

CYLINDRICAL VECTOR WAVE FUNCTION REPRESENTATIONS OF THE DYADIC GREEN'S FUNCTIONS FOR CYLINDRICAL MULTILAYERED GYROTROPIC BIANISOTROPIC MEDIA

E. L. Tan and S. Y. Tan

School of Electrical and Electronic Engineering
Nanyang Technological University
Singapore 639798, Republic of Singapore

- 1. Introduction**
- 2. Eigenfunction Expansions of Electromagnetic Fields**
- 3. Unbounded Dyadic Green's Functions**
- 4. Scattered Dyadic Green's Functions**
- 5. Application to a Coated Conducting Cylinder**
- 6. Conclusion**

Appendices

References

1. INTRODUCTION

For several decades, the dyadic Green's function technique has been one of the basic tools for analyzing electromagnetic radiation, propagation and scattering problems [1–4]. Many researchers have shown considerable interest in constructing and applying the dyadic Green's functions for various layered media of different geometries. Among the canonical problems which admit closed-form (spectral domain) solutions, the dyadic Green's functions for planar stratified media have so far received the greatest attention and have been the most well-developed up to present. These planar layered media include examples for isotropic [1–4], chiral/biisotropic [5] and even more complex media [6–10]. For other geometries such as cylindrical multilayered structures, their

material constituents are mostly restricted to simpler isotropic [1–4, 11] and chiral/biisotropic [12, 13] types owing to the complexity of more general media. Recently, the dyadic Green's functions for unbounded lossless reciprocal uniaxial bianisotropic media have been formulated in terms of commonly employed cylindrical vector wave functions [14, 15]. Such representation provides the framework for studying multilayered structures composed of these media. However, the solutions did not include explicit singularities required for complete expansion in source regions [1]. These singularities have been taken into account in the dyadic Green's function solution for unbounded gyroelectric medium [16].

In this paper, we present a rigorous and concise formulation of the complete eigenfunction expansions of the dyadic Green's functions for cylindrical multilayered media. The media may consist of any number of layers bounded by optional impedance/admittance walls. The material constituents of the media layers are allowed to be of the most general coaxially gyrotropic bianisotropic type [17]. These materials thus embrace the gyroelectric and reciprocal uniaxial bianisotropic media mentioned above. Both electric and magnetic dyadic Green's functions attributed to arbitrary field and source locations are derived simultaneously. Making use of the principle of scattering superposition, these dyadics are decomposed into unbounded and scattered parts. The unbounded parts are expanded following the approach described in [18]. In particular, based on the theory of distributions, the singularities and discontinuities are deduced directly from Maxwell equations cast in dyadic forms. Moreover, the discontinuity relations obtained are fully utilized to construct the eigenfunction expansions outside the source point. These eigenfunctions are expressed in terms of an alternative more expedient set of cylindrical vector functions, which have been exploited earlier in [19] stating their relationships to those commonly employed ones. The scattered parts which account for the scattered waves caused by layer interfaces and/or walls are composed of scattering coefficient matrices. Applying the effective reflection and transmission concepts illustrated in [4, 5, 20], these scattering coefficients are determined in terms of global reflection and transmission matrices without cumbersome operations while providing good physical insights into the scattering mechanism. Corresponding to the impedance/admittance boundary walls, the global reflection matrices are related directly to the wall impedance/admittance dyadics. To illustrate the application

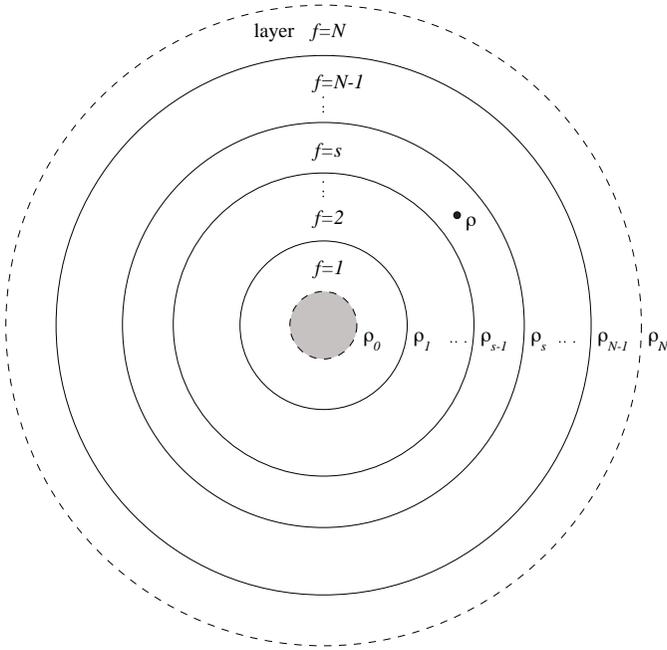


Figure 1. Geometry of cylindrical multilayered gyrotropic bianisotropic media. The media may be bounded by optional impedance/admittance walls at $\rho = \rho_0$ and $\rho = \rho_N$. Point source is impressed at $\bar{r} = \bar{r}'$ ($\rho = \rho'$) within layer s .

of general expressions obtained, the configuration of a perfect conducting cylinder coated with gyrotropic bianisotropic medium is considered. Throughout the following analysis, $e^{-i\omega t}$ time dependence is assumed and suppressed.

2. EIGENFUNCTION EXPANSIONS OF ELECTROMAGNETIC FIELDS

Consider a circular cylindrical multilayered structure comprising N layers of gyrotropic bianisotropic media, Fig. 1. The core center is assumed to coincide with the \hat{z} axis of our cylindrical coordinate system $(\hat{\rho}, \hat{\phi}, \hat{z})$, which is also the common distinguished gyrotropic axis of the media constituents. The outermost (labeled N) and innermost (labeled 1) material layers can be extended to infinity and origin

or bounded by impedance/admittance walls at their outer and inner boundaries respectively. These optional boundary walls as well as the outer interface of each layer are denoted by $\rho = \rho_f$ ($f = 0, 1, \dots, N$). Within each material layer f ($f = 1, 2, \dots, N$), the medium is homogeneous and characterized by constitutive equations [2]

$$\overline{D}_f = \overline{\epsilon}_f \cdot \overline{E}_f + \overline{\xi}_f \cdot \overline{H}_f \quad (1)$$

$$\overline{B}_f = \overline{\zeta}_f \cdot \overline{E}_f + \overline{\mu}_f \cdot \overline{H}_f \quad (2)$$

where the constitutive dyadics/pseudodyadics take the form

$$\overline{\epsilon}_f = \epsilon_{tf} \overline{I}_{t_z} + \epsilon_{af} \hat{z} \times \overline{I} + \epsilon_{zf} \hat{z} \hat{z} \quad (3)$$

$$\overline{\mu}_f = \mu_{tf} \overline{I}_{t_z} + \mu_{af} \hat{z} \times \overline{I} + \mu_{zf} \hat{z} \hat{z} \quad (4)$$

$$\overline{\xi}_f = \xi_{tf} \overline{I}_{t_z} + \xi_{af} \hat{z} \times \overline{I} + \xi_{zf} \hat{z} \hat{z} \quad (5)$$

$$\overline{\zeta}_f = \zeta_{tf} \overline{I}_{t_z} + \zeta_{af} \hat{z} \times \overline{I} + \zeta_{zf} \hat{z} \hat{z}. \quad (6)$$

\overline{I} is the idemfactor and \overline{I}_{t_z} denotes its transverse-to- \hat{z} part, i.e., $\overline{I}_{t_z} = \overline{I} - \hat{z} \hat{z}$. Here, we are dealing with the most general coaxially gyrotropic bianisotropic medium which embraces many composite materials proposed recently for variety of potential applications such as anti-reflection coatings, phase shifters, polarization transformers, etc. [21, 22].

In general, the electric and magnetic fields in each medium layer consist of superposition of four sets of eigenfunctions. Each eigenfunction can be represented in angular spectrum domain in the form [15] ($j = 1, 2, 3, 4$)

$$\overline{E}_{jf} = a_{ejf} \overline{M}_{jf} + b_{ejf} \overline{N}_{jf} + c_{ejf} \overline{L}_{jf} \quad (7)$$

$$\overline{H}_{jf} = a_{hjf} \overline{M}_{jf} + b_{hjf} \overline{N}_{jf} + c_{hjf} \overline{L}_{jf} \quad (8)$$

where \overline{M} , \overline{N} , \overline{L} are the commonly employed cylindrical vector wave functions defined in, e.g., [4, p. 394]. Subscript jf refers to the radial wave number λ_{jf} in these wave functions. abc are the expansion coefficients to be determined from source-free Maxwell equations. Alternatively, one can make use of a more expedient set of cylindrical vector functions defined transverse and longitudinal to \hat{z} as

$$\overline{m} = \left[\hat{\rho} \frac{in}{\rho} Z_n(\lambda\rho) - \hat{\phi} \frac{\partial}{\partial \rho} Z_n(\lambda\rho) \right] e^{in\phi + iz} \quad (9)$$

$$\bar{n} = \left[\hat{\rho} \frac{\partial}{\partial \rho} Z_n(\lambda \rho) + \hat{\phi} \frac{in}{\rho} Z_n(\lambda \rho) \right] e^{in\phi + ihz} \quad (10)$$

$$\bar{l} = \hat{z} Z_n(\lambda \rho) e^{in\phi + ihz} \quad (11)$$

where $Z_n(\lambda \rho)$ denotes the Bessel function of appropriate kind depending on λ . The properties and identities for these vector functions along with their relationships to those commonly employed above have been discussed in [19]. Moreover, for $Z_n(\lambda \rho)$ chosen to be Hankel function of the first kind, i.e., $Z_n(\lambda \rho) \equiv H_n^{(1)}(\lambda \rho)$, the asymptotic expressions of \bar{m} , \bar{n} and \bar{l} (superscripted H) correspond directly to $\hat{\phi}$, $\hat{\rho}$, and \hat{z} components respectively as

$$\bar{m}^H \sim -i\lambda \sqrt{\frac{2}{\pi\lambda\rho}} e^{i\lambda\rho + ihz + in(\phi - \pi/2) - i\pi/4} \hat{\phi} \quad (12)$$

$$\bar{n}^H \sim i\lambda \sqrt{\frac{2}{\pi\lambda\rho}} e^{i\lambda\rho + ihz + in(\phi - \pi/2) - i\pi/4} \hat{\rho} \quad (13)$$

$$\bar{l}^H \sim \sqrt{\frac{2}{\pi\lambda\rho}} e^{i\lambda\rho + ihz + in(\phi - \pi/2) - i\pi/4} \hat{z} \quad (14)$$

These expressions have been derived using the well-known asymptotic value of Hankel function for large argument, as encountered in the case of far field consideration [1, 2]. Writing in terms of this set of vector functions, the eigenfunctions in each layer become

$$\bar{E}_{jf} = A_{ejf} \bar{m}_{jf} + B_{ejf} \bar{n}_{jf} + C_{ejf} \bar{l}_{jf} \quad (15)$$

$$\bar{H}_{jf} = A_{hjf} \bar{m}_{jf} + B_{hjf} \bar{n}_{jf} + C_{hjf} \bar{l}_{jf} \quad (16)$$

The ABC expansion coefficients are to be determined from source-free Maxwell equations as well. In particular, for the electric field eigenfunctions, their coefficients satisfy a homogeneous matrix equation

$$\begin{bmatrix} H_{tf} + \lambda_{jf}^2 \alpha_{zf} & -H_{af} & L_{tf} \\ H_{af} & H_{tf} & L_{af} \\ L'_{tf} & -L'_{af} & \alpha_{tf} - \omega^2 \delta_{zf} / \lambda_{jf}^2 \end{bmatrix} \cdot \begin{bmatrix} A_{ejf} \\ B_{ejf} \\ C_{ejf} \end{bmatrix} = \bar{0}. \quad (17)$$

The steps leading to this matrix have been detailed in [19]. For convenience, the elements are given explicitly in Appendix A. To cater for nontrivial solutions of (17), it is necessary that the determinant vanishes resulting in a biquadratic dispersion equation in λ_{jf} . Using this

λ_{jf} , one can then solve for the corresponding homogeneous solutions [19]. With reference to (9)–(16), we note that λ_{1f} , λ_{2f} ($\text{Im}\lambda_{2f} > 0$) should be associated with Hankel function $H_n^{(1)}(\lambda\rho)$ and λ_{3f} , λ_{4f} ($\lambda_{3f} = -\lambda_{2f}$) with Bessel function $J_n(\lambda\rho)$. (Occasionally, superscripts would be attached as in (12)–(14) for clarity: $Z_n^H(\lambda_{2f}\rho) \equiv H_n^{(1)}(\lambda_{2f}\rho)$, $Z_n^J(\lambda_{2f}\rho) \equiv J_n(\lambda_{2f}\rho)$.) Such an association yields regular solutions at $\rho \rightarrow \infty$ and $\rho \rightarrow 0$ in the outermost and innermost layers respectively.

In an alternative manner, the above homogeneous equation can be transformed into an eigensystem equation protruding the eigenvalues λ_{jf}^2 , e.g.,

$$\begin{bmatrix} -\frac{H_{tf}^2 + H_{af}^2}{\alpha_{zf} H_{tf}} & -\frac{H_{tf} L_{tf} + H_{af} L_{af}}{\alpha_{zf} H_{tf}} \\ \frac{(H_{tf}^2 + H_{af}^2)(H_{tf} L'_{tf} + H_{af} L'_{af})}{\alpha_{zf} H_{tf} (\alpha_{tf} H_{tf} + L_{af} L'_{af})} & \frac{(H_{tf} L_{tf} + H_{af} L_{af})(H_{tf} L'_{tf} + H_{af} L'_{af}) + \omega^2 \alpha_{zf} \delta_{zf} H_{tf}^2}{\alpha_{zf} H_{tf} (\alpha_{tf} H_{tf} + L_{af} L'_{af})} \end{bmatrix} \cdot \begin{bmatrix} A_{ejf} \\ C_{ejf} \end{bmatrix} = \lambda_{jf}^2 \begin{bmatrix} A_{ejf} \\ C_{ejf} \end{bmatrix}. \tag{18}$$

Although we have chosen here the unknowns to be A_{ejf} and C_{ejf} , one can equally well consider other coefficients. For example, the more commonly encountered ones would be C_{ejf} and C_{hjf} [23–25], which involve both electric and magnetic coefficient types. Here, we have demonstrated the connections between the eigensystem and homogeneous system originated from second order Helmholtz-like (electric only) operator. Note that this link has been facilitated much through the use of vector wave functions which embed most differential operators leaving only algebraic parameters (e.g., A_{ejf}) to be dealt with even in the transverse domain. For any choice of unknown coefficients, the corresponding eigenvalues and eigenvectors can be evaluated easily since the matrix size is only 2×2 . Assuming two linearly independent set of eigenvectors ($A_{e_{2f}}$, $C_{e_{2f}}$) have been obtained for λ_{2f} , the other expansion coefficients can be found readily as

$$B_{ejf} = -[H_{af} A_{ejf} + L_{af} C_{ejf}] / H_{tf} \tag{19}$$

$$A_{hjf} = [(L'_{tf} - i\omega\alpha_{zf}\xi_{zf}) A_{ejf} - L'_{af} B_{ejf} + \alpha_{tf} C_{ejf}] / (i\omega) \tag{20}$$

$$B_{hjf} = [L'_{af} A_{ejf} + (L'_{tf} - i\omega\alpha_{zf}\xi_{zf}) B_{ejf} + \alpha_{af} C_{ejf}] / (i\omega) \tag{21}$$

$$C_{hjf} = [\lambda_{jf}^2 \alpha_{zf} A_{ejf} - i\omega\alpha_{zf}\zeta_{zf} C_{ejf}] / (i\omega). \tag{22}$$

Taking into account the same set of coefficients for λ_{3f} but recall their different kinds of Bessel functions associated, we have thus expressed the eigenfields directly as linear combinations of \bar{m} , \bar{n} , \bar{l} in (15)–(16). If desired, one can cast the ABC representations into those abc 's in (7)–(8) by considering the relationships between the two sets of cylindrical vector functions [19]. Having obtained both electric and magnetic eigenfunctions, we are ready to construct the expansions of dyadic Green's functions in terms of these eigenfunctions.

3. UNBOUNDED DYADIC GREEN'S FUNCTIONS

Assuming an electric current source \bar{J}_s is impressed at $\bar{r} = \bar{r}'$ within layer s ($s = 1, 2, \dots, N$). Due to linearity of Maxwell equations, the electric and magnetic fields in layer f can be related directly to the source in layer s via

$$\bar{E}_f(\bar{r}) = \iiint_{V'} dv' \bar{\bar{G}}_e^{(fs)}(\bar{r}, \bar{r}') \cdot \bar{J}_s(\bar{r}') \tag{23}$$

$$\bar{H}_f(\bar{r}) = \iiint_{V'} dv' \bar{\bar{G}}_m^{(fs)}(\bar{r}, \bar{r}') \cdot \bar{J}_s(\bar{r}') \tag{24}$$

where $\bar{\bar{G}}_e^{(fs)}$ and $\bar{\bar{G}}_m^{(fs)}$ are respectively the layered-media electric and magnetic dyadic Green's functions. According to the principle of scattering superposition, the dyadic Green's functions can be decomposed into unbounded and scattered parts as

$$\bar{\bar{G}}_e^{(fs)} = \bar{\bar{G}}_{e0s} \delta_{fs} + \bar{\bar{G}}_{eS}^{(fs)} \tag{25}$$

$$\bar{\bar{G}}_m^{(fs)} = \bar{\bar{G}}_{m0s} \delta_{fs} + \bar{\bar{G}}_{mS}^{(fs)} \tag{26}$$

where δ_{fs} is the Kronecker delta symbol. The unbounded dyadic Green's functions $\bar{\bar{G}}_{e0s}$ and $\bar{\bar{G}}_{m0s}$ represent the contribution from the direct waves excited by primary sources in an unbounded medium. The scattered dyadic Green's functions $\bar{\bar{G}}_{eS}^{(fs)}$ and $\bar{\bar{G}}_{mS}^{(fs)}$ represent the additional contribution from the scattered waves caused by layer interfaces and/or boundary walls. We shall concentrate first on the unbounded parts in this section. Since the dyadic Green's functions

represent the field responses attributed to a point source, they can be expanded into three regions along $\hat{\rho}$ direction as

$$\overline{\overline{G}}_{e0s} = \overline{\overline{G}}_{e0s}^0 \delta(\rho' - \rho) + \overline{\overline{G}}_{e0s}^> U(\rho - \rho') + \overline{\overline{G}}_{e0s}^< U(\rho' - \rho) \quad (27)$$

$$\overline{\overline{G}}_{m0s} = \overline{\overline{G}}_{m0s}^0 \delta(\rho' - \rho) + \overline{\overline{G}}_{m0s}^> U(\rho - \rho') + \overline{\overline{G}}_{m0s}^< U(\rho' - \rho) \quad (28)$$

The 0, > and < parts along with one-dimensional Dirac delta function $\delta(\rho' - \rho)$ and Heaviside unit step functions $U(\pm\rho \mp \rho')$ correspond respectively to $\rho = \rho'$ (source region), $\rho > \rho'$ and $\rho < \rho'$. Substituting (27)–(28) into Maxwell equations cast in dyadic forms, we carry out the derivative operations in the sense of distributions following the approach described in [18]. Then, we arrive at various important deductions as follows.

Corresponding to $\frac{\partial}{\partial \rho'} \delta(\rho' - \rho)$ in the Maxwell dyadic equations, we have [18]

$$\overline{\overline{G}}_{e0s}^0 = \frac{1}{i\omega} g_{es}^0 \delta_t \hat{\rho} \hat{\rho}', \quad g_{es}^0 = \left. \frac{\overline{\overline{\mu}}_s : \hat{\rho} \hat{\rho}}{(\overline{\overline{\epsilon}}_s : \hat{\rho} \hat{\rho})(\overline{\overline{\mu}}_s : \hat{\rho} \hat{\rho}) - (\overline{\overline{\xi}}_s : \hat{\rho} \hat{\rho})(\overline{\overline{\zeta}}_s : \hat{\rho} \hat{\rho})} \right|_{\overline{\overline{r}} = \overline{\overline{r}'}} \quad (29)$$

$$\overline{\overline{G}}_{m0s}^0 = \frac{1}{i\omega} g_{ms}^0 \delta_t \hat{\rho} \hat{\rho}', \quad g_{ms}^0 = \left. \frac{-\overline{\overline{\zeta}}_s : \hat{\rho} \hat{\rho}}{(\overline{\overline{\epsilon}}_s : \hat{\rho} \hat{\rho})(\overline{\overline{\mu}}_s : \hat{\rho} \hat{\rho}) - (\overline{\overline{\xi}}_s : \hat{\rho} \hat{\rho})(\overline{\overline{\zeta}}_s : \hat{\rho} \hat{\rho})} \right|_{\overline{\overline{r}} = \overline{\overline{r}'}} \quad (30)$$

Here, δ_t is the two-dimensional transverse Dirac delta function defined by

$$\delta_t = \frac{1}{\rho'} \delta(\phi' - \phi) \delta(z' - z). \quad (31)$$

Hence, similar to the case of [18], we have obtained in a simple manner the source point dyadic delta function terms required for both $\overline{\overline{G}}_{e0s}$ and $\overline{\overline{G}}_{m0s}$ simultaneously. These singularities are seen to recover that of gyroelectric medium (electric) dyadic Green’s function in [16], especially when we express them in terms of the gyrotropic components of (3)–(6):

$$\overline{\overline{G}}_{e0s}^0 \delta(\rho' - \rho) = \frac{1}{i\omega} g_{es}^0 \delta(\overline{\overline{r}'} - \overline{\overline{r}}) \hat{\rho} \hat{\rho}', \quad g_{es}^0 = \frac{\mu_{ts}}{\epsilon_{ts} \mu_{ts} - \xi_{ts} \zeta_{ts}} \quad (32)$$

$$\overline{\overline{G}}_{m0s}^0 \delta(\rho' - \rho) = \frac{1}{i\omega} g_{ms}^0 \delta(\overline{\overline{r}'} - \overline{\overline{r}}) \hat{\rho} \hat{\rho}', \quad g_{ms}^0 = \frac{-\zeta_{ts}}{\epsilon_{ts} \mu_{ts} - \xi_{ts} \zeta_{ts}} \quad (33)$$

Note that our results have been derived without recourse to any integration but follows directly from Maxwell equations treated in the distribution sense.

Corresponding to $U(\pm\rho \mp \rho')$ in the Maxwell dyadic equations, we have

$$\nabla \times \overline{\overline{G}}_{e0s}^{\geq} = i\omega(\overline{\overline{\zeta}}_s \cdot \overline{\overline{G}}_{e0s}^{\geq} + \overline{\overline{\mu}}_s \cdot \overline{\overline{G}}_{m0s}^{\geq}) \quad (34)$$

$$\nabla \times \overline{\overline{G}}_{m0s}^{\geq} = -i\omega(\overline{\overline{\epsilon}}_s \cdot \overline{\overline{G}}_{e0s}^{\geq} + \overline{\overline{\xi}}_s \cdot \overline{\overline{G}}_{m0s}^{\geq}) . \quad (35)$$

These homogeneous equations anticipate the homogeneous solutions which can be represented using \overline{E}_{js} , \overline{H}_{js} from (15)–(16) for the field antecedents:

$$\overline{\overline{G}}_{e0s}^{\geq} = \int_h \sum_n \overline{E}_{1s} \overline{S}'_{1s} + \overline{E}_{2s} \overline{S}'_{2s} = \int_h \sum_n \overline{E}_{1s}^H \overline{S}'_{1s} + \overline{E}_{2s}^H \overline{S}'_{2s} \quad (36)$$

$$\overline{\overline{G}}_{m0s}^{\geq} = \int_h \sum_n \overline{H}_{1s} \overline{S}'_{1s} + \overline{H}_{2s} \overline{S}'_{2s} = \int_h \sum_n \overline{H}_{1s}^H \overline{S}'_{1s} + \overline{H}_{2s}^H \overline{S}'_{2s} . \quad (37)$$

$\int_h \sum_n$ implies $\int_{-\infty}^{\infty} dh \sum_{n=-\infty}^{\infty}$ and \overline{S}'_{js} are the source consequents to be determined as functions of source (primed) coordinates. In accordance with our previous reminder, \overline{E}_{1s} and \overline{H}_{1s} with $H_n^{(1)}(\lambda_{1s}\rho)$ correspond to $\rho > \rho'$, while \overline{E}_{3s} and \overline{H}_{3s} with $J_n(\lambda_{1s}\rho)$ correspond to $\rho < \rho'$ respectively.

Corresponding to $\delta(\rho' - \rho)$ in the Maxwell dyadic equations, we have for $\rho = \rho'$ [18]

$$\begin{aligned} \hat{\rho} \times (\overline{\overline{G}}_{e0s}^> - \overline{\overline{G}}_{e0s}^<) &= -\frac{1}{i\omega} g_{es}^0 \nabla_t \delta_t \times \hat{\rho} \hat{\rho}' \\ &\quad + \overline{\overline{I}}_{t\rho} \cdot (g_{es}^0 \overline{\overline{\zeta}}_s + g_{ms}^0 \overline{\overline{\mu}}_s) \cdot \hat{\rho} \hat{\rho}' \delta_t \end{aligned} \quad (38)$$

$$\begin{aligned} \hat{\rho} \times (\overline{\overline{G}}_{m0s}^> - \overline{\overline{G}}_{m0s}^<) &= \overline{\overline{I}}_{t\rho} \delta_t - \frac{1}{i\omega} g_{ms}^0 \nabla_t \delta_t \times \hat{\rho} \hat{\rho}' \\ &\quad - \overline{\overline{I}}_{t\rho} \cdot (g_{es}^0 \overline{\overline{\epsilon}}_s + g_{ms}^0 \overline{\overline{\xi}}_s) \cdot \hat{\rho} \hat{\rho}' \delta_t. \end{aligned} \quad (39)$$

$\overline{\overline{I}}_{t\rho}$ is the transverse-to- $\hat{\rho}$ part of idemfactor, $\overline{\overline{I}}_{t\rho} = \overline{\overline{I}} - \hat{\rho} \hat{\rho}$ (not to be confused with $\overline{\overline{I}}_{tz}$ in (3)–(6)) and ∇_t is the gradient operator taken with respect to transverse coordinates. Expressing explicitly in terms of the gyrotropic components, they read

$$\hat{\rho} \times (\overline{\overline{G}}_{e0s}^> - \overline{\overline{G}}_{e0s}^<) = -\frac{1}{i\omega} g_{es}^0 \nabla_t \delta_t \times \hat{\rho} \hat{\rho}' + (g_{es}^0 \zeta_{as} + g_{ms}^0 \mu_{as}) \hat{\phi} \hat{\rho}' \delta_t \quad (40)$$

$$\hat{\rho} \times (\overline{G}_{m0s}^> - \overline{G}_{m0s}^<) = \overline{I}_{t\rho} \delta_t - \frac{1}{i\omega} g_{ms}^0 \nabla_t \delta_t \times \hat{\rho} \hat{\rho}' - (g_{es}^0 \epsilon_{as} + g_{ms}^0 \xi_{as}) \hat{\phi} \hat{\rho}' \delta_t \quad (41)$$

Equations (40)–(41) describe the discontinuities present in the eigenfunction expansions of the dyadic Green's functions for gyrotropic bianisotropic media. These discontinuity relations constitute the fundamental equations from which \overline{S}'_{js} and hence the dyadic Green's functions can be solved explicitly. Substituting (36)–(37) into (40)–(41) and extracting the transverse-to- $\hat{\rho}$ components with the aid of two linearly independent transverse vector functions \overline{v}_{t1} , \overline{v}_{t2} , we get

$$\begin{bmatrix} \overline{E}_s^> & -\overline{E}_s^< \\ \overline{H}_s^> & -\overline{H}_s^< \end{bmatrix}_{\rho=\rho'} \cdot \begin{bmatrix} \overline{S}_s^> \\ \overline{S}_s^< \end{bmatrix} = \begin{bmatrix} \overline{\rho}_{es} \\ \overline{V}_t + \overline{\rho}_{ms} \end{bmatrix}. \quad (42)$$

The submatrices/subvectors in this expression are defined as

$$\overline{E}_s^{\geq} = \int_h \sum_n \int_{S_\rho} \begin{bmatrix} \overline{v}_{t1} \\ \overline{v}_{t2} \end{bmatrix} \cdot \left[\hat{\rho} \times \overline{E}_{1s}^H, \hat{\rho} \times \overline{E}_{2s}^H \right] \quad (43)$$

$$\overline{H}_s^{\geq} = \int_h \sum_n \int_{S_\rho} \begin{bmatrix} \overline{v}_{t1} \\ \overline{v}_{t2} \end{bmatrix} \cdot \left[\hat{\rho} \times \overline{H}_{1s}^H, \hat{\rho} \times \overline{H}_{2s}^H \right] \quad (44)$$

$$\overline{S}_s^{\geq} = \begin{bmatrix} \overline{S}'_{3s} \\ \overline{S}'_{4s} \end{bmatrix}, \quad \overline{V}_t = \int_{S_\rho} \begin{bmatrix} \overline{v}_{t1} \\ \overline{v}_{t2} \end{bmatrix} \delta_t \quad (45)$$

$$\overline{\rho}_{es} = \int_{S_\rho} \left[\begin{array}{l} \left[\frac{1}{i\omega} g_{es}^0 \nabla_t \cdot (\hat{\rho} \times \overline{v}_{t1}) + (g_{es}^0 \zeta_{as} + g_{ms}^0 \mu_{as}) \overline{v}_{t1} \cdot \hat{\phi} \right] \hat{\rho}' \\ \left[\frac{1}{i\omega} g_{es}^0 \nabla_t \cdot (\hat{\rho} \times \overline{v}_{t2}) + (g_{es}^0 \zeta_{as} + g_{ms}^0 \mu_{as}) \overline{v}_{t2} \cdot \hat{\phi} \right] \hat{\rho}' \end{array} \right] \delta_t \quad (46)$$

$$\overline{\rho}_{ms} = \int_{S_\rho} \left[\begin{array}{l} \left[\frac{1}{i\omega} g_{ms}^0 \nabla_t \cdot (\hat{\rho} \times \overline{v}_{t1}) - (g_{es}^0 \epsilon_{as} + g_{ms}^0 \xi_{as}) \overline{v}_{t1} \cdot \hat{\phi} \right] \hat{\rho}' \\ \left[\frac{1}{i\omega} g_{ms}^0 \nabla_t \cdot (\hat{\rho} \times \overline{v}_{t2}) - (g_{es}^0 \epsilon_{as} + g_{ms}^0 \xi_{as}) \overline{v}_{t2} \cdot \hat{\phi} \right] \hat{\rho}' \end{array} \right] \delta_t \quad (47)$$

and \int_{S_ρ} implies $\int_{-\infty}^{\infty} dz \int_0^{2\pi} d\phi$. Solving equation (42), we arrive at

$$\begin{aligned} \overline{S}_s^{\geq} = \pm \left\{ \left[\overline{E}_s^{\geq} - \overline{E}_s^{\leq} \cdot \left(\overline{H}_s^{\leq} \right)^{-1} \cdot \overline{H}_s^{\geq} \right]^{-1} \cdot \overline{\rho}_{es} \right. \\ \left. + \left[\overline{H}_s^{\geq} - \overline{H}_s^{\leq} \cdot \left(\overline{E}_s^{\leq} \right)^{-1} \cdot \overline{E}_s^{\geq} \right]^{-1} \cdot [\overline{\rho}_{ms} + \overline{V}_t] \right\}_{\rho=\rho'} \end{aligned} \quad (48)$$

and hence all dyads in (36)–(37) have been determined. Together with (32)–(33), we have thus obtained the complete dyadic Green's functions for unbounded gyrotropic bianisotropic medium (27)–(28).

The solution (48) has been expressed in compact form ready to be dot-multiplied with sources of arbitrary orientation by taking into account all possible current components. If desired, these source consequents can be written explicitly in terms of cylindrical vector wave functions. To be specific, let us choose the transverse vector functions to be

$$\overline{v}_{t1} = \frac{1}{4\pi^2} e^{-in'\phi - ih'z} \hat{\phi} \quad (49)$$

$$\overline{v}_{t2} = \frac{1}{4\pi^2} e^{-in'\phi - ih'z} \hat{z} . \quad (50)$$

Then, \overline{E}_s^{\geq} and \overline{H}_s^{\geq} in (43)–(44) become (omitting the primes for n' , h')

$$\overline{E}_s^{\geq} = \begin{bmatrix} -C_{e1s} Z_n^H(\lambda_{1s}\rho) & -C_{e2s} Z_n^H(\lambda_{2s}\rho) \\ -A_{e1s} \frac{\partial}{\partial \rho} Z_n^H(\lambda_{1s}\rho) & -A_{e2s} \frac{\partial}{\partial \rho} Z_n^H(\lambda_{2s}\rho) \\ + B_{e1s} \frac{in}{\rho} Z_n^H(\lambda_{1s}\rho) & + B_{e2s} \frac{in}{\rho} Z_n^H(\lambda_{2s}\rho) \end{bmatrix} \quad (51)$$

$$\overline{H}_s^{\geq} = \begin{bmatrix} -C_{h1s} Z_n^H(\lambda_{1s}\rho) & -C_{h2s} Z_n^H(\lambda_{2s}\rho) \\ -A_{h1s} \frac{\partial}{\partial \rho} Z_n^H(\lambda_{1s}\rho) & -A_{h2s} \frac{\partial}{\partial \rho} Z_n^H(\lambda_{2s}\rho) \\ + B_{h1s} \frac{in}{\rho} Z_n^H(\lambda_{1s}\rho) & + B_{h2s} \frac{in}{\rho} Z_n^H(\lambda_{2s}\rho) \end{bmatrix} \quad (52)$$

Multiplying out the matrix elements and employing the Wronskian relation for Bessel-Hankel functions, (48) yields for example ($j = 1$)

$$\begin{aligned} \bar{S}'_{1s} = & -\frac{i}{8\pi} \left[S_{\rho\partial 1s} \frac{\partial}{\partial \rho'} J_n(\lambda_{1s}\rho') \hat{\rho}' + S_{\rho J 1s} \frac{in}{\rho'} J_n(\lambda_{1s}\rho') \hat{\rho}' \right. \\ & + S_{\phi\partial 1s} \frac{\partial}{\partial \rho'} J_n(\lambda_{1s}\rho') \hat{\phi}' + S_{\phi J 1s} \frac{in}{\rho'} J_n(\lambda_{1s}\rho') \hat{\phi}' \\ & \left. + S_{z 1s} J_n(\lambda_{1s}\rho') \hat{z}' \right] e^{-in\phi' - ihz'}. \end{aligned} \quad (53)$$

The various factors S 's are associated with respective terms dictated by their subscripts. For convenience, their expressions are given explicitly in Appendix B. The corresponding factors for $j = 2$ can be deduced from those for $j = 1$ via simple swapping of $1 \leftrightarrow 2$ in the subscripts of (B1)–(B7), while those for $j = 3$ and $j = 4$ should defer in the associated Bessel function kind as well. Considering the relationships among the expansion coefficients, one can show that

$$S_{\phi J 1s} = -S_{\rho\partial 1s} \quad (54)$$

$$S_{\rho J 1s} = S_{\phi\partial 1s}. \quad (55)$$

It is now expedient to introduce the vector wave functions in terms of source coordinates (reversing the sign of i in (9)–(11))

$$\bar{m}^C = \left[-\hat{\rho}' \frac{in}{\rho'} Z_n(\lambda\rho') - \hat{\phi}' \frac{\partial}{\partial \rho'} Z_n(\lambda\rho') \right] e^{-in\phi' - ihz'} \quad (56)$$

$$\bar{n}^C = \left[\hat{\rho}' \frac{\partial}{\partial \rho'} Z_n(\lambda\rho') - \hat{\phi}' \frac{in}{\rho'} Z_n(\lambda\rho') \right] e^{-in\phi' - ihz'} \quad (57)$$

$$\bar{l}^C = \hat{z}' Z_n(\lambda\rho') e^{-in\phi' - ihz'}. \quad (58)$$

Then we can write (53) as

$$\bar{S}'_{1s} = A_{s1s} \bar{m}_{1s}^{CJ} + B_{s1s} \bar{n}_{1s}^{CJ} + C_{s1s} \bar{l}_{1s}^{CJ} \quad (59)$$

with the expansion coefficients follow readily from (54)–(55)

$$A_{s1s} = \frac{i}{8\pi} \bar{S}_{\phi\partial 1s} \quad (60)$$

$$B_{s1s} = -\frac{i}{8\pi} \bar{S}_{\rho\partial 1s} \quad (61)$$

$$C_{s1s} = -\frac{i}{8\pi} \bar{S}_{z 1s}. \quad (62)$$

Equation (59) has been expressed directly as linear combination of \overline{m}^C , \overline{n}^C , \overline{l}^C functions without having to introduce a complementary medium as required by the modified reciprocity theorem [2]. These results include those of lossless reciprocal uniaxial bianisotropic media derived earlier using contour integration or residue method in [14]. To compute the far field radiation of an electric dipole, one can follow the saddle point method outlined in [16, 26] by noting the asymptotic expressions in (12)–(14). For an alternative representation analogous to the above, one can cast the \overline{m}^C , \overline{n}^C , \overline{l}^C and ABC 's into the more commonly employed notations \overline{M}^C , \overline{N}^C , \overline{L}^C and abc 's. Notice that the algebraic operations in (48) merely involve matrices of order 2×2 , namely those $\overline{E}_s^{\gtrless}$ and $\overline{H}_s^{\gtrless}$. In actuality, these matrices not only allow us to solve explicitly for those source consequents but also constitute the key ingredients from which one can derive the reflection and transmission matrices. These matrices will be used to construct the scattered dyadic Green's functions to be discussed below.

4. SCATTERED DYADIC GREEN'S FUNCTIONS

To facilitate the formulation of scattered dyadic Green's functions, we make use of the effective reflection and transmission concepts as illustrated in [4, 5, 20]. We first define the local and global reflection and transmission matrices for cylindrical multilayered gyrotropic bianisotropic media. The local matrices correspond to reflection and transmission at the interface of a two-layered structure while the global matrices also take account of multiple-reflections for media with more layers/walls. As asserted in [20], these matrices can be conveniently expressed in terms of the key matrices $\overline{E}_f^{\gtrless}$ and $\overline{H}_f^{\gtrless}$ (of layer f) defined in (43)–(44). In particular, the local (superscripted l) reflection and transmission matrices are given explicitly by

$$\begin{aligned} \overline{R}_{f,f\pm 1}^l &= \left[\overline{E}_f^{\lessgtr} - \overline{E}_{f\pm 1}^{\gtrless} \cdot \left(\overline{H}_{f\pm 1}^{\gtrless} \right)^{-1} \cdot \overline{H}_f^{\lessgtr} \right]^{-1} \\ &\cdot \left[\overline{E}_{f\pm 1}^{\gtrless} \cdot \left(\overline{H}_{f\pm 1}^{\gtrless} \right)^{-1} \cdot \overline{H}_f^{\gtrless} - \overline{E}_f^{\gtrless} \right] \Big|_{\rho=\rho_{\min}(f,f\pm 1)} \end{aligned} \quad (63)$$

$$\begin{aligned} \overline{T}_{f,f\pm 1}^l &= \left[\overline{E}_{f\pm 1}^{\gtrless} - \overline{E}_f^{\lessgtr} \cdot \left(\overline{H}_f^{\lessgtr} \right)^{-1} \cdot \overline{H}_{f\pm 1}^{\gtrless} \right]^{-1} \\ &\cdot \left[\overline{E}_f^{\gtrless} - \overline{E}_f^{\lessgtr} \cdot \left(\overline{H}_f^{\lessgtr} \right)^{-1} \cdot \overline{H}_f^{\gtrless} \right] \Big|_{\rho=\rho_{\min}(f,f\pm 1)} \end{aligned} \quad (64)$$

where $\rho = \rho_{\min(f, f\pm 1)}$ indicates that all ρ 's are to be replaced by the interface of layers f and $f + 1$ or $f - 1$, that is $\rho = \rho_f$ or $\rho = \rho_{f-1}$ according to our convention (see Fig. 1). On the other hand, the global (superscripted g) matrices can be written in terms of the local ones as

$$\overline{\overline{R}}_{f, f\pm 1}^g = \overline{\overline{R}}_{f, f\pm 1}^l + \overline{\overline{T}}_{f\pm 1, f}^l \cdot \overline{\overline{R}}_{f\pm 1, f\pm 2}^g \cdot \overline{\overline{S}}_{f, f\pm 1} \quad (65)$$

$$\overline{\overline{T}}_{f, f\pm 1}^g = \overline{\overline{S}}_{f\pm 1, f\pm 2} \cdots \overline{\overline{S}}_{f\pm 1, f\pm 2} \cdot \overline{\overline{S}}_{f, f\pm 1}, \quad l = 1, 2, \dots, \quad N - f \quad (66)$$

$$\overline{\overline{S}}_{f, f\pm 1} = \left[\overline{\overline{I}} - \overline{\overline{R}}_{f\pm 1, f}^l \cdot \overline{\overline{R}}_{f\pm 1, f\pm 2}^g \right]^{-1} \cdot \overline{\overline{T}}_{f, f\pm 1}^l \quad (67)$$

Notice that all the local and global matrices are of size 2×2 only.

Equations (65)–(66) can be applied recursively to find the global reflection and transmission matrices for all layers provided the initial matrices $\overline{\overline{R}}_{N, N+1}^g$ and $\overline{\overline{R}}_{1, 0}^g$ have been specified. When the boundary walls are not present, both $\overline{\overline{R}}_{N, N+1}^g$ and $\overline{\overline{R}}_{1, 0}^g$ are zero matrices since there is no wave reflected from infinity and origin in the outermost and innermost layers respectively. On the other hand, if there exist impedance/admittance walls at $\rho = \rho_N$ and/or $\rho = \rho_0$ as depicted in Fig. 1, these matrices can be related directly to the wall impedance/admittance dyadics. Specifically, letting the impedance walls at $\rho = \rho_N$ and $\rho = \rho_0$ be characterized respectively by $\overline{\overline{Z}}_N$ and $\overline{\overline{Z}}_0$ in spectral domain as

$$\hat{\rho} \times \overline{\overline{E}}_{jN} = \overline{\overline{Z}}_N \cdot \overline{\overline{H}}_{jN}, \quad \rho = \rho_N \quad (68)$$

$$\hat{\rho} \times \overline{\overline{E}}_{j1} = \overline{\overline{Z}}_0 \cdot \overline{\overline{H}}_{j1}, \quad \rho = \rho_0, \quad (69)$$

the global reflection matrices can be found to be

$$\overline{\overline{R}}_{N, N+1}^g(\overline{\overline{Z}}_N) = - \left[\overline{\overline{E}}_N^< - \overline{\overline{H}}_N^<(\overline{\overline{Z}}_N) \right]^{-1} \cdot \left[\overline{\overline{E}}_N^> - \overline{\overline{H}}_N^>(\overline{\overline{Z}}_N) \right] \Big|_{\rho=\rho_N} \quad (70)$$

$$\overline{\overline{R}}_{1, 0}^g(\overline{\overline{Z}}_0) = - \left[\overline{\overline{E}}_1^> - \overline{\overline{H}}_1^>(\overline{\overline{Z}}_0) \right]^{-1} \cdot \left[\overline{\overline{E}}_1^< - \overline{\overline{H}}_1^<(\overline{\overline{Z}}_0) \right] \Big|_{\rho=\rho_0} \quad (71)$$

where

$$\overline{\overline{H}}_{f(\overline{\overline{Z}})}^{\geq} = \int_h \sum_n \int_{S_\rho} \begin{bmatrix} \overline{v}_{t1} \\ \overline{v}_{t2} \end{bmatrix} \cdot \left[\overline{\overline{Z}} \cdot \overline{\overline{H}}_{1f}^H, \overline{\overline{Z}} \cdot \overline{\overline{H}}_{2f}^H \right]. \quad (72)$$

In the case of perfect electric conducting walls, (70)–(71) reduce to

$$\overline{\overline{R}}_{N,N+1}^g(\text{PEC}) = - \left[\overline{\overline{E}}_N^< \right]^{-1} \cdot \overline{\overline{E}}_N^> \Big|_{\rho=\rho_N} \quad (73)$$

$$\overline{\overline{R}}_{1,0}^g(\text{PEC}) = - \left[\overline{\overline{E}}_1^> \right]^{-1} \cdot \overline{\overline{E}}_1^< \Big|_{\rho=\rho_0}. \quad (74)$$

By the same token, for the admittance walls characterized by $\overline{\overline{Y}}_N$ and $\overline{\overline{Y}}_0$ as

$$\hat{\rho} \times \overline{\overline{H}}_{jN} = \overline{\overline{Y}}_N \cdot \overline{\overline{E}}_{jN}, \quad \rho = \rho_N \quad (75)$$

$$\hat{\rho} \times \overline{\overline{H}}_{j1} = \overline{\overline{Y}}_0 \cdot \overline{\overline{E}}_{j1}, \quad \rho = \rho_0, \quad (76)$$

the global reflection matrices take the form

$$\overline{\overline{R}}_{N,N+1}^g(\overline{\overline{Y}}_N) = - \left[\overline{\overline{H}}_N^< - \overline{\overline{E}}_N^<(\overline{\overline{Y}}_N) \right]^{-1} \cdot \left[\overline{\overline{H}}_N^> - \overline{\overline{E}}_N^<(\overline{\overline{Y}}_N) \right] \Big|_{\rho=\rho_N} \quad (77)$$

$$\overline{\overline{R}}_{1,0}^g(\overline{\overline{Y}}_0) = - \left[\overline{\overline{H}}_1^> - \overline{\overline{E}}_1^>(\overline{\overline{Y}}_0) \right]^{-1} \cdot \left[\overline{\overline{H}}_1^< - \overline{\overline{E}}_1^<(\overline{\overline{Y}}_0) \right] \Big|_{\rho=\rho_0} \quad (78)$$

where

$$\overline{\overline{E}}_{f(\overline{\overline{Y}})}^{\cong} = \int_h \sum_n \int_{S_\rho} \begin{bmatrix} \overline{v}_{t1} \\ \overline{v}_{t2} \end{bmatrix} \cdot \left[\overline{\overline{Y}} \cdot \overline{\overline{E}}_{1f}^H, \overline{\overline{Y}} \cdot \overline{\overline{E}}_{2f}^H \right]. \quad (79)$$

In the case of perfect magnetic conducting walls, we have

$$R_{N,N+1}^g(\text{PMC}) = - \left[\overline{\overline{H}}_N^< \right]^{-1} \cdot \overline{\overline{H}}_N^> \Big|_{\rho=\rho_N} \quad (80)$$

$$\overline{\overline{R}}_{1,0}^g(\text{PMC}) = - \left[\overline{\overline{H}}_1^> \right]^{-1} \cdot \overline{\overline{H}}_1^< \Big|_{\rho=\rho_0}. \quad (81)$$

Notice that most commonly encountered boundary conditions have been lumped together into one 'universal' description via the global reflection matrices. Observe also the (extra) roles of $\overline{\overline{E}}_f^{\cong}$ and $\overline{\overline{H}}_f^{\cong}$ key matrices in characterizing various boundary conditions, particularly in conjunction with PEC and PMC walls, c.f. (73)–(74) and (80)–(81).

With the reflection and transmission matrices available, we proceed to derive the scattered dyadic Green's functions which are composed

of 2×2 scattering coefficient matrices $\overline{\overline{A}}^{(fs)}$, $\overline{\overline{B}}^{(fs)}$, $\overline{\overline{C}}^{(fs)}$, $\overline{\overline{D}}^{(fs)}$ as

$$\begin{aligned} \overline{\overline{G}}_{eS}^{(fs)} = \int_h \sum_n \left\{ \right. & \left[\overline{E}_{1f}, \overline{E}_{2f} \right] \cdot \overline{\overline{A}}^{(fs)} \cdot \left[\overline{S}'_{1s}, \overline{S}'_{2s} \right]^T \\ & + \left[\overline{E}_{1f}, \overline{E}_{2f} \right] \cdot \overline{\overline{B}}^{(fs)} \cdot \left[\overline{S}'_{3s}, \overline{S}'_{4s} \right]^T \\ & + \left[\overline{E}_{3f}, \overline{E}_{4f} \right] \cdot \overline{\overline{C}}^{(fs)} \cdot \left[\overline{S}'_{1s}, \overline{S}'_{2s} \right]^T \\ & \left. + \left[\overline{E}_{3f}, \overline{E}_{4f} \right] \cdot \overline{\overline{D}}^{(fs)} \cdot \left[\overline{S}'_{3s}, \overline{S}'_{4s} \right]^T \right\} \quad (82) \end{aligned}$$

$$\begin{aligned} \overline{\overline{G}}_{mS}^{(fs)} = \int_h \sum_n \left\{ \right. & \left[\overline{H}_{1f}, \overline{H}_{2f} \right] \cdot \overline{\overline{A}}^{(fs)} \cdot \left[\overline{S}'_{1s}, \overline{S}'_{2s} \right]^T \\ & + \left[\overline{H}_{1f}, \overline{H}_{2f} \right] \cdot \overline{\overline{B}}^{(fs)} \cdot \left[\overline{S}'_{3s}, \overline{S}'_{4s} \right]^T \\ & + \left[\overline{H}_{3f}, \overline{H}_{4f} \right] \cdot \overline{\overline{C}}^{(fs)} \cdot \left[\overline{S}'_{1s}, \overline{S}'_{2s} \right]^T \\ & \left. + \left[\overline{H}_{3f}, \overline{H}_{4f} \right] \cdot \overline{\overline{D}}^{(fs)} \cdot \left[\overline{S}'_{3s}, \overline{S}'_{4s} \right]^T \right\}. \quad (83) \end{aligned}$$

The scattering coefficient matrices relate the scattered fields in layer f to the primary fields excited by sources in layer s . Upon tracing the path of scattered and primary waves across interfaces and imposing the corresponding constraint conditions, these matrices can be expressed in compact and convenient form in terms of the global reflection and transmission matrices as [20]

$$\begin{aligned} \overline{\overline{A}}^{(fs)} = & \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \delta_{fs} \\ & + \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} U(f-s) \\ & + \overline{\overline{R}}_{f,f-1}^g \cdot \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s+1}^g U(s-f) \quad (84) \end{aligned}$$

$$\begin{aligned} \overline{\overline{B}}^{(fs)} = & \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s-1}^g \delta_{fs} \\ & + \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s-1}^g U(f-s) \\ & + \overline{\overline{R}}_{f,f-1}^g \cdot \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} U(s-f) \quad (85) \end{aligned}$$

$$\begin{aligned}
 \overline{\overline{C}}^{(fs)} &= \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s+1}^g \cdot \overline{\overline{R}}_{s,s-1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s+1}^g \delta_{fs} \\
 &\quad + \overline{\overline{R}}_{f,f+1}^g \cdot \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} U(f-s) \\
 &\quad + \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s+1}^g \cdot \overline{\overline{R}}_{s,s-1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s+1}^g U(s-f) \quad (86)
 \end{aligned}$$

$$\begin{aligned}
 \overline{\overline{D}}^{(fs)} &= \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s+1}^g \cdot \overline{\overline{R}}_{s,s-1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s+1}^g \cdot \overline{\overline{R}}_{s,s-1}^g \delta_{fs} \\
 &\quad + \overline{\overline{R}}_{f,f+1}^g \cdot \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s-1}^g U(f-s) \\
 &\quad + \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s+1}^g \cdot \overline{\overline{R}}_{s,s-1}^g \right]^{-1} U(s-f) . \quad (87)
 \end{aligned}$$

Here, we have used the Kronecker delta and Heaviside unit step functions to take account of arbitrary field and source locations, i.e., within any layer $f = s$, $f > s$ or $f < s$.

Note that the utilization of reflection and transmission concepts has avoided complicated formulation of the scattering coefficients. Moreover, these expressions also feature much fundamental insights into the scattering mechanism. As an example, let us consider the $\overline{\overline{B}}^{(fs)}$ coefficient matrix which relates the outward-regular field antecedents (\overline{E}_{1f} , \overline{E}_{2f}) with the source consequents generating inward-regular fields (\overline{S}'_{3s} , \overline{S}'_{4s}). For a specific case of field point $f > s$, the expression extracted from (85) reads

$$\overline{\overline{B}}^{(fs)} = \overline{\overline{T}}_{s,f}^g \cdot \left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1} \cdot \overline{\overline{R}}_{s,s-1}^g \quad (88)$$

Referring to the terms starting from the right, we can associate some physical interpretation to each of them as follows: originating from \overline{S}'_{3s} , \overline{S}'_{4s} in layer s , the inward-regular waves first undergo reflection ($\overline{\overline{R}}_{s,s-1}^g$) while approaching layer $s-1$. These waves then bounce back as outward-regular waves and commence multiple-reflections ($\left[\overline{\overline{I}} - \overline{\overline{R}}_{s,s-1}^g \cdot \overline{\overline{R}}_{s,s+1}^g \right]^{-1}$) before being transmitted ($\overline{\overline{T}}_{s,f}^g$) to layer f as \overline{E}_{1f} , \overline{E}_{2f} . From here, one observes immediately the physical significance corresponding to various constituents of the scattered dyadic Green's functions presented in (82)–(87). Together with (27)–(28), we

have thus obtained the complete dyadic Green's functions (25)–(26) pertaining to point source excitation in cylindrical multilayered gyrotropic bianisotropic media.

5. APPLICATION TO A COATED CONDUCTING CYLINDER

Thus far, the general expressions of the dyadic Green's functions have been determined. In order to demonstrate how to simplify the general expressions for some specific cases, let us consider a perfect conducting cylinder coated with a homogeneous layer of gyrotropic bianisotropic medium (layer 1) and embedded in isotropic host medium (layer 2). The cylinder is of radius ρ_0 and the coating thickness is $\rho_1 - \rho_0$ (refer to Fig. 1 with layer 2 extended). This configuration is frequently employed to investigate microstrip structures mounted on cylindrical surfaces [27]–[32]. Both cases of electric point source located within gyrotropic bianisotropic coating as well as within isotropic host space are studied. Since the mathematical expressions for unbounded dyadic Green's functions remain unchanged irrespective of source location, we shall concentrate on the scattered dyadic Green's functions below.

Assume that the electric (\overline{E}_{j1}) and magnetic (\overline{H}_{j1}) eigenfunctions in gyrotropic bianisotropic coating have been expressed in terms of *ABC* coefficients resulting in the $\overline{\overline{E}}_1^{\gtrless}$ and $\overline{\overline{H}}_1^{\gtrless}$ matrices of (51)–(52). For the isotropic medium characterized by $\overline{\overline{\epsilon}}_2 = \epsilon_2 \overline{\overline{I}}$, $\overline{\overline{\mu}}_2 = \mu_2 \overline{\overline{I}}$ and $\overline{\overline{\xi}}_2 = \overline{\overline{\zeta}}_2 = \overline{\overline{0}}$, the $\overline{\overline{E}}_2^{\gtrless}$ and $\overline{\overline{H}}_2^{\gtrless}$ matrices corresponding to common TE and TM modes are (defined in slight different form in [19])

$$\overline{\overline{E}}_2^{\gtrless} = \begin{bmatrix} 0 & -\frac{\lambda_2^2}{k_2} Z_n^H(\lambda_2 \rho) \\ -\frac{\partial}{\partial \rho'} Z_n^H(\lambda_2 \rho) & -\frac{h n}{k_2 \rho} Z_n^H(\lambda_2 \rho) \end{bmatrix} \tag{89}$$

$$\overline{\overline{H}}_2^{\gtrless} = \eta_2 \begin{bmatrix} -\frac{\lambda_2^2}{k_2} Z_n^H(\lambda_2 \rho) & 0 \\ -\frac{h n}{k_2 \rho} Z_n^H(\lambda_2 \rho) & -\frac{\partial}{\partial \rho'} Z_n^H(\lambda_2 \rho) \end{bmatrix} \tag{90}$$

where $\lambda_2^2 = k_2^2 - h^2$, $\eta_2 = \frac{k_2}{i \omega \mu_2}$ and $k_2^2 = \omega^2 \mu_2 \epsilon_2$. Substituting these $\overline{\overline{E}}_f^{\gtrless}$, $\overline{\overline{H}}_f^{\gtrless}$ into the formulas for reflection and transmission matrices and noting that $\overline{\overline{R}}_{2,3}^g = \overline{\overline{0}}$, the final expressions of the scattering

coefficient matrices for various combinations of f and s in (84)–(87) reduce to

$f = 1, s = 1 :$

$$\overline{\overline{B}}^{(11)} = \left[\overline{\overline{T}} - \overline{\overline{R}}_{1,0(\text{PEC})}^g \cdot \overline{\overline{R}}_{1,2}^l \right]^{-1} \cdot \overline{\overline{R}}_{1,0(\text{PEC})}^g \quad (91)$$

$$\overline{\overline{A}}^{(11)} = \overline{\overline{B}}^{(11)} \cdot \overline{\overline{R}}_{1,2}^l \quad (92)$$

$$\overline{\overline{C}}^{(11)} = \left[\overline{\overline{T}} - \overline{\overline{R}}_{1,2}^l \cdot \overline{\overline{R}}_{1,0(\text{PEC})}^g \right]^{-1} \cdot \overline{\overline{R}}_{1,2}^l \quad (93)$$

$$\overline{\overline{D}}^{(11)} = \overline{\overline{C}}^{(11)} \cdot \overline{\overline{R}}_{1,0(\text{PEC})}^g \quad (94)$$

$f = 2, s = 1 :$

$$\overline{\overline{A}}^{(21)} = \overline{\overline{T}}_{1,2}^l \cdot \left[\overline{\overline{T}} - \overline{\overline{R}}_{1,0(\text{PEC})}^g \cdot \overline{\overline{R}}_{1,2}^l \right]^{-1} \quad (95)$$

$$\overline{\overline{B}}^{(21)} = \overline{\overline{A}}^{(21)} \cdot \overline{\overline{R}}_{1,0(\text{PEC})}^g \quad (96)$$

$$\overline{\overline{C}}^{(21)} = \overline{\overline{D}}^{(21)} = \overline{\overline{0}} \quad (97)$$

$f = 1, s = 2 :$

$$\overline{\overline{D}}^{(12)} = \overline{\overline{T}}_{2,1}^l \quad (98)$$

$$\overline{\overline{B}}^{(12)} = \overline{\overline{R}}_{1,0(\text{PEC})}^g \cdot \overline{\overline{D}}^{(12)} \quad (99)$$

$$\overline{\overline{A}}^{(12)} = \overline{\overline{C}}^{(12)} = \overline{\overline{0}} \quad (100)$$

$f = 2, s = 2 :$

$$\overline{\overline{B}}^{(22)} = \overline{\overline{R}}_{2,1}^g \quad (101)$$

$$\overline{\overline{A}}^{(22)} = \overline{\overline{C}}^{(22)} = \overline{\overline{D}}^{(22)} = \overline{\overline{0}} . \quad (102)$$

Notice from above that once $\overline{\overline{E}}_f^{\gtrless}$ and $\overline{\overline{H}}_f^{\gtrless}$ have been worked out from eigenfunction expansions, the local and global reflection and transmission matrices which incorporate as well various boundary wall conditions can be calculated directly. Then, the scattering coefficient matrices and hence the scattered dyadic Green's functions can be determined readily. Furthermore, recall that the source consequents $\overline{\overline{S}}'_{js}$ associated with unbounded dyadic Green's functions can also be expressed in terms of $\overline{\overline{E}}_s^{\gtrless}$ and $\overline{\overline{H}}_s^{\gtrless}$ (of layer s), c.f. (48). Therefore,

it has become apparent that the most important step in solving the dyadic Green's functions for multilayered media is in determining the 2×2 $\overline{\overline{E}}^{\lessgtr}$ and $\overline{\overline{H}}^{\lessgtr}$ for each layer, which amounts to finding the ABC coefficients for electromagnetic eigenfields in source-free regions. Having determined both unbounded and scattered parts, we have thus obtained the complete dyadic Green's functions for all field and source locations in our coated cylinder configuration. For media with more number of layers/walls, the general expressions given in the previous section can be applied in the similar manner.

6. CONCLUSION

This paper has presented a rigorous and concise formulation of the complete eigenfunction expansions of the dyadic Green's functions for cylindrical multilayered gyrotropic bianisotropic media. Both electric and magnetic dyadic Green's functions for arbitrary field and source locations have been derived simultaneously making use of the principle of scattering superposition. Based on the theory of distributions, the singularities and discontinuities associated with unbounded dyadic Green's functions have been deduced directly from Maxwell equations cast in dyadic forms. Using the discontinuity relations obtained, the eigenfunction expansions outside the source point have been constructed in terms of an alternative more expedient set of cylindrical vector functions. The scattered dyadic Green's functions have been determined by applying the concepts of effective reflection and transmission of outward-regular and inward-regular waves. This approach has avoided cumbersome operations and has also provided good physical insights into the scattering mechanism. Moreover, the scattering coefficient matrices have been expressed in compact and convenient forms involving global reflection and transmission matrices of size 2×2 only. Corresponding to the impedance/admittance boundary walls, the global reflection matrices have been related directly to the wall impedance/admittance dyadics. Throughout the formulation, much recognition has been given to the roles of $\overline{\overline{E}}^{\lessgtr}$ and $\overline{\overline{H}}^{\lessgtr}$ being the key ingredients of source consequents, reflection and transmission matrices, and boundary wall characterizations, thus asserting the important step of studying source-free eigenfields in solving dyadic Green's functions. To illustrate the application of general expressions obtained, the configuration of a perfect conducting cylinder coated with gyrotropic

bianisotropic medium has been considered. As the materials treated in this paper are of the most general gyrotropic form, the results presented here can be applied to many specific cases of recently proposed composites, as well as to geometries having arbitrary number of layers with or without impedance/admittance boundary walls.

APPENDIX A

The matrix elements in (17) are defined with the following notations:

$$\begin{aligned}
 H_{tf} = & \{h^2\mu_{tf} + \omega h [\mu_{tf}(\xi_{af} - \zeta_{af}) - \mu_{af}(\xi_{tf} - \zeta_{tf})] \\
 & - \omega^2 [\epsilon_{tf}(\mu_{tf}^2 + \mu_{af}^2) - \mu_{tf}(\xi_{tf}\zeta_{tf} - \xi_{af}\zeta_{af}) \\
 & - \mu_{af}(\xi_{tf}\zeta_{af} + \zeta_{tf}\xi_{af})]\} / (\mu_{tf}^2 + \mu_{af}^2) \quad (A1)
 \end{aligned}$$

$$\begin{aligned}
 H_{af} = & \{-h^2\mu_{af} - \omega h [\mu_{af}(\xi_{af} - \zeta_{af}) + \mu_{tf}(\xi_{tf} - \zeta_{tf})] \\
 & - \omega^2 [\epsilon_{af}(\mu_{tf}^2 + \mu_{af}^2) + \mu_{af}(\xi_{tf}\zeta_{tf} - \xi_{af}\zeta_{af}) \\
 & - \mu_{tf}(\xi_{tf}\zeta_{af} + \zeta_{tf}\xi_{af})]\} / (\mu_{tf}^2 + \mu_{af}^2) \quad (A2)
 \end{aligned}$$

$$\begin{aligned}
 L_{tf} = & [ih\mu_{af} + i\omega(\mu_{tf}\xi_{tf} + \mu_{af}\xi_{af})] / (\mu_{tf}^2 + \mu_{af}^2) \\
 & - i\omega\zeta_{zf}/\mu_{zf} \quad (A3)
 \end{aligned}$$

$$L_{af} = [ih\mu_{tf} + i\omega(\mu_{tf}\xi_{af} - \mu_{af}\xi_{tf})] / (\mu_{tf}^2 + \mu_{af}^2) \quad (A4)$$

$$\begin{aligned}
 L'_{tf} = & [ih\mu_{af} - i\omega(\mu_{tf}\zeta_{tf} + \mu_{af}\zeta_{af})] / (\mu_{tf}^2 + \mu_{af}^2) \\
 & + i\omega\xi_{zf}/\mu_{zf} \quad (A5)
 \end{aligned}$$

$$L'_{af} = [ih\mu_{tf} - i\omega(\mu_{tf}\zeta_{af} - \mu_{af}\zeta_{tf})] / (\mu_{tf}^2 + \mu_{af}^2) \quad (A6)$$

$$\alpha_{tf} = \mu_{tf} / (\mu_{tf}^2 + \mu_{af}^2) \quad (A7)$$

$$\alpha_{af} = -\mu_{af} / (\mu_{tf}^2 + \mu_{af}^2) \quad (A8)$$

$$\alpha_{zf} = 1/\mu_{zf} \quad (A9)$$

$$\delta_{zf} = \epsilon_{zf} - \xi_{zf}\zeta_{zf}/\mu_{zf} \cdot \quad (A10)$$

APPENDIX B

The source consequent factors in (53) are given explicitly by

$$S_{\rho\partial 1s} = [C_{h2s}\phi_{es} - C_{e2s}\phi_{ms}]/\Delta_C \quad (\text{B1})$$

$$\begin{aligned} S_{\rho J1s} = & \{ [A_{e2s}(B_{h1s}C_{h2s} - B_{h2s}C_{h1s}) \\ & - A_{h2s}(B_{e1s}C_{h2s} - B_{e2s}C_{h1s})] \phi_{es} \\ & + [A_{h2s}(B_{e1s}C_{e2s} - B_{e2s}C_{e1s}) \\ & - A_{e2s}(B_{h1s}C_{e2s} - B_{h2s}C_{e1s})] \phi_{ms} \} / (\Delta_A \Delta_C) \\ & + [-A_{h2s}g_{es}^0 + A_{e2s}g_{ms}^0] / (i\omega \Delta_A) \end{aligned} \quad (\text{B2})$$

$$S_{\phi\partial 1s} = -C_{e2s}/\Delta_C \quad (\text{B3})$$

$$\begin{aligned} S_{\phi J1s} = & [A_{h2s}(B_{e1s}C_{e2s} - B_{e2s}C_{e1s}) \\ & - A_{e2s}(B_{h1s}C_{e2s} - B_{h2s}C_{e1s})] / (\Delta_A \Delta_C) \end{aligned} \quad (\text{B4})$$

$$S_{z1s} = A_{e2s}/\Delta_A \quad (\text{B5})$$

$$\Delta_C = C_{e1s}C_{h2s} - C_{e2s}C_{h1s} \quad (\text{B6})$$

$$\Delta_A = A_{e1s}A_{h2s} - A_{e2s}A_{h1s} \quad (\text{B7})$$

$$\phi_{es} = g_{es}^0 \zeta_{as} + g_{ms}^0 \mu_{as} - (h/\omega) g_{es}^0 \quad (\text{B8})$$

$$\phi_{ms} = - [g_{es}^0 \epsilon_{as} + g_{ms}^0 \xi_{as} + (h/\omega) g_{ms}^0] . \quad (\text{B9})$$

REFERENCES

1. Tai, C. T., *Dyadic Green's Functions in Electromagnetic Theory*, IEEE Press, New York, 1993.
2. Kong, J. A., *Electromagnetic Wave Theory*, Wiley, New York, 1990.
3. Collin, R. E., *Field Theory of Guided Waves*, IEEE Press, New York, 1991.
4. Chew, W. C., *Waves and Fields in Inhomogeneous Media*, Van Nostrand, New York, 1990.
5. Ali, S. M., T. M. Habashy, and J. A. Kong, "Spectral-domain dyadic Green's function in layered chiral media," *J. Opt. Soc. Am. A*, Vol. 9, No. 3, 413–423, 1992.
6. Lee, J. K., and J. A. Kong, "Dyadic Green's functions for layered anisotropic medium," *Electromag.*, Vol. 3, 111–130, 1983.

7. Habashy, T. M., S. M. Ali, J. A. Kong, and M. D. Grossi, "Dyadic Green's functions in a planar stratified arbitrary magnetized linear plasma," *Radio Sci.*, Vol. 26, No. 3, 701–715, 1991.
8. Barkeshli, S., "Electromagnetic dyadic Green's functions for multilayered symmetric gyroelectric media," *Radio Sci.*, Vol. 28, No. 1, 23–36, 1993.
9. Mudaliar S., and J. K. Lee, "Dyadic Green's functions for a two-layer biaxially anisotropic medium," *J. Electrom. Waves Applicat.*, Vol. 10, No. 7, 909–923, 1996.
10. Zhuck, N. P., and A. S. Omar, "Scalarization approach in the theory of em waves in a plane-layered bianisotropic medium: general formulations," *Progress In Electromagnetics Research*, PIER 12, 219–249, 1996.
11. Xiang, Z., and Y. Lu, "Electromagnetic dyadic Green's function in cylindrically multilayered media," *IEEE Trans. Microwave Theory Tech.*, Vol. 44, No. 4, 614–621, 1996.
12. Yin, W., and W. Wang, "Dyadic Green's function of cylindrical multilayered chiral media and its applications," *J. Electrom. Waves Applicat.*, Vol. 7, No. 7, 1005–1027, 1993.
13. Li, L. W., P. S. Kooi, M. S. Leong, and T. S. Yeo, "Analytic representation of scattering dyadic Green's functions' coefficients for cylindrically multilayered chiral media," *J. Electrom. Waves Applicat.*, Vol. 9, No. 9, 1207–1221, 1993.
14. Cheng, D., and W. Ren, "Green dyadics in reciprocal uniaxial bianisotropic media by cylindrical vector wave functions," *Phys. Rev. E.*, Vol. 54, No. 3, 2917–2924, 1996.
15. Cheng, D., "Field representations in reciprocal uniaxial bianisotropic media by cylindrical vector wave functions," *Microwave Opt. Technol. Lett.*, Vol. 13, No. 6, 358–363, 1996.
16. Cottis, P. G., "Radiation in an unbounded gyroelectric medium," *J. Electrom. Waves Applicat.*, Vol. 12, No. 2, 249–268, 1998.
17. Lindell, I. V., "Decomposition of electromagnetic fields in bianisotropic media," *J. Electrom. Waves Applicat.*, Vol. 11, No. 5, 645–657, 1997.
18. Tan, E. L., and S. Y. Tan, "On the eigenfunction expansions of the dyadic Green's functions for bianisotropic media," *Progress In Electromagnetics Research*, PIER 20, 227–247, 1998.
19. Tan, E. L., and S. Y. Tan, "Cylindrical vector wave function representations of electromagnetic fields in gyrotropic bianisotropic media," *J. Electrom. Waves Applicat.*, Vol. 13, 1461–1476, 1999.

20. Tan, E. L., and S. Y. Tan, "A unified representation of the dyadic Green's functions for planar, cylindrical and spherical multilayered biisotropic media," *Progress In Electromagnetics Research*, PIER 20, 75–100, 1998.
21. Lindell, I. V., A. H. Sihvola, S. A. Tretyakov, and A. J. Viitanen, *Electromagnetic Waves in Chiral and Bi-Isotropic Media*, Artech House, London, 1994.
22. Cory, H., "Chiral devices – an overview of canonical problems," *J. Electrom. Waves Applicat.*, Vol. 9, No. 5/6, 805–829, 1995.
23. Pelet, P., and N. Engheta, "The theory of chirowaveguides," *IEEE Trans. Antennas Propagat.*, Vol. 38, No. 1, 90–98, 1990.
24. Shen, Z. X., "Scattering by a circular impedance cylinder coated with a composite anisotropic chiral sheath," *J. Electrom. Waves Applicat.*, Vol. 8, No. 12, 1625–1644, 1994.
25. Konistis, K., and J. L. Tsalamengas, "Plane wave scattering by an array of bianisotropic cylinders enclosed by another one in an unbounded bianisotropic space: oblique incidence," *J. Electrom. Waves Applicat.*, Vol. 11, No. 8, 1073–1090, 1997.
26. Felsen, L. B., and N. Marcuvitz, *Radiation and Scattering of Waves*, Prentice-Hall, New Jersey, 1973.
27. Ali, S. M., T. M. Habashy, J. F. Kiang, and J. A. Kong "Resonance in cylindrical-rectangular and wraparound microstrip structures," *IEEE Trans. Microwave Theory Tech.*, Vol. 37, No. 11, 1773–1783, 1989.
28. Habashy, T. M., S. M. Ali, and J. A. Kong "Input impedance and radiation pattern of cylindrical-rectangular and wraparound Mmicrostrip antennas," *IEEE Trans. Antennas Propagat.*, Vol. 38, No. 5, 722–731, 1990.
29. Fonseca, S. B. A., and A. J. Giarola, "Analysis of microstrip wraparound antennas using dyadic Green's functions," *IEEE Trans. Antennas Propagat.*, Vol. 31, No. 2, 248–253, 1983.
30. Ashkenazy, J., S. Shtrikman, and D. Treves, "Electric surface current model for the analysis of microstrip antennas on cylindrical bodies," *IEEE Trans. Antennas Propagat.*, Vol. 33, No. 3, 295–300, 1985.
31. Alexopoulos, N. G., and A. Nakatani, "Cylindrical substrate microstrip line characterization," *IEEE Trans. Microwave Theory Tech.*, Vol. 35, No. 9, 843–849, 1987.
32. Nakatani, A., and N. G. Alexopoulos, "Coupled microstrip lines on a cylindrical substrate," *IEEE Trans. Microwave Theory Tech.*, Vol. 35, No. 12, 1392–1398, 1987.