Progress In Electromagnetics Research, PIER 9, 137–156, 1994

WAVE PROPAGATION IN A STRATIFIED CHIRAL SLAB WITH MULTIPLE DISCONTINUITIES: OBLIQUE INCIDENEC

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1. Introduction

In view of its possible usefulness in a variety of applications, wave propagation, radiation and guidance in chiral media have received considerable attention recently, cf. [1–4]. Electromagnetic chirality addresses the effects of handedness in electrodynamics. A chiral object is a three-dimensional object that has a specific handedness, i.e. whose mirror image cannot be made to coincide with the original object by means of translations and rotations. A (homogeneous and isotropic) chiral medium is a macroscopically continuous medium composed of microscopic chiral objects that are uniformly distributed and randomly oriented. When a linearly polarized electromagnetic wave is normally incident on a chiral slab, two propagating modes with different phase velocities are generated in the medium. After propagation through the slab the polarization of the transmitted field is rotated with respect to the polarization of the incident field [5].

In the present paper, we consider a time-harmonic electromagnetic plane wave *obliquely* incident on a stratified chiral slab with multiple discontinuities in the parameters. Our approach is based on an identification of down-going and up-going eigenmodes. This idea, which is often referred to as wave-splitting, has been used to solve direct and inverse scattering problems for inhomogeneous media in the time-domain (cf. e.g. [6–13] and earlier references given there; obliquely incident transient plane waves are treated in [14–16]). Recently this approach has been applied to stratified complex media in the case of *normally incident waves* in the frequency-domain (cf. [17–19]). We emphasize that the approach is not limited to piecewise constant stratifications and it can be applied effectively to stratified media whose properties vary in a general way as a function of depth. Up- and down-going eigenmodes derived in the present paper are used to rewrite Maxwell's equations. The solution for the reflection coefficient matrix is given for both a multilayered structure and a general stratified slab (for the later case the solution is obtained by solving a Riccati equation). Two Green function matrices are introduced to express the internal eigenmodes in terms of the down-going incident modes. Ordinary differential equations (ODEs) for the Green functions are given together with the boundary conditions. The boundary values of the Green functions are related to the reflection and transmission coefficient matrices, i.e., the Green functions give the scattered fields as well as the internal fields. Numerical results are presented for a case in which the parameters vary continuously, except for an interior finite discontinuity and finite discontinuities at the slab interfaces.

2. Formulation

Consider an electromagnetic plane wave with harmonic time dependence $\exp(jwt)$ which is obliquely incident on a stratified chiral slab. The slab occupies the region $0 \le z \le l$ and the media on either side of this slab are homogeneous. The plane wave impinges on the slab from the upper side $z < 0$.

The constitutive relations for the stratified (reciprocal) chiral medium are taken to be (cf. e.g. [20])

$$\bar{\mathbf{D}}(z) = \epsilon(z)\bar{\mathbf{E}}(z) - j\kappa(z)\sqrt{\epsilon_0\mu_0}\,\bar{\mathbf{H}}(z), \tag{1}$$

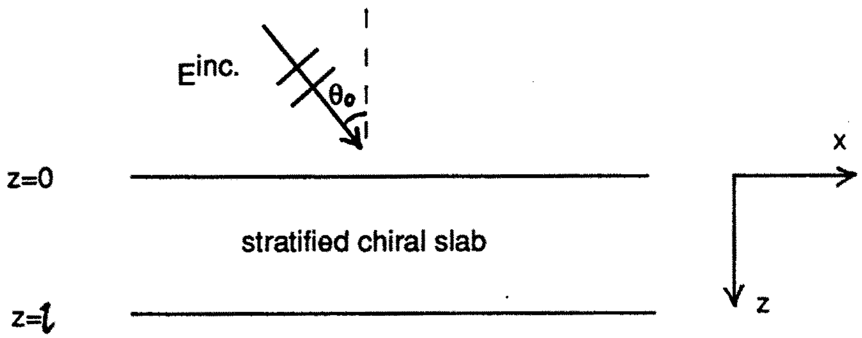

Figure 1. The scattering configuration.

$$\bar{\mathbf{B}}(z) = \mu(z)\bar{\mathbf{H}}(z) + j\kappa(z)\sqrt{\epsilon_0\mu_0}\,\bar{\mathbf{E}}(z), \tag{2}$$

where the permittivity $\epsilon(z)$, the permeability $\mu(z)$, and the chirality parameter $\kappa(z)$ (dimensionless) vary with the depth z. The constants ϵ_0, μ_0 are the permittivity and permeability in vacuum, respectively. The parameter κ describes the degree of chirality. For practical cases, κ is small and satisfies $\kappa^2 < (\epsilon\mu)/(\epsilon_0\mu_0)$. Note that the material parameters ϵ, μ and κ may be frequency-dependent (however, the frequency-dependence of the parameters is suppressed in their arguments for simplicity of notation). For a lossless medium, all three parameters are real numbers. Assume that the wave vector $\bar{\mathbf{k}}_0$ of the incident wave is parallel to the $xz-$ plane and makes an angle θ_0 with the $z-$ axis. The scattering geometry is illustrated in Fig. 1.

Maxwell's equations in a Cartesian coordinate system are

$$\nabla \times \bar{\mathbf{E}} = -j\omega\bar{\mathbf{B}}, \qquad \nabla \times \bar{\mathbf{H}} = j\omega\bar{\mathbf{D}}, \tag{3}$$

where

$$\bar{\mathbf{E}} = (E_1, E_2, E_3), \qquad \bar{\mathbf{H}} = (H_1, H_2, H_3).$$

The x and t dependence of the electromagnetic fields is $e^{j(wt-s_0x)}$, where the propagation constant s_0 is given by

$$s_0 = k_0 \sin \theta_0, \tag{4}$$

and where k_0 be the wave number of the incident wave. Note that since the incident plane is assumed to be the $xz-$ plane, the fields are $y-$ independent. From the third components of Maxwell's equations and the constitutive relations (1) and (2), one can express the third components of $\bar{\mathbf{E}}$ and $\bar{\mathbf{H}}$ in terms of the field components in the xy plane as follows

$$\begin{bmatrix} E_3 \\ H_3 \end{bmatrix} = P \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}, \tag{5}$$

where

$$\mathbf{E} = \begin{bmatrix} E_1 \\ E_2 \end{bmatrix}, \qquad \mathbf{H} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix},$$

and where the 2×4 matrix P is given by

$$P = \frac{1}{\epsilon\mu - \kappa^2 \epsilon_0 \mu_0} \left(\frac{s_0}{\omega} \right) \begin{bmatrix} 0 & j\kappa\sqrt{\epsilon_0\mu_0} & 0 & -\mu \\ 0 & \epsilon & 0 & j\kappa\sqrt{\epsilon_0\mu_0} \end{bmatrix} \tag{6}$$

Adopting the constitutive relations (1) and (2), the first two components (in the following referred to as the tangential components) of Maxwell's equations can be rewritten in the following form

$$\partial_z \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = j\omega \begin{bmatrix} 0 & -j\kappa\sqrt{\epsilon_0\mu_0} & 0 & -\mu \\ j\kappa\sqrt{\epsilon_0\mu_0} & 0 & \mu & 0 \\ 0 & \epsilon & 0 & -j\kappa\sqrt{\epsilon_0\mu_0} \\ -\epsilon & 0 & j\kappa\sqrt{\epsilon_0\mu_0} & 0 \end{bmatrix} \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}$$

$$-js_0 \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} E_3 \\ H_3 \end{bmatrix} \tag{7}$$

Substituting Eq. (5) into Eq. (7) yields

$$\partial_z \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = W \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}, \tag{8}$$

where

$$W = j\omega \left(\begin{bmatrix} 0 & -j\kappa\sqrt{\epsilon_0\mu_0} & 0 & -\mu \\ j\kappa\sqrt{\epsilon_0\mu_0} & 0 & \mu & 0 \\ 0 & \epsilon & 0 & -j\kappa\sqrt{\epsilon_0\mu_0} \\ -\epsilon & 0 & j\kappa\sqrt{\epsilon_0\mu_0} & 0 \end{bmatrix} - q \begin{bmatrix} 0 & j\kappa\sqrt{\epsilon_0\mu_0} & 0 & -\mu \\ 0 & 0 & 0 & 0 \\ 0 & \epsilon & 0 & j\kappa\sqrt{\epsilon_0\mu_0} \\ 0 & 0 & 0 & 0 \end{bmatrix} \right), \tag{9}$$

with

$$q = \frac{1}{\epsilon\mu - \kappa^2\epsilon_0\mu_0} \left(\frac{s_0^2}{\omega^2} \right). \tag{10}$$

Eq. (8) expresses Maxwell's equations for a chiral medium in terms of the tangential electric and magnetic fields.

3. Up- and Down-Going Modes in a Homogeneous Chiral Medium

In this section we identify the down-going modes $\mathbf{E}^+$ (propagating in the positive z direction) and up-going modes $\mathbf{E}^-$ (propagating in the negative z direction) in a homogeneous chiral medium characterized by the parameters ϵ, μ and κ (these parameters are $z-$ independent in this section) in the xyz coordinate system introduced in the previous section. For a plane wave mode (denoted ϕ) with harmonic time dependence $\exp(jwt)$, the down-going condition is $\partial_z\phi = -jk_z\phi$, and the up-going condition is $\partial_z\phi = jk_z\phi$ (k_z is the z component of the wave vector). Let

$$\begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix} = T \begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}, \tag{11}$$

where

$$\mathbf{E}^+ = \begin{bmatrix} E_1^+ \\ E_2^+ \end{bmatrix}, \qquad \mathbf{E}^- = \begin{bmatrix} E_1^- \\ E_2^- \end{bmatrix},$$

and where the transformation matrix T will be determined later. For a homogeneous medium, the matrix T is independent of z. Therefore, from Eqs. (8) and (11) one has

$$\partial_z \begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix} = TWT^{-1} \begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix},$$

where T^{-1} is the inverse of T. In order to allow a physical interpretation of $\mathbf{E}^+$ and $\mathbf{E}^-$ as down-going and up-going eigenmodes, respectively, TWT^{-1} should be diagonal. From the usual process of diagonalizing a matrix, it follows that the diagonal elements are the eigenvalues of the matrix W and the matrix T^{-1} consists of the eigenvectors of the matrix W (the eigenvectors should be ordered appropriately so that $\mathbf{E}^+$ and $\mathbf{E}^-$ have the physical interpretation as down-going and up-going modes, respectively). Performing the indicated derivation, one obtains the following diagonalized system

$$\partial_z \begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix} = \begin{bmatrix} -j\omega\lambda_1 & 0 & 0 & 0 \\ 0 & -j\omega\lambda_2 & 0 & 0 \\ 0 & 0 & j\omega\lambda_1 & 0 \\ 0 & 0 & 0 & j\omega\lambda_2 \end{bmatrix} \begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix}, \tag{12}$$

where

$$\lambda_1 = \left[\left(\sqrt{\epsilon\mu} + \kappa\sqrt{\epsilon_0\mu_0} \right)^2 - s_0^2/\omega^2 \right]^{1/2}, \tag{13}$$

$$\lambda_2 = \left[\left(\sqrt{\epsilon\mu} - \kappa\sqrt{\epsilon_0\mu_0} \right)^2 - s_0^2/\omega^2 \right]^{1/2}, \tag{14}$$

and where one takes the square roots that give λ_i, $i = 1, 2$, positive real parts. The matrix T is given by

$$T = \frac{1}{2}\begin{bmatrix} 1 & j/\nu_1 & -j\sqrt{\mu/\epsilon} & \sqrt{\mu/\epsilon}/\nu_1 \\ j/\nu_2 & 1 & -\sqrt{\mu/\epsilon}/\nu_2 & j\sqrt{\mu/\epsilon} \\ 1 & -j/\nu_1 & -j\sqrt{\mu/\epsilon} & -\sqrt{\mu/\epsilon}/\nu_1 \\ -j/\nu_2 & 1 & \sqrt{\mu/\epsilon}/\nu_2 & j\sqrt{\mu/\epsilon} \end{bmatrix}, \tag{15}$$

and its inverse is

$$T^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -j\nu_2 & 1 & j\nu_2 \\ -j\nu_1 & 1 & j\nu_1 & 1 \\ j\sqrt{\epsilon/\mu} & -\sqrt{\epsilon/\mu}\nu_2 & j\sqrt{\epsilon/\mu} & \sqrt{\epsilon/\mu}\nu_2 \\ \sqrt{\epsilon/\mu}\nu_1 & -j\sqrt{\epsilon/\mu} & -\sqrt{\epsilon/\mu}\nu_1 & -j\sqrt{\epsilon/\mu} \end{bmatrix}, \tag{16}$$

where

$$\nu_1 = \frac{\sqrt{\epsilon\mu} + \kappa\sqrt{\epsilon_0\mu_0}}{\lambda_1}, \tag{17}$$

$$\nu_2 = \frac{(\sqrt{\epsilon\mu} - \kappa\sqrt{\epsilon_0\mu_0}) - q(\sqrt{\epsilon\mu} + \kappa\sqrt{\epsilon_0\mu_0})}{\lambda_2}. \tag{18}$$

Note that these eigenmodes are circularly polarized. The total electromagnetic fields are thus decomposed into the down- and up-going modes as follows

$$\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix} = T^{-1}\begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix}. \tag{19}$$

For normal incidence, i.e. $\theta_0 = 0$, all the results in this section reduce to the ones given in [17]. In the following sections these eigenmodes are used to solve the propagation and scattering problem for a stratified chiral slab between homogeneous chiral half-spaces excited by an obliquely incident plane wave.

4. Multilayered Chiral Slab

We wish to treat the influence of both finite discontinuities and of a continuous variation of the medium parameters in the slab. The case of a finite number of piecewise constant layers then serves as a suitable context in which to introduce the reflection, transmission and Green matrices, together with the equations which they satisfy. This is done in the present section. The generalization to continuously varying parameter case is given in the next section.

Consider the scattering problem for a chiral slab consisting of N homogeneous layers. Denote the positions of the interfaces by $z_0(=0)$, z_1, z_2, $\dots, z_{N-1}, z_N(=l)$. A reflection coefficient matrix, denoted $\mathbf{r}(z)$ (note that the frequency-dependence of all the fields and scattering coefficients will not be given in their arguments of them for simplicity of notation), associated with the internal eigenmodes at depth z is introduced as follows (cf. [17])

$$\mathbf{E}^-(z) = \mathbf{r}(z)\mathbf{E}^+(z) \equiv \begin{bmatrix} r_{11}(z) & r_{12}(z) \\ r_{21}(z) & r_{22}(z) \end{bmatrix} \mathbf{E}^+(z). \tag{20}$$

From the above definition one notices that $\mathbf{r}(0^-)$ is the physical reflection coefficient matrix for the whole slab.

Substituting Eq. (20) into Eq. (12), one obtains the following equation for the $\mathbf{r}$ matrix within each layer

$$\partial_z \mathbf{r} = j\omega(\lambda\mathbf{r} + \mathbf{r}\lambda), \qquad z_i < z < z_{i+1}, \tag{21}$$

where

$$\lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \tag{22}$$

Integrating Eq. (21) from z_{i+1}^- to z_i^+, yields

$$\mathbf{r}(z_i^+) = \begin{bmatrix} r_{11}(z_{i+1}^-)e^{-2j\omega\lambda_1(z_{i+1} - z_i)} & r_{12}(z_{i+1}^-)e^{-j\omega(\lambda_1 + \lambda_2)(z_{i+1} - z_i)} \\ r_{21}(z_{i+1}^-)e^{-i\omega(\lambda_1 + \lambda_2)(z_{i+1} - z_i)} & r_{22}(z_{i+1}^-)e^{-2j\omega\lambda_2(z_{i+1} - z_i)} \end{bmatrix} \tag{23}$$

Let

$$T(z)T^{-1}(z') \equiv \begin{bmatrix} A(z,z') & B(z,z') \\ C(z,z') & D(z,z') \end{bmatrix}, \tag{24}$$

where the 2×2 matrices A, B, C and D are

$$A(z,z') = \frac{1}{4}\begin{bmatrix} [1+\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][1+\frac{\nu_1(z')}{\nu_1(z)}] & j[1-\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][\frac{1}{\nu_1(z)-\nu_2(z')}] \\ j[1-\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][\frac{1}{\nu_2(z)-\nu_1(z')}] & [1+\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][1+\frac{\nu_2(z')}{\nu_2(z)}] \end{bmatrix}, \tag{25}$$

$$B(z,z') = \frac{1}{4}\begin{bmatrix} [1+\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][1-\frac{\nu_1(z')}{\nu_1(z)}] & j[1-\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][\frac{1}{\nu_1(z)+\nu_2(z')}] \\ j[1-\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][\frac{1}{\nu_2(z)+\nu_1(z')}] & [1+\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][1-\frac{\nu_2(z')}{\nu_2(z)}] \end{bmatrix}, \tag{26}$$

$$C(z,z') = \frac{1}{4}\begin{bmatrix} [1+\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][1-\frac{\nu_1(z')}{\nu_1(z)}] & -j[1-\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][\frac{1}{\nu_1(z)+\nu_2(z')}] \\ -j[1-\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][\frac{1}{\nu_2(z)+\nu_1(z')}] & [1+\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][1-\frac{\nu_2(z')}{\nu_2(z)}] \end{bmatrix}, \tag{27}$$

$$D(z,z') = \frac{1}{4}\begin{bmatrix} [1+\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][1+\frac{\nu_1(z')}{\nu_1(z)}] & -j[1-\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][\frac{1}{\nu_1(z)-\nu_2(z')}] \\ -j[1-\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][\frac{1}{\nu_2(z)-\nu_1(z')}] & [1+\sqrt{\frac{\epsilon(z')/\mu(z')}{\epsilon(z)/\mu(z)}}][1+\frac{\nu_2(z')}{\nu_2(z)}] \end{bmatrix}. \tag{28}$$

From the continuity of the tangential electric fields $\mathbf{E}(z)$ and magnetic fields $\mathbf{H}(z)$, the relations between the reflection coefficient matrices across an interface z_i, $i = 0, 1, 2, \dots, N$ are obtained as (17)

$$\mathbf{r}(z_i^-) = [C(z_i^-, z_i^+) + D(z_i^-, z_i^+)\mathbf{r}(z_i^+)] \\ [A(z_i^-, z_i^+) + B(z_i^-, z_i^+)\mathbf{r}(z_i^+)]^{-1}. \tag{29}$$

Since there is no up-going wave at $z = l^+ (\equiv z_N^+)$, one has the following boundary condition

$$\mathbf{r}(l^+) = 0. \tag{30}$$

Therefore, one obtains a solution for the reflection coefficient matrix $\mathbf{r}(0^-)$
through Eqs. (23)–(30), i.e., starting from $z = z_N^+$ (cf. Eq. (30), using Eq. (29) at each interface and using Eq. (23) within each layer.

To calculate the internal fields, we introduce two matrices $\mathbf{g}^+(z)$ and $\mathbf{g}^-(z)$ of Green functions ([17, 11]) to express the internal eigenmodes $\mathbf{E}^\pm$ in terms of the down-going incident modes $\mathbf{E}^+(0^-)$,

$$\begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix}(z) = \begin{bmatrix} \mathbf{g}^+(z)\mathbf{E}^+(0^-) \\ \mathbf{g}^-(z)\mathbf{E}^+(0^-) \end{bmatrix}, \tag{31}$$

where

$$\mathbf{g}^+(z) = \begin{bmatrix} g_{11}^+(z) & g_{12}^+(z) \\ g_{21}^+(z) & g_{22}^+(z) \end{bmatrix}, \quad \mathbf{g}^-(z) = \begin{bmatrix} g_{11}^-(z) & g_{12}^-(z) \\ g_{21}^-(z) & g_{22}^-(z) \end{bmatrix}. \tag{32}$$

Substituting Eq. (31) into Eq. (12), one obtains the following equations for the matrices $\mathbf{g}^\pm$ within each layer

$$\partial_z \mathbf{g}^+ = -j\omega\lambda\mathbf{g}^+, \qquad z_i < z < z_{i+1}, \tag{33}$$

$$\partial_z \mathbf{g}^- = j\omega\lambda\mathbf{g}^-, \qquad z_i < z < z_{i+1}, \tag{34}$$

which gives

$$\mathbf{g}^+(z_{i+1}^-) = \begin{bmatrix} g_{11}^+(z_i^+)e^{-j\omega\lambda_1(z_{i+1}-z_i)} & g_{12}^+(z_i^+) \\ g_{21}^+(z_i^+) & g_{22}^+(z_i^+)e^{-j\omega\lambda_2(z_{i+1}-z_i)} \end{bmatrix}, \tag{35}$$

$$\mathbf{g}^-(z_{i+1}^-) = \begin{bmatrix} g_{11}^-(z_i^+)e^{j\omega\lambda_1(z_{i+1}-z_i)} & g_{12}^-(z_i^+) \\ g_{21}^-(z_i^+) & g_{22}^-(z_i^+)e^{j\omega\lambda_2(z_{i+1}-z_i)} \end{bmatrix} \tag{36}$$

with $i = 0, 1, 2, ..., N-1$. ¿From the definitions of the Green function and reflection coefficient matrices, one obtains the boundary values for Green function matrices

$$\mathbf{g}^+(0^-) = I, \tag{37}$$

$$\mathbf{g}^-(0^-) = \mathbf{r}(0^-), \tag{38}$$

where I is the 2×2 unit matrix. From the continuity of the tangential electric fields $\mathbf{E}(z)$ and magnetic fields $\mathbf{H}(z)$, one obtains the relation between the Green function matrices across an interface z_i, $i = 0, 1, 2, ..., N$ (cf. [17])

$$\mathbf{g}^+(z_i^+) = A(z_i^+, z_i^-)\mathbf{g}^+(z_i^-) + B(z_i^+, z_i^-)\mathbf{g}^-(z_i^-), \tag{39}$$

$$\mathbf{g}^-(z_i^+) = C(z_i^+, z_i^-)\mathbf{g}^+(z_i^-) + D(z_i^+, z_i^-)\mathbf{g}^-(z_i^-). \tag{40}$$

Therefore, if the reflection coefficient matrix $\mathbf{r}(0^-)$ is known, one obtains the solution for the Green functions $g_{ij}^{\pm}(z)$, $i, j = 1, 2$, using Eqs. (35)–(40).

In particular, the boundary values of $\mathbf{g}^+$ at $z = l^+$ give the transmission coefficient matrix, i.e.

$$\mathbf{g}^+(l^+) = \mathbf{t}, \tag{41}$$

where the transmission coefficient matrix $\mathbf{t}$ is defined by

$$\mathbf{E}^+(l^+) = \mathbf{t}\mathbf{E}^+(0^-) \equiv \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \mathbf{E}^+(0^-). \tag{42}$$

Hence, according to Eqs. (41), (42) and (19) the transmitted fields $\mathbf{E}$ and $\mathbf{H}$ in the homogeneous region $z > l$ can be obtained from a knowledge of $\mathbf{g}^+(l^+)$.

5. General Stratification

In this section we derive the equations which give the solution for the general case in which the parameters $\epsilon(z)$, $\mu(z)$ and $\kappa(z)$ are piecewise continuous and piecewise continuously differentiable (they may have multiple discontinuities).

From Eqs. (8) and (11), one obtains the equations for the eigenmodes $\mathbf{E}^{\pm}$ in a continuously stratified chiral medium

$$\partial_z \begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix} = TWT^{-1} \begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix} + (\partial_z T)T^{-1} \begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix}$$

$$= \begin{bmatrix} -j\omega\lambda + \alpha_1 & \beta_1 \\ \gamma_1 & j\omega\lambda + \delta_1 \end{bmatrix} \begin{bmatrix} \mathbf{E}^+ \\ \mathbf{E}^- \end{bmatrix}, \tag{43}$$

where the 2×2 matrices α_1, β_1, γ_1 and δ_1 are obtained from

$$\begin{bmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{bmatrix} \equiv (\partial_z T)T^{-1}. \tag{44}$$

Note that $\mathbf{E}^+$ and $\mathbf{E}^-$ only have physical meaning as down- and up-going eigenmodes, respectively, in a homogeneous chiral medium but not in an inhomogeneous region (TWT^{-1} is not diagonal in an inhomogeneous region, cf. (43)). Inside the stratified slab Eq. (11) is nevertheless a useful change of basis for Maxwell's equations from the variables $\mathbf{E}$, $\mathbf{H}$ to $\mathbf{E}^{\pm}$.

Substituting Eq. (20) into Eq. (43), one obtains the following ODE for the $\mathbf{r}$ matrix (cf. [17])

$$\partial_z \mathbf{r} = j\omega(\lambda\mathbf{r} + \mathbf{r}\lambda) + \gamma_1 + (\delta_1\mathbf{r} - \mathbf{r}\alpha_1) - \mathbf{r}\beta_1\mathbf{r}. \tag{45}$$

Eq. (45) is a matrix differential equation of Riccati type (the Riccati equation for the stratified reciprocal chiral medium, has also been derived through some other approaches, cf. [21]; it is a consequence of the properties of the Redheffer star product for the scattering operator for a plane-stratified medium, cf. [22]). Thus, one obtains a solution for $\mathbf{r}(0^-)$ by integrating Eq. (45) along the $-z$ direction starting from

$z = l^+$ (cf. Eq. (30)). In particular, one needs to use Eq. (29) to treat any parameter mismatch at a plane.

The ODEs for the Green function matrices are

$$\partial_z \mathbf{g}^+ =(-j\omega\lambda + \alpha_1)\mathbf{g}^+ + \beta_1 \mathbf{g}^-, \quad (46)$$

$$\partial_z \mathbf{g}^- =\gamma_1 \mathbf{g}^+ + (j\omega\lambda + \delta_1)\mathbf{g}^-. \quad (47)$$

The solution for the Green functions $g_{ij}^{\pm}(z)$, $i,j = 1,2$, is obtained by integrating Eqs. (46) and (47) along the z direction starting from $z = 0^-$ (cf. Eqs. (37), (38)). In particular, Eqs. (39) and (40) are used to treat any parameter mismatch at a plane. We refer to [18] and [19] for comments on the numerical solution of the Riccati equation (45) and the ODEs (46), (47), as well as for some numerical results (which refer to a normal incidence case; however, the structure of the equations is the same).

6. Discussion and Numerical Results

A wave-splitting approach to the propagation and scattering problem for a stratified chiral slab excited by an obliquely incident plane wave has been presented, which applies to the general case of multiple discontinuities, contained with continuous variation of the parameters between the discontinuities. Multilayered models with very many thin layers may conceivably also be used to approximate such situations, but a method which directly addresses the problem of non-constant medium parameters should be attractive both in terms of the insight it provides and the resulting numerical effectiveness. Up- and down-going eigenmodes were identified together with their dispersion equations. Above the inhomogeneous slab ($z < 0$), the down-going and up-going eigenmodes are related to the incident and reflected fields, respectively, as

$$\begin{bmatrix} \mathbf{E}^+(0^-) \\ 0 \end{bmatrix} = T(0^-) \begin{bmatrix} \mathbf{E}^{\text{inc.}}(0^-) \\ \mathbf{H}^{\text{inc.}}(0^-) \end{bmatrix}, \quad (48)$$

$$\begin{bmatrix} 0 \\ \mathbf{E}^-(0^-) \end{bmatrix} = T(0^-) \begin{bmatrix} \mathbf{E}^{\text{refl.}}(0^-) \\ \mathbf{H}^{\text{refl.}}(0^-) \end{bmatrix}. \quad (49)$$

In particular, one has

$$\begin{aligned}\begin{bmatrix}\mathbf{E}^{\text{refl.}}(0^-)\\ \mathbf{H}^{\text{refl.}}(0^-)\end{bmatrix} &= T^{-1}(0^-)\begin{bmatrix}0\\ \mathbf{E}^-(0^-)\end{bmatrix}\\ &= T^{-1}(0^-)\begin{bmatrix}0\\ \mathbf{r}(0^-)\mathbf{E}^+(0^-)\end{bmatrix}\\ &= T^{-1}(0^-)\begin{bmatrix}0 & 0\\ \mathbf{r}(0^-) & 0\end{bmatrix}\begin{bmatrix}\mathbf{E}^+(0^-)\\ 0\end{bmatrix}\\ &= T^{-1}(0^-)\begin{bmatrix}0 & 0\\ \mathbf{r}(0^-) & 0\end{bmatrix} T(0^-)\begin{bmatrix}\mathbf{E}^{\text{inc.}}(0^-)\\ \mathbf{H}^{\text{inc.}}(0^-)\end{bmatrix}. \end{aligned} \tag{50}$$

Similarly,

$$\begin{bmatrix}\mathbf{E}^{\text{tr.}}(l^+)\\ \mathbf{H}^{\text{tr.}}(l^+)\end{bmatrix} = T^{-1}(l^+)\begin{bmatrix}\mathbf{t} & 0\\ 0 & 0\end{bmatrix} T(0^-)\begin{bmatrix}\mathbf{E}^{\text{inc.}}(0^-)\\ \mathbf{H}^{\text{inc.}}(0^-)\end{bmatrix}. \tag{51}$$

Therefore, once the scattering coefficient matrices $\mathbf{r}(0^-)$ and $\mathbf{t}$ are calculated, the scattered fields are known from Eqs. (50) and (51).

If the homogeneous regions outside the stratified slab is vacuum, i.e. $\kappa = 0$, when $z < 0$ or $z > l$, then the two pairs of eigenmodes in these regions have the same eigenvalues, i.e. $\lambda_1 = \lambda_2 (= \lambda_0)$, and any combination of the eigenmodes E_1^+ and E_2^+ (or E_1^- and E_2^-) is a down-going (or up-going) eigenmode. In particular, one can then choose the transformation matrix for the eigenmodes in such a way that the associated split eigenmodes correspond to up- and down-going TM and TE modes, respectively. Namely, in stead of putting $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\kappa = 0$ in the transformation matrix T given by Eq. (15), one considers

$$\tilde{T} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & \frac{1}{\nu_0}\sqrt{\mu_0/\epsilon_0} \\ 0 & 1 & -\nu_0\sqrt{\mu_0/\epsilon_0} & 0 \\ 1 & 0 & 0 & -\frac{1}{\nu_0}\sqrt{\mu_0/\epsilon_0} \\ 0 & 1 & \nu_0\sqrt{\mu_0/\epsilon_0} & 0 \end{bmatrix}, \quad z < 0, \tag{52}$$

where $\nu_0 = \sqrt{\mu_0\epsilon_0}/\lambda_0 = 1/\cos\theta_0$, and define the propagating eigenmodes in the region $z < 0$ or $z > l$ as

$$\begin{bmatrix} \mathbf{F}^+ \\ \mathbf{F}^- \end{bmatrix} = \tilde{T}\begin{bmatrix} \mathbf{E} \\ \mathbf{H} \end{bmatrix}, \quad z < 0, \text{ or } z > l. \tag{53}$$

Then linearly polarized $\mathbf{F}^+$ and $\mathbf{F}^-$ also have physical meaning as down- and up-going eigenmodes, respectively, in the vacuum. As can easily be checked, the F_1^+, F_1^- modes are down- and up-going TM modes, respectively, and the F_2^+, F_2^- modes are down- and up-going TE modes, respectively ($F_i^\pm$, $i = 1,2$, give the amplitudes of the tangential components of the electric field for these modes). As is usual, the reflection coefficients for TM and TE modes are defined as

$$r_{XY} = \frac{E_{TX}^r}{E_{TY}^i}, \quad X, Y = E \text{ or } M, \tag{54}$$

where E_{TE}^r (E_{TE}^i) is the amplitude of the reflected (incident) electric field for TE mode, etc.. r_{MM}, r_{EM} are the co- and cross-polarized reflection coefficients for TM mode incidence, respectively, and r_{EE}, r_{ME} are the co- and cross-polarized reflection coefficients for TE mode incidence, respectively. It is easy to see that

$$r_{MM} = \tilde{r}_{11}, \quad r_{ME} = \frac{\text{-}\tilde{r}_{12}}{\cos\theta_0}, \quad r_{EM} = \tilde{r}_{21}\cos\theta_0, \quad r_{EE} = \tilde{r}_{22}, \tag{55}$$

where $\tilde{r}_{ij}$, $i,j = 1,2$, are defined by

$$\mathbf{F}^-(0^-) = \begin{bmatrix} \tilde{r}_{11} & \tilde{r}_{12} \\ \tilde{r}_{21} & \tilde{r}_{22} \end{bmatrix} \mathbf{F}^+(0^-), \tag{56}$$

and are related to the reflection coefficient matrix defined in the previous sections through the formula

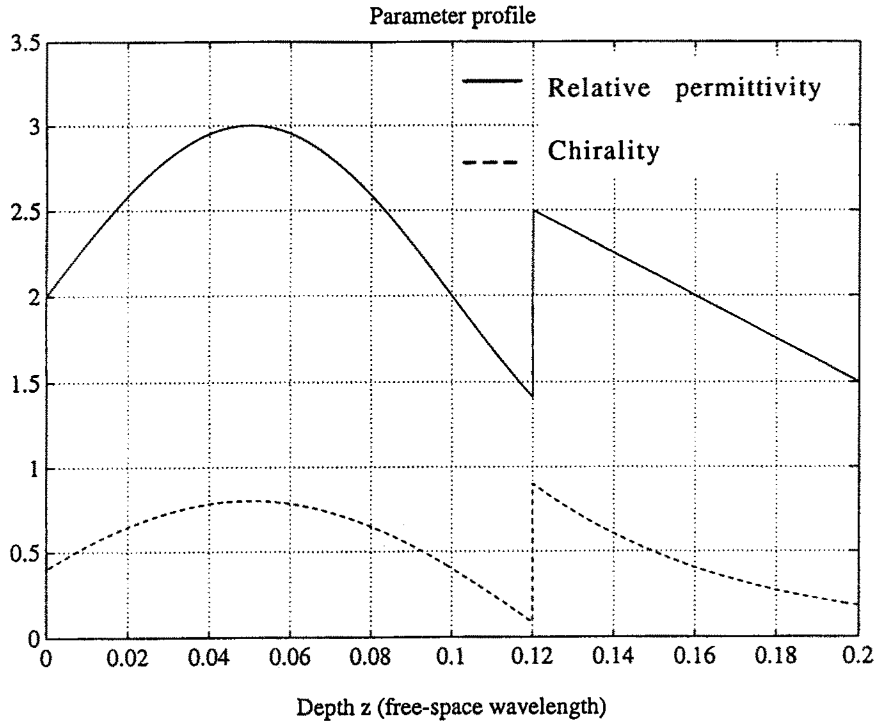

Figure 2. The permittivity and chirality parameter for a stratified chiral slab.

$$\begin{bmatrix} \tilde{r}_{11} & \tilde{r}_{12} \\ \tilde{r}_{21} & \tilde{r}_{22} \end{bmatrix} = [c + d\mathbf{r}(0^-)]\,[a + b\mathbf{r}(0^-)]^{-1}, \tag{57}$$

where the 2×2 matrices a, b, c and d are obtained from

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \equiv \tilde{T}T^{-1}(0^-). \tag{58}$$

Similarly, the co- and cross-polarized transmission coefficients for TE and TM mode incidences can be obtained from

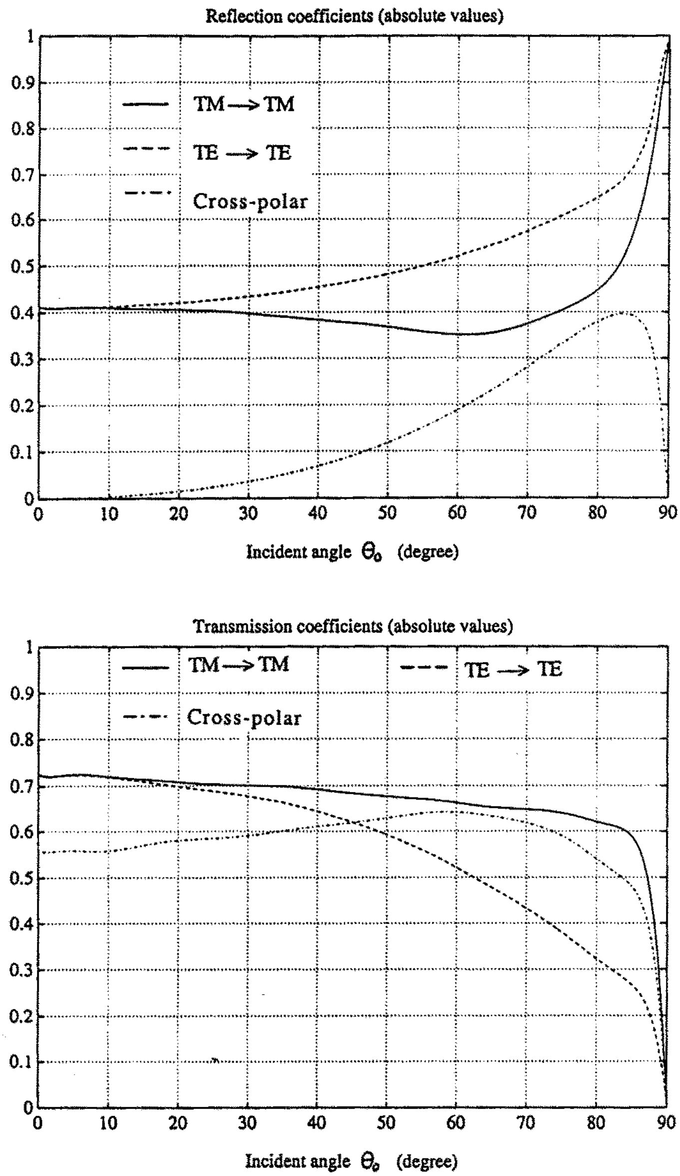

Figure 3. The co- and cross-polarized reflection and transmission coefficients for TE and TM modes as functions of the incident angle θ_0 for the stratified chiral slab given in Fig. 2 (the permeability is a constant μ_0). The thickness of the slab l is equal to 0.2 wavelength in free space (1 wavelength $= 2\pi/(\sqrt{\epsilon_0\mu_0}\omega)$).

$$\begin{bmatrix} t_{MM} & t_{ME}\cos\theta_0 \\ \frac{t_{EM}}{\cos\theta_0} & t_{EE} \end{bmatrix} = \tilde{a}\mathbf{t}\,[a + b\mathbf{r}(0^-)]^{-1}, \tag{59}$$

where the 2×2 matrix $\tilde{a}$ is obtained from

$$\begin{bmatrix} \tilde{a} & \tilde{b} \\ \tilde{c} & \tilde{d} \end{bmatrix} \equiv \tilde{T}T^{-1}(l^+). \tag{60}$$

These relations provide the connection between the formalism developed in Sections 2 – 5 and what constitutes a natural choice of modes in a non-chiral exterior medium. The co- and cross-polarized reflection and transmission coefficients for TE and TM modes as functions of the incident angle θ_0 are plotted in Figs. 3(a) and 3(b) for the stratified chiral slab given in Fig. 2 (the relative permittivity $\epsilon_r = [2+\sin(10\pi z/\lambda_0)]H(z)H(0.12\lambda_0 - z) + [4 - 12.5z/\lambda_0]H(z - 0.12\lambda_0)$, the chirality $\kappa = 0.8\sin[(\pi/6)(40z/\lambda_0 + 1)]H(z)H(0.12\lambda_0 - z) + 0.9\exp[-20(z/\lambda_0 - 0.12)]H(z - 0.12\lambda_0)$, where $\lambda_0 = 2\pi/(\sqrt{\epsilon_0\mu_0}\omega)$ is the wavelength in free space and $H(z)$ is the Heaviside step function vanishing for $z < 0$; the permeability is a constant μ_0). The thickness of the slab l is equal to $0.2\lambda_0$. Numerical results show that

$$r_{EM} = r_{ME}, \qquad t_{EM} = -t_{ME}, \tag{61}$$

which is due to the reciprocity of the medium.

Several aspects of the formalism presented here are under further development. One application of the case of a non-chiral exterior medium is to the computation of the Brewster angle for complex media [24] (the Brewster angle is the incident angle at which there is no reflected power for a certain polarized mode, cf. [23]). The approach can also be used to solve the scattering problem for a dipole radiating above a stratified chiral slab, which will result in a similar problem as the one for the obliquely incident plane wave after making a spatial Fourier transform. The present approach is also useful to the study of propagation in parallel-plate wave-guiding structures involving chiral media.

Acknowledgments

The partial support of the Swedish Research Council for Engineering Sciences is gratefully acknowledged.

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