing exterior derivatives of its components are linearly independent. This illustrates, why we can assume $\phi \in \widetilde{DP}^l_{k+1}(\widehat{T}) \cap X^l_k(\widehat{T}) = Y^l_k(\widehat{T})$. Finally

$$\eta = \phi + (\eta - \phi) \quad , \quad \phi \in Y^l_k(\widehat{T}) \, , \, \eta - \phi \in \widetilde{HN}^l_{k+1}(\widehat{T}) \, .$$

In terms of dimensions, (10) means

$$\dim \widetilde{DP}^l_{k+1}(\widehat{T}) = \dim Y^l_k(\widehat{T}) + \dim \widetilde{HN}^l_{k+1}(\widehat{T}) \, . \tag{12}$$

We point out that by (11), (16), and since applying the exterior derivative twice results in zero

$$\widetilde{HN}^l_{k+1}(\widehat{T}) = d(\widetilde{DP}^{l-1}_{k+2}(\widehat{T})) \, .$$

Thanks to the well–known relationship between the rank and the dimension of the null space of the linear mapping $d : \widetilde{DP}^{l-1}_{k+2}(\widehat{T}) \mapsto \widetilde{HN}^l_{k+1}(\widehat{T})$, we can establish

$$\dim \widetilde{HN}^l_{k+1}(\widehat{T}) = \dim \widetilde{DP}^{l-1}_{k+2}(\widehat{T}) - \dim \widetilde{HN}^{l-1}_{k+2}(\widehat{T}) \, . \tag{13}$$

That $DP^l_{k+1}(\widehat{T}) = \widetilde{DP}^l_{k+1}(\widehat{T}) \oplus DP^l_k(\widehat{T})$ needs no explanation. Then, plugging (12) into (9), we get

$$\dim X^l_k(\widehat{T}) = \dim DP^l_{k+1}(\widehat{T}) - \dim \widetilde{HN}^l_{k+1}(\widehat{T}) \, . \tag{14}$$

We rewrite (14) by means of (13) and then rely on (14) itself with $l-1$ instead of l. These involved manipulations yield the recursion

$$\begin{gathered} \dim X^l_k(\widehat{T}) = \dim DP^l_{k+1}(\widehat{T}) - \dim \widetilde{DP}^{l-1}_{k+2}(\widehat{T}) + \\ \dim DP^{l-1}_{k+2}(\widehat{T}) - \dim X^{l-1}_{k+1}(\widehat{T}) \, . \end{gathered}$$

Combinatorics teaches us

$$\begin{aligned} \dim DP^k_l(\widehat{T}) &= \dim \triangle^l(\mathbb{R}^n) \cdot \dim P_k(\widehat{T}) = \binom{n}{l}\binom{n+k}{k} \, , \\ \dim \widetilde{DP}^k_l(\widehat{T}) &= \dim \triangle^l(\mathbb{R}^n) \cdot \dim \tilde{P}_k(\widehat{T}) = \binom{n}{l}\binom{n+k-1}{k} \, . \end{aligned}$$

Using quite a few identities for binomial coefficients, the final result is

$$\begin{aligned}\dim X_k^l(\widehat{T}) &= \binom{n+1}{l}\binom{n+k+1}{n} - \dim X_{k+1}^{l-1}(\widehat{T}) \\ &= \sum_{i=l}^{0}(-1)^{l-i}\binom{n+1}{i}\binom{n+l-i+k+1}{n} \\ &= \sum_{i=l}^{n}\binom{n+1}{i+1}\binom{i}{i-l}\binom{k+l}{i}\,.\end{aligned}$$

This formula is universally valid, if we stick to the convention that for any $\binom{n}{k} = 0$, if $k < 0$ or $k > n$, $n, k \in \mathbb{Z}$. Concrete dimensions of local spaces in three dimensions are given in table 1.

k	0	1	2	3	4	5	General formula
$l = 1$	6	20	45	84	140	216	$\frac{1}{2}(k+1)(k+3)(k+4)$
$l = 2$	4	15	36	70	120	189	$\frac{1}{2}(k+4)(k+2)(k+1)$

Table 1. Dimensions of higher order spaces of edge ($l = 1$) and face ($l = 2$) elements, c.f. [22, 16].

Remark. The space $X_k^l(\widehat{T})$ is by no means completely determined by (4), (5), and (8). *Remark.* We have seen that

$$DP_{k+1}^l(\widehat{T}) = X_k^l(\widehat{T}) + dX_{k+1}^{l-1}(\widehat{T})\ ,$$

i.e. full polynomial local spaces can be achieved by incorporating derivatives of functions belonging to local spaces of the next higher order. The construction of “second families” of discrete differential forms [23] is based on this fact.

It is nice to know $\dim X_k^l(\widehat{T})$, but we still cannot be certain that the requirements can be satisfied at all. All doubts will be crushed, as soon as we have constructed specimens of $X_k^l(\widehat{T})$. The reader will agree that (4) is the most challenging demand. Therefore, any serious attack on the construction of $X_k^l(\widehat{T})$ will focus on (4), which amounts to Poincaré’s lemma for the local spaces of discrete differential forms. It is worth while studying the proof of theorem 2.1. We find that it is constructive and uses the so-called Poincaré-mapping $k_{\mathbf{a}} : D_0^l(\Omega) \mapsto$

$D_1^{l-1}(\Omega)$, $0 < l \leq n$, for $\mathbf{x} \in \Omega$ and $\mathbf{v}_i \in \mathbb{R}^n$ defined by

$$k_{\mathbf{a}}(\omega)(\mathbf{x})(\mathbf{v}_1, .., \mathbf{v}_{l-1}) := \int_0^1 t^{l-1} \omega(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))(\mathbf{x} - \mathbf{a}, \mathbf{v}_1, .., \mathbf{v}_{l-1})\, dt \, , \tag{15}$$

to get a potential for the closed form ω. For $\omega \in D_1^l(\Omega)$, $k_{\mathbf{a}}(\omega)$ is a valid $l-1$–form and satisfies (cf. [11], Formula 2.13.2)

$$d(k_{\mathbf{a}}(\omega)) + k_{\mathbf{a}}(d\omega) = \omega \, . \tag{16}$$

Sloppily speaking, the Poincaré mapping is a sort of partial right inverse of the exterior derivative, as $d \circ k_{\mathbf{a}} \circ d = d$.

It is immediately clear that the Poincaré-mapping takes polynomial forms to other polynomial forms. Straightforward, but tedious computations reveal the details (cf. lemma 5 in [18]):

Lemma 3.1 *For $\omega \in \widetilde{DP}_k^l(\widehat{T})$, $k > 0$, we know that $k_{\mathbf{a}}(\omega) \in DP_{k+1}^{l-1}(\widehat{T})$. In the special case $\mathbf{a} = 0$, the even stronger assertion $k_0(\omega) \in \widetilde{DP}_{k+1}^{l-1}(\widehat{T})$ holds true.*

This motivates a tentative definition of $X_l^k(\widehat{T})$ by

$$X_l^k(\widehat{T}) := DP_k^l(\widehat{T}) + k_{\mathbf{a}}(DP_k^{l+1}(\widehat{T})) \quad , \mathbf{a} \in \widehat{T} \, .$$

Strictly speaking, this is not a valid definition before we have not shown that the choice of $\mathbf{a} \in \widehat{T}$ does not matter. First recall that for $p \in P_k(\widehat{T})$ the difference $p(\cdot) - p(\cdot - \mathbf{c})$ belongs to $P_{k-1}(\widehat{T})$ for any $\mathbf{c} \in \mathbb{R}^n$. Therefore we can decompose $\omega \in DP_k^l(\widehat{T})$ into

$$\omega(\mathbf{a} + t(\mathbf{x} - \mathbf{a})) = \omega(t\mathbf{x}) + \pi(t\mathbf{x}) \, ,$$

where $\pi \in DP_{k-1}^l(\widehat{T})$. Using the definition of the potential mapping and the token $\bullet$ for $l-1$ vectors from $\mathbb{R}^n$, we get

$$\begin{aligned} k_{\mathbf{a}}(\omega)(\mathbf{x})(\bullet) &= -\int_0^1 t^{l-1} \underbrace{\omega(\mathbf{a} + t(\mathbf{x} - \mathbf{a}))(\mathbf{a}, \bullet)}_{\in DP_k^{l-1}(\widehat{T})} \, dt \\ &\quad + \int_0^1 t^{l-1} \left(\omega(t\mathbf{x}) + \pi(t\mathbf{x})\right)(\mathbf{x}, \bullet) \, dt \\ &= k_0(\omega)(\mathbf{x})(\bullet) + \eta(\mathbf{x})(\bullet) \, , \end{aligned}$$

with an $\eta \in DP_k^{l-1}(\widehat{T})$ according to Lemma (3.1). In short, $k_{\mathbf{a}}(\omega) - k_0(\omega) \in DP_k^{l-1}(\widehat{T})$ and the above definition makes sense. Alas, it fails to meet the requirements: Let us introduce the space of **h**omogeneous polynomial differential forms belonging to the **n**ullspace of the exterior derivative

$$\widetilde{HN}_k^l(S) := \{\omega \in \widetilde{DP}_k^l(S);\, d\omega = 0\} \;.$$

For $0 < l \leq n-2$, $k \geq 1$, pick $\eta \in \widetilde{HN}_{k-1}^{l+2}(\widehat{T})$, set $\mu := k_0(\eta)$ and conclude from (16) that $\mu = k_0(d\mu)$. In addition, lemma (3.1) tells us that $\mu \in \widetilde{DP}_k^{l+1}(\widehat{T})$. Next, set $\omega := k_0(\mu)$, which means $d\omega = dk_0(\mu) = \mu - k_0(d\mu) = 0$ and $\omega \in \widetilde{DP}_{k+1}^l(\widehat{T})$. Moreover, $\eta \neq 0$ involves $\omega \neq 0$, as the Poincaré-mapping is injective. On the other hand, we want to have $\omega \in d(DP_{k+1}^{l-1}(\widehat{T})) \subset DP_k^l(\widehat{T})$, which fails for $\eta \neq 0$ as $DP_k^l(\widehat{T}) \cap \widetilde{DP}_{k+1}^l(\widehat{T}) = \{0\}$. At least one of the requirements is violated, if $\widetilde{HN}_{k-1}^{l+2}(\widehat{T}) \neq \{0\}$, which can be verified by elementary calculations, since $\widetilde{HN}_{k-1}^{l+2}(\widehat{T}) = d\widetilde{DP}_k^{l+1}(\widehat{T})$.

The tentative definition supplies a space that is simply too large. Carefully inspecting its flaws, one finds that only closed forms should be admitted as arguments to the Poincaré-mapping. This leads us to the improved definition

$$\boxed{X_k^l(\widehat{T}) := DP_k^l(\widehat{T}) + k_{\mathbf{a}}\left(\widetilde{HN}_k^{l+1}(\widehat{T})\right)} \tag{17}$$

The point $\mathbf{a} \in \widehat{T}$ is arbitrary, but it does not make a difference, anyway. By lemma (3.1) it is obvious that (8) holds. Then, (3) is an immediate consequence. To verify (4) keep in mind that we could have given the equivalent definition

$$X_k^l(\widehat{T}) := DP_k^l(\widehat{T}) + k_{\mathbf{a}}\left(\{\omega \in DP_k^{l+1}(\widehat{T}),\, d\omega = 0\}\right) \;.$$

We also observe that thanks to lemma (3.1) the splitting

$$X_k^l(\widehat{T}) := DP_k^l(\widehat{T}) \oplus k_0\left(\widetilde{HN}_k^{l+1}(\widehat{T})\right) \tag{18}$$

is direct, as homogeneous polynomials of different degree are linearly independent. Let $\omega \in X_k^l(\widehat{T})$ with $d\omega = 0$ be split according to (18) into $\omega = \omega_0 + \widetilde{\omega}$. Since $\widetilde{\omega} \in \widetilde{DP}_{k+1}^l(\widehat{T})$ and $\omega_0 \in DP_k^l(\widehat{T})$, the linear independence of homogeneous polynomials of different degree combined with lemma (3.1) shows that $d\widetilde{\omega} = 0$. But $\widetilde{\omega} = k_0(\eta)$ for $\eta \in \widetilde{HN}_k^{l+1}(\widehat{T})$ and so (16) can be employed to get

$$\eta = dk_0(\eta) + k_0(d\eta) = d\widetilde{\omega} + k_0(0) = 0 \quad \Rightarrow \quad \widetilde{\omega} = 0 \, .$$

Eventually we have found $\omega = \omega_0 \in DP_k^l(\widehat{T})$ and arrive at

$$\omega = dk_0(\omega) \in d\left(X_k^{l-1}(\widehat{T})\right) \, .$$

To confirm the trace property (5) we consider one $n-1$-dimensional subsimplex F, a face of $\widehat{T}$. Induction with respect to n will settle everything else. Pick $\mathbf{a}$ in (17) as a vertex of F and note that $\mathbf{x} - \mathbf{a}$ belongs to the tangent hyperplane of F. Thus, by the very definition of the trace of a differential form

$$\mathbf{t}_F(k_{\mathbf{a}}(\omega)) = k_{\mathbf{a}}(\mathbf{t}_F\omega) \quad \forall \omega \in D_0^l(\widehat{T}) \, .$$

Since obviously $\mathbf{t}_F(DP_k^l(\widehat{T})) = DP_k^l(F)$, the identity $X_k^l(F) = \mathbf{t}_F X_k^l(\widehat{T})$ follows.

We now scrutinize a few special cases of local ansatz spaces generated by formula (17): For $l = n$ we get $X_k^l(\widehat{T}) = DP_k^l(\widehat{T})$, since $\widetilde{HN}_k^{l+1}(\widehat{T}) = \{0\}$. Also in the case $l = 0$ we end up with complete polynomial spaces: For any $\omega \in D_1^0(\widehat{T})$ with $\omega(0) = 0$ we have $\omega = k_0(d\omega)$ by (16). Obviously this involves

$$k_0(\widetilde{HN}_k^1(\widehat{T})) = \widetilde{DP}_{k+1}^0(\widehat{T}) \, ,$$

as $d\omega \in \widetilde{HN}_k^1(\widehat{T})$ for all $\omega \in \widetilde{DP}_{k+1}^0(\widehat{T})$. An easy consequence is that in the case $l = 0$

$$X_k^0(\widehat{T}) = DP_k^0(\widehat{T}) + \widetilde{DP}_{k+1}^0(\widehat{T}) = DP_{k+1}^0(\widehat{T}) \, ,$$

i.e. the full space of polynomials of degree $\leq k+1$.

Example. An explicit construction of the local space $X_1^1(\widehat{T})$ in two dimensions can be carried out as follows: Using affine coordinates $\{x_1, x_2\}$ we can write

$$\widetilde{DP}_1^2(\widehat{T}) = \{\omega(\mathbf{x}) = (\alpha_1 x_1 + \alpha_2 x_2) dx_1 \wedge dx_2, \, \alpha_1, \alpha_2 \in \mathbb{R}\} \, .$$

Note that in this case $\widetilde{HN}_1^2(\widehat{T}) = \widetilde{DP}_1^2(\widehat{T})$. If this did not hold, the condition $d\omega = 0$ would introduce some linear constraints for the coefficients α_1 and α_2. For $\omega = (\alpha_1 x_1 + \alpha_2 x_2) dx_1 \wedge dx_2$ simple computations yield

$$\begin{aligned} k_0(\omega)(\mathbf{x}) &= \int_0^1 t(\alpha_1 t x_1 + \alpha_2 t x_2)(dx_1 \wedge dx_2)(\mathbf{x}, \cdot)\, dt \\ &= \frac{1}{3}(\alpha_1 x_1 + \alpha_2 x_2)(x_1 dx_2 - x_2 dx_1) \; . \end{aligned} \tag{19}$$

This means according to (17)

$$X_1^1(\widehat{T}) = \left\{ \begin{array}{l} \omega =(\gamma_1 + \beta_{11} x_1 + \beta_{12} x_2 - \alpha_1 x_1 x_2 - \alpha_2 x_2^2) dx_1 + \\ \qquad + (\gamma_2 + \beta_{21} x_1 + \beta_{22} x_2 + \alpha_1 x_1^2 - \alpha_2 x_1 x_2) dx_2, \\ \qquad \gamma_i, \alpha_i, \beta_{ij} \in \mathbb{R} \end{array} \right\} .$$

Remark. Simple counterexamples show that in general

$$X_k^l(\widehat{T}) \neq \underbrace{X_0^0(\widehat{T}) \wedge \ldots \wedge X_0^0(\widehat{T})}_{k-1\,\text{times}} \wedge X_0^l(\widehat{T}) \; .$$

As a technical tool we are going to need the local spaces of discrete differential forms with vanishing trace on $\partial\widehat{T}$

$$H_k^l(\widehat{T}) := \{\omega \in X_k^l(\widehat{T}),\, \mathbf{t}_{\partial\widehat{T}}\omega = 0\} \; .$$

It can be confirmed (cf. [18, Lemma 17]) that the local exact sequence property applies to these spaces, as well

$$\{\omega \in H_k^l(\widehat{T}),\, d\omega = 0\} = d\, H_k^{l-1}(\widehat{T}) \quad , \quad 0 \leq l < n \; , \tag{20}$$

$$\{\omega \in X_k^n(\widehat{T}),\, \int_{\widehat{T}} \omega = 0\} = d\, H_k^{n-1}(\widehat{T}) \; . \tag{21}$$

4. DEGREES OF FREEDOM

The previous section provided us with with local spaces $X_k^l(T)$ of l-forms on the simplices T of a triangulation T_h of Ω. The corresponding global space of finite dimension is

$$X_k^l(T_h, \Omega) := \{\omega \in D^l(\Omega),\, \omega_{|T} \in X_k^l(T)\, \forall T \in T_h\} \; .$$

This innocent looking definition conceals a so-called patch condition, that is, a recipe how two polynomial l-forms have to be glued together at interelement faces $T \cap T'$, $T, T' \in \mathcal{T}_h$, to produce a valid form in $\mathcal{D}^l(\Omega)$. Let me elucidate this by means of some intuitive arguments: Recall that valid l-forms are distinguished by providing meaningful integrals over orientable l-manifolds. Now, consider a smooth l-manifold S, $0 \leq l < n$, that intersects an interelement face F. Problems crop up when the intersection is not a set of measure zero in S, which happens when S somewhere "runs parallel" to F. There the tangent spaces for S are contained in the tangent hyperplane of F. To render the integral of $\omega \in \mathcal{X}^l_k(\mathcal{T}_h, \Omega)$ over S well defined, it is sufficient that $\omega_{|F \cap S}$ provides unique values when applied to tangent vectors of F. As S is arbitrary, the condition is also necessary.

Definition 4.1*A piecewise smooth differential form on $\mathcal{T}_h$ satisfies the* patch condition*, if its traces onto interelement faces from both sides agree.*

We have d arrived at the equivalent definition

$$\mathcal{X}^l_k(\mathcal{T}_h, \Omega) := \{\omega \in \bigotimes_{T \in \mathcal{T}_h} \mathcal{X}^l_k(T),\ \omega \text{ satisfies the patch condition}\}\ .$$

Two issues are looming large: Firstly, forcing the traces to agree implies linear constraints, which might actually curtail the local spaces. In other words, we have to make sure that for any $T \in \mathcal{T}_h$ and $\mu \in \mathcal{X}^l_k(T)$ there is an $\omega \in \mathcal{X}^l_k(\mathcal{T}_h, \Omega)$ such that $\omega_{|T} = \mu$. Secondly, the resulting space $\mathcal{X}^l_k(\mathcal{T}_h, \Omega)$ might not be computationally efficient in the sense that a basis of $\mathcal{X}^l_k(\mathcal{T}_h, \Omega)$ invariably consists of functions with global supports. This would thwart the use of $\mathcal{X}^l_k(\mathcal{T}_h, \Omega)$ for a finite element scheme.

It is the crucial role of suitable degrees of freedom (d.o.f.) to bring about a satisfactory settlement of these issues. In general, degrees of freedom are a basis of the dual space $\mathcal{X}^l_k(\mathcal{T}_h, \Omega)'$. In the spirit of the previous section, we confine ourselves to an affine equivalent construction and first specify the degrees of freedom on the reference simplex $\widehat{T}$ only: We look for a set $\Xi^l_k(\widehat{T}) := \{\hat{\kappa}_1, \ldots, \hat{\kappa}_{N_{k,l}}\}$, $N_{k,l} = \dim \mathcal{X}^l_k(\widehat{T})$, $l \in \{0, \ldots, n\}$, $k \in \mathbb{N}_0$, of linear forms

$$\hat{\kappa}_i : \mathcal{X}^l_k(\widehat{T}) \mapsto \mathbb{R}\ , \quad i \in \{1, \ldots, N_{k,l}\}$$

that satisfies three fundamental requirements:

(i) The set $\Xi^l_k(\widehat{T})$ has to be a basis of the dual space $\mathcal{X}^l_k(\widehat{T})'$, a property that is called *unisolvence.*

(ii) The functionals $\hat{\kappa}_i$ have to fit our policy to exploit affine equivalence in that they remain invariant with respect to the pullback of differential forms. This makes sure that the arbitrary choice of the reference element has no impact.

(iii) We demand a certain *locality*: Let $M_i(\widehat{T})$ stand for the set of all sub-simplices of $\widehat{T}$ of dimension i, $0 \leq i \leq n$, spanned by $i+1$ vertices of $\widehat{T}$. Then for each $S \in M_i(\widehat{T})$, $l \leq i \leq n$, we must be able to find sets $\Xi^l_k(S) \subset \Xi^l_k(\widehat{T})$ such that for $\omega \in X^l_k(\widehat{T})$

$$\mathbf{t}_S\omega = 0 \quad \Longleftrightarrow \quad \kappa(\omega) = 0 \quad \forall \kappa \in \Xi^l_k(S) \; . \tag{22}$$

To see, why the third condition enables us to localize the degrees of freedom, I point out that

$$\Xi^l_k(S_1) \cap \Xi^l_k(S_2) = \Xi^l_k(S_1 \cap S_2) \quad , \quad S_1, S_2 \in M(\widehat{T}) \; , \tag{23}$$

where $M(\widehat{T})$ is the set of all sub-simplices of $\widehat{T}$. This equation arises from combining the locality condition of the d.o.f. with the splitting

$$\begin{aligned} &\{\omega \in X^l_k(\widehat{T}),\, \mathbf{t}_{S_1 \cap S_2}\, \omega = 0\} \\ &= \{\omega \in X^l_k(\widehat{T}),\, \mathbf{t}_{S_1}\omega = 0\} + \{\omega \in X^l_k(\widehat{T}),\, \mathbf{t}_{S_2}\omega = 0\} \; . \end{aligned}$$

This is proved by using a basis for the space of l-forms on $\widehat{T}$ that is based on barycentric coordinate functionals. From (23) we conclude that $\Xi^l_k(S_1) \subset \Xi^l_k(S_2)$, if $S_1 \subset S_2$.

A closer scrutiny reveals that the third requirement means that each degree of freedom on the reference element is associated with an unique sub-simplex, on which it is said to be *supported*. We collect the d.o.f. supported on $S \in M(\widehat{T})$ in the set

$$\Upsilon^l_k(S) := \Xi^l_k(S) \setminus \left(\bigcup_{S' \subset S, S' \neq S} \Xi^l_k(S') \right) \; .$$

Vice versa, the trace of a discrete form onto S is uniquely determined by the degrees of freedom in $\Xi^l_k(S)$.

Locality makes it possible to turn local degrees of freedom into global ones by the following procedure: For each i-dimensional sub-simplex $S \in M_i(T_h)$ of the global triangulation pick some adjacent element $T_S \in T_h$ and define the set $\Xi^l_k(S, T_h)$ of global degrees of freedom associated with S by

$$\Upsilon^l_k(S, T_h) := \{\omega \in X^l_k(T_h, \Omega) \mapsto \widehat{\kappa}(\mathbf{\Phi}^*_{T_S}\omega),\, \widehat{\kappa} \in \Upsilon^l_k(\widehat{S})\} \; , \tag{24}$$

where $\mathbf{\Phi}_{T_S} : T_S \mapsto \widehat{T}$ is affine and $\widehat{S} = \mathbf{\Phi}_{T_S}(S) \in M_i(\widehat{T})$. Note that thanks to the patch condition and affine invariance the selection of T_S is irrelevant. Thus, we obtain the set of global degrees of freedom

$$\Xi^l_k(T_h, \Omega) := \bigcup_{S \in M_i(T_h), l \leq i \leq n} \Upsilon^l_k(S, T_h) \ .$$

It is clear that they are also locally supported on sub-simplices of T_h. In sum, any set of degrees of freedom on $\widehat{T}$ that fulfills 1.–3. generates a proper set of global degrees of freedom.

It turns out that the patch condition is also necessary for $\Upsilon^l_k(S, T_h)$ to be well defined. Consider the face $S := T_1 \cap T_2$, $T_1, T_2 \in T_h$ with related affine mappings $\mathbf{\Phi}_1 : T_1 \mapsto \widehat{T}$, $\mathbf{\Phi}_2 : T_2 \mapsto \widehat{T}$, then, if (24) is to be meaningful

$$\widehat{\kappa}(\mathbf{\Phi}_1^* \omega_1 - \mathbf{\Phi}_2^* \omega_2) = 0 \quad \forall \widehat{\kappa} \in \Xi^l_k(\mathbf{\Phi}_1(S)) \ ,$$

where $\omega_1 := \omega_{|T_1}$, $\omega_2 := \omega_{|T_2}$. As a consequence

$$\mathbf{t}_{\mathbf{\Phi}_1(S)}(\mathbf{\Phi}_1^* \omega_1 - \mathbf{\Phi}_2^* \omega_2) = 0 \ ,$$

and the traces have to be same, since $\mathbf{\Phi}_{1|S} = \mathbf{\Phi}_{2|S}$. The gist is that meaningful global degrees of freedom already guarantee the patch condition.

All these considerations would be futile, unless suitable local degrees of freedom can be found. I am going to give a positive answer to this question by providing some specimens of $\Xi^l_k(\widehat{T})$. Let me first motivate the construction: The general form of the functionals is immediate clear, as natural linear forms on spaces of differential forms are given by integrals. Using appropriate integrals we also get the invariance under pullback for free. Locality suggests that the d.o.f. should be based on integrals over sub-simplices of $\widehat{T}$. However, how can we integrate an l-form over S, $S \in M_i$, if $i > l$? Weighting with suitable $i - l$-forms provides the answer. Via these considerations I arrived at the following description of degrees of freedom:

DEFINITION 4.2 (Degrees of freedom) *For $S \in M_i(\widehat{T})$, $0 \leq i \leq n$, let $\{\eta_{1,S}, \dots, \eta_{d,S}\}$, $d = d(k, l, S) := \dim DP^{i-l}_{k-i+l}(S)$, denote a basis of $DP^{i-l}_{k-i+l}(S)$. Then we can choose the linear forms $\widehat{\kappa}_{m,S} : X^l_k(\widehat{T}) \mapsto \mathbb{R}$ given by*

$$\widehat{\kappa}_{m,S}(\omega) := \int_S \omega \wedge \eta_{m,S} \ , \quad S \in M_i(\widehat{T}) \, , \ l \leq i \leq n \, , \ 1 \leq m \leq d(k, l, S)$$

as degrees of freedom belonging to $X^l_k(\widehat{T})$*.*

If we take unisolvence for granted, the essential locality of these degrees of freedom is quickly established: For a sub-simplex S simply choose

$$\Upsilon^l_k(S) := \{\widehat{\kappa}_{m,S},\, m = 1, \dots, d(k,l,S)\} \quad , \quad \Xi^l_k(S) := \bigcup_{S' \subset S} \Upsilon^l_k(S') \; .$$

Note that for $S' \subset S$ the values $\widehat{\kappa}_{m,S'}(\omega)$ are determined by $\mathbf{t}_S\omega$. Yet the union of all $\widehat{\kappa} \in \Xi^l_k(S)$ exactly matches a set of d.o.f. that definition (4.2) proposes for $X^l_k(S)$ (taking into account affine equivalence). Now the fairly obscure requirement (5) comes into play. It ensures $\mathbf{t}_S X^l_k(\widehat{T}) = X^l_k(S)$. Therefore, unisolvence, which is now assumed to hold for $\Xi^l_k(S)$, enforces $\mathbf{t}_S\omega = 0$.

Let me remark that definition (4.2) permits us to compute the number of d.o.f. supported on $S \in M_i(\widehat{T})$, $l \leq i \leq n$, easily:

$$\sharp\Upsilon^l_k(S) = \dim DP^{i-l}_{k-i+l}(S) = \binom{i}{i-l}\binom{k+l}{i} \tag{25}$$

It remains to settle the very issue of unisolvence. We start with a technical lemma, whose proof can be looked up in [18]:

LEMMA 4.3 *Any* $\omega \in H^l_k(\widehat{T})$ *with vanishing differential* $d\omega$*, which satisfies* $\int_{\widehat{T}} \omega \wedge \eta = 0$ *for all* $\eta \in DP^{n-l}_{k-(n-l)}(\widehat{T})$ *has to be identically 0.*

THEOREM 4.4 (Unisolvence of degrees of freedom) *The degrees of freedom supplied by Definition (4.2) form a dual basis of* $X^l_k(\widehat{T})$*.*

Proof. A simple counting argument (cf. (25)) reveals that the number of degrees of freedom from definition (4.2) agrees with the dimension of $X^l_k(\widehat{T})$ computed in the previous section. It remains to be shown that they are linearly independent, i.e., that

$$\omega \in X^l_k(\widehat{T}) \quad , \quad \kappa(\omega) = 0 \quad \forall \kappa \in \Xi^l_k(\widehat{T}) \quad \Rightarrow \quad \omega = 0 \; .$$

We employ a "double induction" argument with respect to the dimension n (increasing) and the order l, $0 \leq l \leq n$, of the differential form (decreasing).

(I) For $n = 1$ the assertion of the lemma is trivial. For arbitrary n and $l = n$ we have $X^n_k(\widehat{T}) = DP^n_k(\widehat{T})$ and only one kind of degree of freedom remaining, namely those of the form

$$\int_{\widehat{T}} \omega \wedge \eta \quad \text{for} \quad \eta \in DP^0_k(\widehat{T}) \; . \tag{26}$$

For $\omega = \varphi\, dx_1 \wedge \ldots \wedge dx_n$, $\varphi \in P_k(\widehat{T})$, pick $\eta = \varphi$. Then (26) is equal to $\int_{\widehat{T}} \varphi^2\, d\mathbf{x}$. Thus the assumption of the lemma immediately implies $\varphi = 0$ and $\omega = 0$.

(II) Now, we admit general $n \in \mathbb{N}$ and $l \in \{0, \ldots, n-1\}$. Assume that the lemma holds true for differential forms of order $l+1$ and in any dimension smaller than n.

For any lower dimensional simplex $\widehat{S} \in M_i(\widehat{T})$, $l+1 \leq i \leq n$, integration by parts (1) establishes the equality

$$\int\limits_{\widehat{S}} d\omega \wedge \eta = \int\limits_{\widehat{S}} d(\omega \wedge \eta) - (-1)^l \int\limits_{\widehat{S}} \omega \wedge d\eta \ ,$$

with $\eta \in DP^{i-l-1}_{k-i+l+1}(\widehat{S})$. The second term must vanish, since, by Lemma (3.1), $d\eta \in DP^{i-l}_{k-i+l}(\widehat{S})$, which makes it belong to the space spanned by the "test polynomials" (weights) in Definition (4.2). To the first term we apply Stokes' theorem and we get

$$\int\limits_{\widehat{S}} d(\omega \wedge \eta) = \int\limits_{\partial\widehat{S}} \omega \wedge \eta \ .$$

Again, we have recovered a right hand side that can be written as a weighted sum of values of degrees of freedom. Hence, the first term must be zero, too.

By construction $d\omega \in X^{l+1}_k(\widehat{T})$. Above, we have shown

$$\int\limits_{\widehat{S}} d\omega \wedge \eta = 0 \quad \forall \eta \in DP^{i-l-1}_{k-i+l+1}(\widehat{S}) \quad , \forall \widehat{S} \in M_i(\widehat{T}),\, l+1 \leq i \leq n \ .$$

By the induction assumption with respect to l, this enforces $d\omega = 0$. This means for our particular choice of local ansatz spaces that $\omega \in DP^l_k(\widehat{T})$.

Requirement (5) tells us that for $F \in M_{n-1}(\widehat{T})$ $\mathbf{t}_F\omega \in X^l_k(F)$. As I pointed out before, the degrees of freedom for $X^l_k(\widehat{T})$ that belong to a face F are suitable degrees of freedom for $X^l_k(F)$. Relying on the induction assumption for $n-1$, we see that $\mathbf{t}_{\partial\widehat{T}}\omega = 0$.

In sum, ω complies with all assumptions of Lemma (4.3). We infer that $\omega = 0$. This completes one step of the induction. □

Remark. Often degrees of freedom of "interpolatory type" are introduced that rely on point values of vector representatives of discrete differential forms [16, 15]. They can be read as using special quadraturs schemes for the evaluation of weighted integrals like those occurring in definition (4.2). However, these degrees of freedom are no longer affine equivalent.

The previous theorem establishes the existence of at least one viable set of degrees of freedom on $\widehat{T}$. There is still some leeway left as to the choice of $\Xi^l_k(\widehat{T})$ and this goes beyond merely fixing the test forms $\eta_{m,S}$ in definition (4.2). Set $\Upsilon^l_k(S) = \{\kappa^S_1, \ldots, \kappa^S_d\}$, $d = d(k,l,S)$ from (25), and define

$$\check{\kappa}^S_i := \sum_{j=1}^{d} a^S_{ij}\kappa^S_j + \eta \quad , \quad \eta \in \operatorname{Span}\left\{ \bigcup_{S'\subset S, S'\neq S} \Xi^l_k(S') \right\} ,$$

where $\mathbf{A}^S := (a^S_{ij}) \in \mathbb{R}^{d,d}$ is to be regular. Please note that this matrix takes into account switching from one basis of $DP^{i-l}_{k-i+l}(S)$ to another in definition 4. Then the new set of degrees of freedom

$$\check{\Xi}^l_k(\widehat{T}) := \{\check{\kappa}^S_j, 1 \leq j \leq d(k,l,S),\, S \in M_i(\widehat{T}),\, l \leq i \leq n\}$$

meets all requirements, in particular locality as

$$\operatorname{Span}\left\{\Xi^l_k(S)\right\} = \operatorname{Span}\left\{\check{\Xi}^l_k(S)\right\} \quad \forall S \in M_i(\widehat{T}),\, l \leq i \leq n \,. \tag{27}$$

The entire transformation can be represented by a regular square matrix $\mathbf{A}$ with diagonal blocks $\mathbf{A}^S$. If the d.o.f. are arranged according to the dimension of the sub-simplices that support them, this matrix becomes block-triangular. Observe that $\sharp\Upsilon^l_k(S) = \sharp\check{\Upsilon}^l_k(S)$ and so (25) represents an invariant for possible sets of d.o.f. on $\widehat{T}$. The numbers for $n = 3$ are listed in table 2. A message is that by no means degrees of freedom can be supported on $S \in M_i(\widehat{T})$, if $i > l + k$.

Example. Consider second order edge elements, i.e. $n = 3$, $l = 1$, $k = 2$. On each edge three degrees of freedom are located, whereas each face holds six of them and three belong to the tetrahedron. An admissible transformation of degrees of freedom can be done in the following fashion: Incorporate into the degrees of freedom on faces those on the edges and enhance the volume-d.o.f.s by any contribution of other d.o.f.

Given global degrees of freedom, *canonical interpolation operators* (also called nodal projectors)

$$\Pi^l_k : D^l_0(\Omega) \mapsto X^l_k(\Omega; T_h)$$

are declared by assigning to a continuous differential form that unique discrete form with the same nodal values

$$\kappa(\omega - \Pi^l_k\omega) = 0 \quad \forall \kappa \in \Xi^l_k(T_h, \Omega) \,.$$

k		0	1	2	3	4	5	6
$l=0$	vertices	1	1	1	1	1	1	1
	edges	0	1	2	3	4	5	6
	faces	0	0	1	3	6	10	15
	cell	0	0	0	1	4	10	20
$l=1$	edges	1	2	3	4	5	6	7
	faces	0	2	6	12	20	30	42
	cell	0	0	3	12	30	60	105
$l=2$	faces	1	3	6	10	15	21	28
	cell	0	3	12	30	60	105	168

Table 2. Numbers of d.o.f. associated with subsimplices for tetrahedral discrete differential forms.

Owing to (27), any suitable set $\Xi^l_k(T_h,\Omega)$ yields the same projector as they all span the same space of functionals. So Π^l_k is well defined even if we are sloppy about the concrete set $\Xi^l_k(T_h,\Omega)$. Locality of degrees of freedom makes the nodal interpolation operator commute with the trace onto a collection of sub-simplices. For the reference simplex and the associated local projector $\widehat{\Pi}^l_k$ this can be stated as

$$\mathbf{t}_S \circ \widehat{\Pi}^l_k = \widehat{\Pi}^l_k(S) \circ \mathbf{t}_S \quad , \forall S \in M(\widehat{T}) \, , \tag{28}$$

where $\widehat{\Pi}^l_k(S)$ is the nodal projector $\widehat{\Pi}^l_k(S) : D^l_0(S) \mapsto X^l_k(S)$.

Another important consequence of the choice of the global degrees of freedom is the “commuting diagram property”:

THEOREM 4.5 (Commuting diagram property) *Given the above definitions of the spaces and the degrees of freedom the following diagram commutes:*

$$\begin{array}{ccc} D^l_1(\Omega) & \xrightarrow{d} & D^{l+1}_0(\Omega) \\ \Pi^l_k \big\downarrow & & \big\downarrow \Pi^{l+1}_k \\ X^l_k(\Omega;T_h) & \xrightarrow{d} & X^{l+1}_k(\Omega;T_h) \end{array}$$

Proof. To begin with, we remark that the statement of the theorem is purely local and affine invariant. Hence, proving it for the reference element $\widehat{T}$ and the associated nodal projection $\widehat{\Pi}^l_k$ will do. We use the set $\Xi^l_k(\widehat{T})$ provided by definition (4.2).

For $\omega \in D^l_\infty(\widehat{T})$, $0 \leq l < n$, we set $\pi := \omega - \widehat{\Pi}^l_k \omega$. By the definition of the nodal interpolation operator we see that for $o \in M_i(\widehat{T})$, $l \leq i \leq n$,

$$\int_o \pi \wedge \eta = 0 \quad \text{for all } \eta \in DP^{i-l}_{k-i+l}(o)\ . \tag{29}$$

The remainder of the proof relies on the same ideas as the proof of Theorem 4.4: Integrating by parts we get for $\eta \in DP^{i-l-1}_{k-i+l+1}(o)$, $l+1 \leq i \leq n$, $o \in M_i(\widehat{T})$

$$\int_o d\pi \wedge \eta = \int_o d\,(\pi \wedge \eta)\ -\ (-1)^l \int_o \pi \wedge d\eta\ .$$

The second integral evaluates to zero as $d\eta \in DP^{i-l}_{k-i+l}(o)$. The first term is eligible for an application of Stokes' theorem:

$$\int_o d\,(\pi \wedge \eta) = \int_{\partial o} \pi \wedge \eta$$

Obviously $\partial o \in M_{i-1}(\hat{T})$ for $o \in M_i(\hat{T})$, so that (29) also forces the first term to vanish. Thus follows

$$\widehat{\Pi}^{l+1}_k\, d\left(\omega - \widehat{\Pi}^l_k\, \omega\right) = 0\ ,$$

and, since $d\left(\widehat{\Pi}^l_k\, \omega\right) \in X^{l+1}_k$, we obtain through the unisolvence of the degrees of freedom

$$\widehat{\Pi}^{l+1}_k\ (d\omega) = d\left(\widehat{\Pi}^l_k\, \omega\right)\ . \tag{30}$$

Taking into account that nodal values are left unchanged by canonical affine transformations, we get (30) for every element and, finally, for the entire finite element spaces. □

An immediate consequence of this theorem should be mentioned:

COROLLARY 4.6 *The nodal interpolation operators preserve the kernels of the external derivative, i.e.*

$$d\omega = 0 \ \textit{ for } \omega \in X^l_k(\Omega, T_h) \quad \Longrightarrow \quad d\left(\Pi^l_k\, \omega\right) = 0$$

A second corollary to theorem (4.5) has to do with the *p-hierarchical splitting* of higher order finite element spaces. It is naturally induced by the family of nodal interpolation operators parameterized by the polynomial order k. Denote the p-hierarchical components by

$$\widetilde{X}^l_k(\Omega; T_h) := \left(\Pi^l_k - \Pi^l_{k-1}\right) X^l_k(\Omega; T_h) \qquad (k \geq 1) . \tag{31}$$

COROLLARY 4.7 *With the notations introduced above we have*

$$d\, \widetilde{X}^l_k(\Omega; T_h) \subset \widetilde{X}^{l+1}_k(\Omega; T_h) ,$$

i.e. the exterior derivative respects the p-hierarchical splitting.

5. HIERARCHICAL BASES

Remember that fixing the set $\Xi^l_k(\widehat{T})$ amounts to fixing a basis $B^l_k := \{\beta_\kappa,\, \kappa \in \Xi^l_k(\widehat{T})\}$ of $X^l_k(\widehat{T})$ by duality, i.e. $\kappa'(\beta_\kappa) = \delta_{\kappa,\kappa'}$, where δ is Kronecker's symbol. In the sequel I am taking for granted that $\Xi^l_k(\widehat{T})$ is at our disposal.

Above we have investigated admissible transformations of $\Xi^l_k(\widehat{T}) \to \check{\Xi}^l_k(\widehat{T})$, which directly translate into changes of bases $B^l_k \to \check{B}^l_k$. To explain the mechanism, let me pick some sub-simplex S of $\widehat{T}$. We have seen that any transformation leaves $\mathrm{Span}\left\{\Xi^l_k(S)\right\}$ invariant. A basis form β_κ belonging to a degree of freedom $\kappa \in \Xi^l_k(\widehat{T})$ will therefore be mapped into another form $\check{\beta}$ that satisfies

$$\forall S \in M(\widehat{T}),\, \kappa \notin \Xi^l_k(S) : \quad \forall \kappa' \in \Xi^l_k(S) : \quad \kappa'(\check{\beta}) = 0 .$$

Equivalently, due to (23),

$$S \in M(\widehat{T}) \quad , \quad \kappa \in \Upsilon^l_k(S) \quad \Rightarrow \quad \mathbf{t}_{S'}\beta_\kappa = 0 \quad \forall S' \in M(\widehat{T}),\, S \not\subset S' .$$

After all, we have found invariant subspaces

$$Y^l_k(S) := \{\omega \in X^l_k(\widehat{T}),\, \mathbf{t}_{S'}\omega = 0 \quad \forall S' \in M(\widehat{T}),\, S \not\subset S'\}$$

with respect to legal changes of bases. A similar insight can be gained from the matrix description of the transformation $\Xi^l_k(\widehat{T}) \to \check{\Xi}^l_k(\widehat{T})$ through the matrix $\mathbf{A}$ (see Sect. 4). If the bases inherit the ordering of the degrees of freedom, the corresponding change of bases is described by the transposed matrix $\mathbf{A}^T$. Recall that $\mathbf{A}$ could be rearranged to become block-triangular. Eventually, this means that a basis form β_κ, $\kappa \in \Upsilon^l_k(S)$ can undergo only modifications through adding contributions from $Y^l_k(S)$.

Corollary 5.1. *Under admissible transformations the set* $\{\beta_\kappa,\, \kappa \in \Upsilon^l_k(S)\}$, $S \in M(\widehat{T})$, *is converted into a set of representatives of a basis of the quotient space*

$$\bar{Y}^l_k(S) := Y^l_k(S)/\{\omega \in Y^l_k(S),\, \mathbf{t}_S\omega = 0\} .$$

Example. Let us take a look at higher order edge elements, i.e. $n = 3$, $l = 1$, $k \geq 2$. A local basis according to definition (4.2) covers functions associated with edges, faces, and the entire reference element. During a legal change of bases, the functions at a single edge can be combined with each other and augmented by any contributions from functions belonging to adjacent faces and $\widehat{T}$. Basis functions at a face, apart from mixing them, can only receive contributions from the interior. Finally, the interior basis functions can only be reshuffled among themselves.

In addition, the union of all sets of representatives of bases of $\bar{Y}^l_k(S)$, $S \in M(\widehat{T})$, will yield a valid basis of $X^l_k(\widehat{T})$. As $\bar{Y}^l_k(S)$ is isomorphic to $H^l_k(S) := \mathbf{t}_S Y^l_k(S)$ we only have to come up with bases of $H^l_k(S)$, $S \in M(\widehat{T})$ and some extension procedure in order to find a basis of $X^l_k(\widehat{T})$.

Example. Barycentric coordinates are a convenient tool for stating discrete differential forms [16]. They do not violate the coordinate-free setting, because they are a completely affine concept. Writing $\lambda_0, \ldots, \lambda_n$ for the the barycentric coordinate functions with respect to $\widehat{T}$, we find for $k, l \in \mathbb{N}_0$, $k + l \geq n$,

$$\{\omega \in DP^l_{k+1}(\widehat{T}),\, \mathbf{t}_{\partial\widehat{T}}\omega = 0\} = \operatorname{Span}\left\{\begin{array}{l} p_I(\lambda_0, \ldots, \lambda_n)\lambda_{I'}d\lambda_I, \\ I' \cup I = \{0, \ldots, n\}, \\ \sharp I = l,\, p_I \in \tilde{\boldsymbol{P}}_{k+l-n}(\mathbb{R}^{n+1}) \end{array}\right\}, \tag{32}$$

where $\lambda_{I'} = \prod_{j\in I'} \lambda_j$, $d\lambda_I = \lambda_{i_1} \wedge \ldots \wedge \lambda_{i_l}$. Hence, we can write $\gamma \in H^l_k(S)$, $S \in M_i(\widehat{T})$, in the form

$$\gamma = \sum_{I \subset \{j_0, \ldots, j_i\}} p_I(\lambda_{j_0}, \ldots, \lambda_{j_i})\lambda_{I'}d\lambda_I$$

where $\lambda_{j_0}, \ldots, \lambda_{j_i}$ are those barycentric coordinates of $\widehat{T}$ that do not vanish on S. This representation instantly provides an extension to a

$\gamma \in Y_k^l$ by simply reinterpreting the barycentric coordinate functions of S as those of $\widehat{T}$.

I aim at finding a p–hierarchical basis for $X_k^l(\widehat{T})$. This means that the basis can be split into subsets spanning $\widetilde{X}_k^l(\widehat{T})$. The latter space is defined according to (31). In light of (28) and the preceding considerations, I first focus on hierarchical bases for $H_k^l(S)$, $S \in M_i(\widehat{T})$, $l \leq i \leq n$. Fix $S \in M_i(\widehat{T})$ and consider the direct p–hierarchical decomposition

$$H_k^l(S) = \sum_{p=0}^{k} \widetilde{H}_p^l(S) \quad , \quad \widetilde{H}_p^l(S) := (\widehat{\Pi}_p^l(S) - \widehat{\Pi}_{p-1}^l(S))H_k^l(S) \;\text{ for }\; p > 0 \;.$$

Then, for $p > 0$ pick a closed form $\omega \in \widetilde{H}_p^l(S)$. If $i = l$, the fact that ω belongs to a genuine hierarchical surplus ($p > 0$) implies $\int_S \omega = 0$. Thus, we can apply (20) which bears out the existence of $\eta' \in H_p^{l-1}(S)$ such that $d\eta' = \omega$. Then, set

$$\eta := (\widehat{\Pi}_p^{l-1}(S) - \widehat{\Pi}_{p-1}^{l-1}(S))\eta' \;.$$

By definition $\eta \in \widetilde{H}_p^{l-1}(S)$ and from corollary (4.7) we learn that

$$d\eta = (\widehat{\Pi}_p^{l-1}(S) - \widehat{\Pi}_{p-1}^{l}(S))\omega = \omega \;.$$

The bottom line is that the exact sequence property even carries over to the higher order components of the p–hierarchical splitting of $H_k^l(S)$. We introduce a further direct splitting

$$\widetilde{H}_p^l(S) = d(\widetilde{H}_p^{l-1}(S)) \oplus \widetilde{C}_p^l(S) = d\,\widetilde{C}_p^{l-1}(S) \oplus \widetilde{C}_p^l(S)$$

without worrying about the details of the complement spaces $\widetilde{C}_k^l$, whose mere existence will be needed. The dimensions of these spaces satisfy the simple recurrence

$$\begin{gathered}\dim \widetilde{C}_p^l(S) + \dim \widetilde{C}_p^{l-1}(S) = \\ \dim \widetilde{H}_p^l(S) = \sharp\Upsilon_p^l(S) - \sharp\Upsilon_{p-1}^l(S) = \binom{i}{l}\binom{p+l-1}{i-1} \;,\end{gathered}$$

and in addition $\dim \widetilde{C}_p^l(S) = 0$, if $l > i$ or $p + l < i$, where i is the dimension of S. For $n = 3$ the dimensions are listed in table 3

In particular, we conclude

p		[0]	1	2	3	4	5	6
$l=0$	vertices	[1]	0	0	0	0	0	0
	edges	[0]	1	1	1	1	1	1
	faces	[0]	0	1	2	3	4	5
	cell	[0]	0	0	1	3	6	10
$l=1$	edges	[1]	0	0	0	0	0	0
	faces	[0]	2	3	4	5	6	7
	cell	[0]	0	3	8	15	24	35
$l=2$	faces	[1]	0	0	0	0	0	0
	cell	[0]	3	6	10	15	21	28

Table 3. Dimensions of $C_p^l(S)$ for a tetraahedron.

- $\dim \widetilde{C}_p^i(S) = 0$, i.e. for $p > 0$ the space $\widetilde{H}_p^i(S)$ contains only closed forms.
- $\dim \widetilde{C}_p^{i-p-1}(S) = 0$, i.e. $\widetilde{H}_p^{i-p}(S)$ does not contain any non-trivial closed forms.

Example. Let us revisit the example in section 1, i.e. $n = 2$, $k = 1$, and $l = 1$. We want to find a basis for $\widetilde{C}_1^1(\widehat{T})$ in this case. The desired basis forms must have non-vanishing contributions from $\widetilde{DP}_2^1(\widehat{T})$ and those contributions have to be in the range of $k_0(\widetilde{HN}_1^2(\widehat{T}))$. Thus, the second order parts of the basis forms have to comply with (19). In addition, traces on $\partial\widehat{T}$ have to be zero. Taking into account (32), this restricts the search to the set

$$\{\omega = A\lambda_1\lambda_2 d\lambda_0 + B\lambda_0\lambda_2 d\lambda_1 + C\lambda_0\lambda_1 d\lambda_2\} \ .$$

Pick $\widehat{T}$ as reference triangle with respect to the chosen coordinates, i.e. $\widehat{T} := \{\mathbf{x} \in \mathbb{R}^2,\ 0 \leq x_1, x_2 \leq 1,\ x_1 + x_2 \leq 1\}$. Then $\lambda_0 = 1 - x_1 - x_2$, $\lambda_1 = x_1$, and $\lambda_2 = x_2$. In the canonical basis for 1-forms we can express ω by

$$\begin{gathered}\omega = (B - (A+B)x_1 - Bx_2)x_2 dx_1 + (C - Cx_1 - (C+A)x_2)x_1 dx_2 \\ A, B, C \in \mathbb{R} \ .\end{gathered}$$

Equating the coefficients for corresponding monomials of order 2, we get (with α_1, α_2 from (19))

$$\frac{1}{3}\alpha_1 = A + B \quad , \quad \frac{1}{3}\alpha_2 = B \quad , \quad \frac{1}{3}\alpha_1 = -C \quad , \quad \frac{1}{3}\alpha_2 = -(C + A) \ .$$

A solution exists for any $\alpha_1, \alpha_2 \in \mathbb{R}$, e.g. $\alpha_1 = 3$, $\alpha_2 = 0$ leads to $A = 1, B = 0, C = -1$, whereas $\alpha_1 = 0$, $\alpha_2 = 3$ means $A = 1, B = -1, C = 0$. Then we arrive at the following two basis forms for $\widetilde{C}_1^1(\widehat{T})$

$$\beta_1 = -x_1 x_2 dx_1 + (x_1 - 1)x_1 dx_2 \quad , \quad \beta_2 = (1 - x_2)x_2 dx_1 - x_2 x_1 dx_2 \; .$$

Here no hierarchical surplus has to be computed, as there are no non-trivial discrete 1-forms of order 0 that are supported in the interior of a triangle.

To continue with the construction of hierarchical bases, we select bases $\{\zeta_{p,1}^{l,S}, \ldots, \zeta_{p,K}^{l,S}\}$, $K = K(l,p,S) := \dim \widetilde{C}_p^l(S)$, of $\widetilde{C}_p^l(S)$. The freedom at this stage will mean that many different instances of p-hierarchical bases are possible. Since $H_k^l(S) := \mathbf{t}_S Y_k^l(S)$ these basis forms can be extended to forms $\{\gamma_{p,1}^{l,S}, \ldots, \gamma_{p,K}^{l,S}\}$ in $Y_k^l(S)$. Finally set

$$\beta_{p,j}^{l,S} := (\widehat{\Pi}_p^l - \widehat{\Pi}_{p-1}^l)\gamma_{p,j}^{l,S} \quad , \; 1 \le j \le K \; .$$

As $Y_k^l(S)$ is invariant under the nodal projectors $\widehat{\Pi}_p^l$, $0 \le p \le k$, the $\beta_{p,j}^{l,S}$ still belong to $Y_k^l(S)$. Thanks to (28) we know $\mathbf{t}_S \beta_{p,j}^{l,S} = \zeta_{p,j}^{l,S}$. In sum, $\beta_{p,1}^{l,S}, \ldots, \beta_{p,K}^{l,S}$ are representatives of linearly independent elements of $\bar{Y}_k^l(S)$ and they respect the hierarchical decomposition.

Now we are in a position to describe the hierarchical bases for $X_k^l(\widehat{T})$ explicitly through

$$B_0^l \cup \left\{ \begin{array}{l} \beta_{p,j}^{l,S}, d\beta_{p,m}^{l,S}, \, 1 \le j \le K(l,p,S), \, 1 \le m \le K(l-1,p,S), \\ S \in M_i(\widehat{T}), \, 0 \le i \le n, \, 1 \le p \le k \end{array} \right\} . \tag{33}$$

Linear independence of the forms is clear from the way they have been constructed. As well, their construction guarantees the locality property. However, can any $\omega \in X_k^l(\widehat{T})$ be expressed as a linear combination of forms from (33)? Start with the p–hierarchical decomposition of ω

$$\omega = \omega_0 + \ldots + \omega_k \quad , \quad \omega_0 := \widehat{\Pi}_0^l \omega \, , \; \omega_p := (\widehat{\Pi}_p^l - \widehat{\Pi}_{p-1}^l)\omega, \, 0 \le p \le k \; .$$

Then pick $S \in M_l(\widehat{T})$ and express $\mathbf{t}_S \omega_p \in \widetilde{X}_p^l(S)$ through $\{\zeta_{p,1}^{l,S}, \ldots, \zeta_{K,p}^{l,S}, d\zeta_{p,1}^{l-1,S}, \ldots, d\zeta_{M,p}^{l-1,S}\}$. Subtracting the same linear combination of $\{\beta_{p,1}^{l,S}, \ldots, \beta_{p,K}^{l,S}, d\beta_{p,1}^{l-1,S}, \ldots, d\beta_{p,M}^{l-1,S}\}$ from ω_p yields

$\omega_p' \in \widetilde{X}_p^l(\widehat{T})$ with vanishing trace on S. Carrying out the same procedure for the other sub-simplices in $M_l(\widehat{T})$ results in a form $\bar{\omega} \in \widetilde{X}_p^l(\widehat{T})$ whose trace on all l-dimensional subsimplices is zero. Choosing $S \in M_{l+1}(\widehat{T})$ this means $\mathbf{t}_S\bar{\omega} \in H_p^l(S)$ and we can repeat the construction for $l+1$-dimensional sub-simplices. This can be done until we reach $S = \widehat{T}$. I point out that a closed ω_p, $p \geq 1$, will finally be represented solely through closed basis forms. This should be remembered as an interesting feature of (33).

As the hierarchical basis (33) complies with all requirements, it induces valid local degrees of freedom and, thus, a corresponding basis of the global space $X_k^l(T_h, \Omega)$. The global basis retains all affine-invariant, say all essential, features of the local hierarchical basis of $X_k^l(\widehat{T})$:

THEOREM 5.2 *Given a closed discrete differential form in the hierarchical surplus space $\widetilde{X}_p^l(T_h, \Omega)$, $p \geq 1$, it can be written as linear combination of closed forms with local support.*

Proof Everything has already been settled by the above considerations. The local contributions are spawned by exterior derivatives of hierarchical basis functions for discrete $l-1$-forms.□

Remark. Recursion lends itself for the practical construction of the hierarchical basis: Build hierarchical bases $H_k^l(\widehat{T})$ for in dimensions $l, \ldots, n-1$ first. Then extend them to elements of the hierarchical basis of $X_k^l(\widehat{T})$. Extension is straightforward, if barycentric coordinates are used to represent the differential forms.

The hierarchical basis is also the key to building spaces of discrete differential forms on T_h, whose polynomial degree may vary from element to element. For each $T \in T_h$ we fix $p_T \in \mathbb{N}_0$, which stands for the desired polynomial degree on that particular element. Then define

$$p_S := \min\{p_T,\ S \subset T\} \quad , \quad S \in M(T_h)\ . \tag{34}$$

as the polynomial degree associated with each sub-simplex S of the mesh. As is the case with any admissible basis, the forms in the global hierarchical basis of $X_k^l(T_h, \Omega)$ belong to exactly one sub-simplex of T_h. The global p–adaptive basis is then given by

$$B_{\{p_T\}}^l := \bigcup_{S \in M(T_h)} \{\text{ Basis } l\text{-forms of degree } \leq p_S \text{ associated with } S\ \}\ .$$

Of course, the global p–adaptive space of discrete l-forms is defined as

$$X_{\{p_T\}}^l(\Omega, T_h) := \operatorname{Span}\left\{B_{\{p_T\}}^l\right\}\ .$$

THEOREM 5.3 *If Ω is contractible, the spaces $X^l_{\{p_T\}}(\Omega, T_h)$, $0 \leq l \leq n$, possess the exact sequence property.*

Proof. If $\omega \in X^l_{\{p_T\}}(\Omega, T_h)$ is closed we can split it into two closed forms according to

$$\omega = \omega_0 + \widetilde{\omega}, \quad \omega_0 = \Pi^l_0 \; .$$

As seen before, the hierarchical surplus $\widetilde{\omega}$ can only comprise closed basis forms of $B^l_{\{p_T\}}$ that are derivatives of hierarchical basis $l-1$-forms of the same polynomial degree and belonging to the same sub-simplex:

$$\exists \eta \in X^{l-1}_{\{p_T\}}(\Omega, T_h) \ : \quad \widetilde{\omega} = d\eta \; .$$

The existence of a discrete potential for lowest order discrete differential forms (Whitney-forms) is a result from topology.□

Remark. It turns out that the existence of discrete potentials is crucial for the discrete compactness property of edge elements (discrete 1-forms) [21, 10, 5]. Discrete compactness is a necessary condition for the convergence of Galerkin finite element solutions of eigenvalue problems.

Of course, $B^l_{\{p_T\}}(\Omega, T_h)$ is not the only possible basis for $X^l_{\{p_T\}}$ and there is ample room for better choices following the prescriptions from the beginning of the paragraph. For finite element applications changing to another basis is even recommended in order to improve the condition number of the resulting system of linear equations [2].

REFERENCES

1. Andersen, L. and J. Volakis, "Development and application of a novel class of hierarchical tangential vector finite elements for electromagnetics," *IEEE Trans. Antennas and Propagation,* Vol. 47, 112–120, 1999.
2. Babuška, I., M. Griebel, and J. Pitkaränta, "The problem of selecting the shape functions for a p-type finite element," *Int. J. Num. Meth. Engin.,* Vol. 28, 1891–1908, 1988.
3. Baldomir, D. and P. Hammond, *Geometry of Electromagnetic Systems,* Clarendon Press, Oxford, 1996.
4. Beck, R., R. Hiptmair, and B. Wohlmuth, "A hierarchical error estimator for eddy current computation," *ENUMATH 99 — Proceedings of the 3rd European Conference on Numerical Mathematics and Advanced Applications,* P. Neittaanmäki and T. Tiihonen (eds.), July 26–30, Jyvskyl, Finland, 110–120, World Scientific, Singapore, 2000.

5. Boffi, D., "Discrete compactness and Fortin operator for edge elements," *Tech. Rep. AM187,* Department of Mathematics, Pennsylvania State University, State College, USA, April 1999. To appear in Numerische Mathematik.
6. Bossavit, A., "Mixed finite elements and the complex of Whitney forms," *The Mathematics of Finite Elements and Applications VI,* J. Whiteman (ed.), 137–144, Academic Press, London, 1988.
7. Bossavit, A., "Whitney forms: A class of finite elements for three-dimensional computations in electromagnetism," *IEE Proc. A,* Vol. 135, 493–500, 1988.
8. Bossavit, A., "On the geometry of electromagnetism IV: "Maxwell's house," *J. Japan Soc. Appl. Electromagnetics & Mech.,* Vol. 6,318–326, 1998.
9. Brenner, S. and R. Scott, *Mathematical Theory of Finite Element Methods,* Texts in Applied Mathematics, Springer-Verlag, New York, 1994.
10. Caorsi, S., P. Fernandes, and M. Raffetto, "On the convergence of Galerkin finite element approximations of electromagnetic eigenproblems," *SIAM J. Numer. Anal.,* To appear.
11. Cartan, H., *Formes Différentielles,* Hermann, Paris, 1967.
12. Ciarlet, P., "The finite element method for elliptic problems," *Studies in Mathematics and its Applications,* Vol. 4, North-Holland, Amsterdam, 1978.
13. Demkowicz, L., P. Monk, L. Vardapetyan, and W. Rachowicz, "De Rham diagram for *hp* finite element spaces," *Tech. Rep.,* 99-06, TICAM, University of Texas, Austin, TX, 1999.
14. Deschamps, G., "Electromagnetics and differential forms," *Proc. IEEE,* Vol. 69, 676–695, 1981.
15. Elmkies, A. and P. Joly, "Eléments finis et condensation de masse pour les équations des Maxwell: le cas 3D," *Tech. Rep.,* 3381, INRIA, Roucquencourt, Domaine de Voluveau, France, May 1998.
16. Graglia, R., D. Wilton, and A. Peterson, "Higher order interpolatory vector bases for computational electromagnetics," *IEEE Trans. Antennas and Propagation,* Vol. 45, 329–342, 1997.
17. Graglia, R., D. Wilton, A. Peterson, and I.-L. Gheorma, "Higher order interpolatory vector bases on prism elements," *IEEE Trans. Antennas and Propagation,* Vol. 46, 442–450, 1998.
18. Hiptmair, R., "Canonical construction of finite elements," *Math. Comp.,* Vol. 68,1325–1346, 1999.
19. Iwaniec, T., "Nonlinear differential forms," "Lectures notes of the International Summer School in Jyvaskyla," 1998 80, University of

Jyväskylä, Department of Mathematics, Jyväskylä, Finland, 1999.

20. Monk, P., "On the p and hp-extension of Nédélec's conforming elements," *J. Comp. Appl. Math.,* Vol. 53, 117–137, 1994.
21. Monk, P. and L. Demkowicz, "Discrete compactness and the approximation of Maxwell's equations in $\mathbb{R}^3$," *Math. Comp.,* 1999, to appear.
22. Nédélec, J., "Mixed finite elements in R^3," *Numer. Math.,* Vol. 35, 315–341, 1980.
23. Nédélec, J., "A new family of mixed finite elements in R^3," *Numer. Math.,* Vol. 50, 57–81, 1986.
24. Peng, G., R. Dyczij-Edlinger, and J.-F. Lee, "Hierarchical methods for solving matrix equations from TVFEMs for microwave components," *IEEE Trans. Mag.,* Vol. 35, 1474–1477, 1998.
25. Raviart, P. A. and J. M. Thomas, "A mixed finite element method for second order elliptic problems," *Springer Lecture Notes in Mathematics,* Vol. 606, 292–315, Springer-Verlag, New York, 1977.
26. Savage, J. and A. Peterson, "Higher order vector finite elements for tetrahedral cells," *IEEE Trans. Microwave Theory and Techniques,* Vol. 44, 874–879, 1996.
27. Whitney, H., *Geometric Integration Theory,* Princeton Univ. Press, Princeton, 1957.
28. Yiailtsis, T. and T. Tsiboukis, "A systematic approach to the construction of higher order vector finite elements for three-dimensional electromagnetic field computation," *COMPEL,* Vol. 14, 49–53, 1995.