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ELECTROMAGNETIC MODELING OF BI-ISOTROPIC MEDIA

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1. Introduction

The present wave of interest in complex materials has reached a stage within the field of microwaves where commercial applications can be expected soon. Of course, new electromagnetic effects in materials have been always of interest for engineers, but particularly in the latest years, novel materials have attracted special attention. Within this class of novel, "exotic" materials one may count, for example, chiral, nonreciprocal, nonlinear, gyrotropic, and high-T_c superconducting materials, which all seem to contain potential for promising applications in microwave and millimeter wave engineering, as well as in the infrared and optical regime.

In the design of composite materials for electromagnetic applications, the aim often is to find the correct way of mixing component phases to realize the desired electric and magnetic macroscopic parameters in the mixture. Having now — or soon — access to new "nonclassical" materials, this task is even more challenging. When playing with these materials, one has to be aware of the fact that there is magnetoelectric or anisotropic coupling within the molecular level of the materials. This shows itself at the phenomenological description in the constitutive relations. Therefore the mixing laws for heterogeneous media become more intertwined and complex than in the classical description of dielectric mixtures. Along with this trend towards more general electromagnetic description, also the design of new, e.g. chiral composites, has to rely on more complicated and interdependent formulas than the early-days design of artificial dielectrics.

Media effects are diverse, and bifurcate in several dimensions. This article will focus on one subclass of novel media, viz. that of bi-isotropic media. Bi-isotropic media are more general than isotropic media due to the fact that magnetoelectric coupling is allowed. A possibility to subdivide bi-isotropic materials is the following classification with five groups:

- ordinary *dielectric* media, consisting microscopically of electrically polarizable entities, which are induced by the electric field
- *magnetic* media, displaying, analogously to dielectric media, magnetic polarization due to an external magnetic field
- reciprocal *chiral* media, that due to their inherent left- or right-handedness exhibit magnetically caused electric dipole moment density and vice versa
- *nonreciprocal* media, where the magnetoelectric effect is in-phase unlike in chiral media
- media characterized by any combination of the four aforementioned effects

The research on chiral media has been especially intense within microwaves in recent years [1], and chirality is an example of novel material effects at best. Nonreciprocity is a comparable effect to chirality from the theoretical point of view but in terms manufacturing real-life samples, nonreciprocal materials are not as advanced. However, electromagnetic analyses promise wonderful potential in applications for naturally nonreciprocal media.

The following sections attempt to present electromagnetic models for heterogeneous bi-isotropic media that can be used in chiral composite design, for example. But in addition to presenting expressions for effective material parameters of bi-isotropic mixtures, this article tries to convey a message that there is profound physical contents hidden (and sometimes even clearly visible) within the mixing laws of bi-isotropic media. One can intuitively interpret the polarizability and macroscopic parameter expressions and gain insight how the properties of the microgeometry of the medium are reflected in macroscopic properties that rule the electromagnetic response of the material. This fact, the expanded physical interpretation, is one reward that one gets after paying the price of facing complicated expressions in bi-isotropic media modeling.

2. Constitutive Parameters

The most general linear, homogeneous, nondiffusive medium responds to electromagnetic excitation according to the following bianisotropic constitutive relations:

$$\bar{D} = \bar{\bar{\epsilon}} \cdot \bar{E} + \bar{\bar{\xi}} \cdot \bar{H} \tag{1}$$

$$\bar{B} = \bar{\bar{\mu}} \cdot \bar{H} + \bar{\bar{\zeta}} \cdot \bar{E} \tag{2}$$

Here, $\bar{E}$ is the electric and $\bar{H}$ the magnetic field strength, $\bar{D}$ is the electric and $\bar{B}$ the magnetic flux density. The material effects are contained in the dyadics $\bar{\bar{\epsilon}}, \bar{\bar{\mu}}, \bar{\bar{\xi}}, \bar{\bar{\zeta}}$ that, due to anisotropy, contain 36 scalar material parameters in maximum.

In this article, however, the emphasis is on isotropic media. Hence the material response is independent of the direction of the vector force of the electric or magnetic field. The parameter dyadics above are multiples of the unit dyadic. Therefore constitutive relations of *bi-isotropic* media can be written as

$$\bar{D} = \epsilon \bar{E} + \xi \bar{H} \tag{3}$$

$$\bar{B} = \mu \bar{H} + \zeta \bar{E} \tag{4}$$

and, redefining the magnetoelectric coupling coefficients ξ, ζ:

$$\bar{D} = \epsilon \bar{E} + (\chi - j\kappa)\sqrt{\mu_0 \epsilon_0}\bar{H} \tag{5}$$

$$\bar{B} = \mu \bar{H} + (\chi + j\kappa)\sqrt{\mu_0 \epsilon_0}\bar{E} \tag{6}$$

The advantage of using relations (5)–6 rather than (3)–(4) is that the material parameters in (5)–(6) have clear physical meaning: the permittivity ϵ is a measure for the electrical polarization induced by the electric field, and permeability μ gives correspondingly the magnitude of the magnetic co-polarizability. The magnetoelectric parameters measure the crosspolarizability propensity of the medium, and are confined very naturally in κ and χ.

The parameter κ contains the degree of chirality, and it is a dimensionless parameter, due to the separation of the factor $\sqrt{\mu_0 \epsilon_0}$ (ϵ_0 and μ_0 are the permittivity and permeability of the vacuum). Chirality is a measure for the handedness of the material. For a left–right symmetric medium[1], $\kappa = 0$.

The remaining material parameter is χ which is a dimensionless quantity for the degree of inherent nonreciprocity in the medium. It is known that externally, nonrecirocity can be created by a static magnetic field in ferrites or plasmas, but in these cases, the medium is anisotropic. The simultaneous requirements of both nonreciprocity and isotropy lead to magnetoelectric coupling, and a nonzero value for χ. An example of naturally nonreciprocal material (although anisotropic) is chromium oxide[2] [2,3].

Honoring scientists that have given contributions to the study of these magnetoelectric media, the reciprocal chiral medium ($\chi = 0, \kappa \neq 0$) can be called *Pasteur medium,* and the nonreciprocal nonchiral medium ($\chi \neq 0, \kappa = 0$) sometimes carries the name *Tellegen medium.* For references about their work as well as other historical contributions to the magnetoelectric material research, *cf.* [4,5].

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[1] A medium which does not differ from its mirror image.

[2] Cr_2O_3

In the simplest case of lossless materials, it can be shown that all the four parameters $\epsilon, \mu, \chi, \kappa$ are real. For lossy media, the material parameters are complex scalars.

The coefficient j in the constitutive relations reflects the fact that the relations implicitly assume a time-harmonic dependence for the fields. The convention for the sign of the imaginary part is according to the hidden time dependence $\exp(j\omega t)$. In addition to relations (5)–(6), there are also other notations and constitutive relations for the material parameters of chiral and bi-isotropic media, see e.g. [6]. This paper will only deal with relations (5)–(6). If there exists a need to translate the conclusions on bi-isotropic media of the present study to other representations, the formulas of [6] can be easily applied for this purpose.

The magnetoelectric material parameters can be measured for example by analyzing the state of polarized radiation that has been reflected from a bi-isotropic interface or transmitted through a bi-isotropic slab. The common belief is that because chirality affects the polarization plane of a propagating wave, transmission methods would be most suitable for measurements of κ. Correspondingly, reflection methods would give most easily the nonreciprocity parameter. Although there exists a simple relation between chirality and the total rotation of the plane of linear polarization, it is true that chirality affects the reflected wave, too[3]. This effect has been shown to be especially strong and would provide an alternative method for spectroscopy of chiral materials [8]. However, the focus in this article is not on measurement techniques but rather on the properties on heterogeneous bi-isotropic media. Let us therefore start by treating the polarizability properties of a single bi-isotropic inclusion.

3. Polarizabilities of Bi-isotropic Inclusions

Taking a look at any sample of material media on the small-scale level, the structure is always seen to display inhomogeneitites. A classical approach is to model the structure by polarizable entities, called in-

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[3] Especially in layered structures the chiral reflection effect can be used advantageously; the numerous stealth-technology-related applications witness this [7].

clusions, or more complicated clusters of inclusions in the case of dense media. On the other hand, the essence of the macroscopic characterization (for example equations (5)–(6)) are the four scalars $\epsilon, \mu, \chi, \kappa$. All polarization phenomena are "hidden" beyond the values of these quantities. In modeling, one needs a macroscopic description of the material. Hence, the average polarization has to be calculated. This is the dipole moment density, wherefore the polarizabilities of the inclusions composing the mixture have to be known. The polarizability, on the other hand, depends on the properties of the inclusions: their volume, shape, and refractive index. Let us focus on the laws that govern these dependencies.

3.1 Bi-isotropic Sphere

In order to calculate the polarizability of a homogeneous bi-isotropic sphere (described by parameters $\epsilon, \mu, \kappa, \chi$), the quasistatic problem of this sphere in vacuum with the presence of an electromagnetic field has to be solved. Although the analysis in this report focuses on time-dependent fields, static field solutions[4] can be used in calculating the polarization densities for small particles. The quasistatic approach is equally valid for magnetoelectric problems as in purely dielectric cases. This question has been discussed thoroughly in [9], and is not repeated here[5]. The quasistatic assumption means that the particle will

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[4] A static field solution for the electric field inside a spherical inclusion in a constant external field is also constant.

[5] It may hurt intuition to use quasistatic approximation in solving bi-isotropic, and particularly chiral problems because it is well known that chiral effects disappear as the frequency ω goes to zero [10]. As a matter of fact, the low-frequency expansion of κ begins with a linear function of frequency: $\kappa(\omega) = \kappa_1 \omega, \ (\omega \to 0)$. This follows from the observation that if one transforms from frequency to time domain by a Fourier integral, the result has to be real. Therefore all j s have to appear in multiples of ω in the expressions. The other way of noting the absence of chiral effects in statics is to use another set of constitutive equations (so called Drude–Born–Fedorov [4] relations), where the chiral parameter is multiplied by the curl of the electric field. In statics, $\nabla \times \bar{E} = 0$. See also [11] where the polarizability expressions of small chiral spheres are derived starting with the full set of Maxwell equations.

be approximated electromagnetically by electric and magnetic dipoles; higher multipoles are ignored. The incident electric and magnetic fields are $\bar{E}, \bar{H}$, which create electric and magnetic polarization $\bar{P}_e, \bar{P}_m$ in the sphere. As a straightforward generalization from the classical dielectric case [12,13], the internal fields can be calculated from the external fields and the polarization inside the sphere in the following way:

$$\begin{pmatrix} \bar{E}_i \\ \bar{H}_i \end{pmatrix} = \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix} - \frac{1}{3} \begin{pmatrix} \bar{P}_e/\epsilon_0 \\ \bar{P}_m/\mu_0 \end{pmatrix} \tag{7}$$

On the other hand, the internal polarization densities are related to the internal fields as

$$\begin{pmatrix} \bar{P}_e \\ \bar{P}_m \end{pmatrix} = \begin{pmatrix} \epsilon - \epsilon_0 & (\chi - j\kappa)\sqrt{\mu_0\epsilon_0} \\ (\chi + j\kappa)\sqrt{\mu_0\epsilon_0} & \mu - \mu_0 \end{pmatrix} \begin{pmatrix} \bar{E}_i \\ \bar{H}_i \end{pmatrix} \tag{8}$$

From these two coupled equations, the internal fields can be solved:

$$\begin{pmatrix} \bar{E}_i \\ \bar{H}_i \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} 3\epsilon_0(\mu + 2\mu_0) & -3\mu_0(\chi - j\kappa)\sqrt{\mu_0\epsilon_0} \\ -3\epsilon_0(\chi + j\kappa)\sqrt{\mu_0\epsilon_0} & 3\mu_0(\epsilon + 2\epsilon_0) \end{pmatrix} \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix} \tag{9}$$

with

$$\Delta = (\mu + 2\mu_0)(\epsilon + 2\epsilon_0) - (\chi^2 + \kappa^2)\mu_0\epsilon_0 \tag{10}$$

Outside the scatterer, the polarization densities have effect that is equal to emanating from electric and magnetic dipole moments $\bar{p}_e, \bar{p}_m$. These dipole moments are proportional to the incident field:

$$\begin{pmatrix} \bar{p}_e \\ \bar{p}_m \end{pmatrix} = \begin{pmatrix} \alpha_{ee} & \alpha_{em} \\ \alpha_{me} & \alpha_{mm} \end{pmatrix} \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix} \tag{11}$$

In the polarizability symbols α_{ij}, there are two indices: the first (i) denotes the polarization type, and the second (j) is for the origin of the polarization. The dipole moments are integrals over the sphere volume V of the polarization, which is constant:

$$\begin{pmatrix} \bar{p}_e \\ \bar{p}_m \end{pmatrix} = \int dV \begin{pmatrix} \bar{P}_e \\ \bar{P}_m \end{pmatrix} = V \begin{pmatrix} \bar{P}_e \\ \bar{P}_m \end{pmatrix} \tag{12}$$

The co- and cross-polarizabilitites can be solved from the previous equations by relating the polarizabilities and the incident fields [9,14]:

$$\alpha_{ee} = 3\epsilon_0 V \frac{(\epsilon - \epsilon_0)(\mu + 2\mu_0) - (\chi^2 + \kappa^2)\mu_0\epsilon_0}{(\mu + 2\mu_0)(\epsilon + 2\epsilon_0) - (\chi^2 + \kappa^2)\mu_0\epsilon_0} \tag{13}$$

$$\alpha_{em} = 3\mu_o\epsilon_0 V \frac{3(\chi - j\kappa)\sqrt{\mu_0\epsilon_0}}{(\mu + 2\mu_0)(\epsilon + 2\epsilon_0) - (\chi^2 + \kappa^2)\mu_0\epsilon_0} \tag{14}$$

$$\alpha_{me} = 3\mu_o\epsilon_0 V \frac{3(\chi + j\kappa)\sqrt{\mu_0\epsilon_0}}{(\mu + 2\mu_0)(\epsilon + 2\epsilon_0) - (\chi^2 + \kappa^2)\mu_0\epsilon_0} \tag{15}$$

$$\alpha_{mm} = 3\mu_0 V \frac{(\mu - \mu_0)(\epsilon + 2\epsilon_0) - (\chi^2 + \kappa^2)\mu_0\epsilon_0}{(\mu + 2\mu_0)(\epsilon + 2\epsilon_0) - (\chi^2 + \kappa^2)\mu_0\epsilon_0} \tag{16}$$

These polarizabilities are needed in the analysis to follow, as the mixing relations are derived. For the nonchiral reciprocal limit $\kappa \to 0$, $\chi \to 0$, these polarizabilities simplify to the well-known expressions

$$\alpha_{ee} = 4\pi\epsilon_0 a^3 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \tag{17}$$

$$\alpha_{mm} = 4\pi\mu_0 a^3 \frac{\mu - \mu_0}{\mu + 2\mu_0} \tag{18}$$

$$\alpha_{me} = \alpha_{em} = 0 \tag{19}$$

where a is the radius of the sphere. Note the decoupling of the electric and magnetic quantities of these results compared with the bi-isotropic case.

3.2 Bi-isotropic Ellipsoid

For ellipsoidal bi-isotropic scatterers, the field analysis can be carried through, because as for spheres, the internal fields are constant in the quasistatic case. The difference is that the polarizability matrix, relating exciting fields and dipole moments, does not consist of four scalars α_{ij} but rather of four dyadics $\bar{\bar{\alpha}}_{ij}$. This is because the ellipsoid depolarizes differently along its three axes. The way this happens is determined by the depolarization dyadic which contains the depolarization factors of the ellipsoid. These depend on the axis ratios: If the semiaxes of the ellipsoid are a, b, c, the depolarization factor in the direction of the a axis is [15,16]

$$N_a = \frac{abc}{2} \int_0^\infty \frac{ds}{(s+a^2)\sqrt{(s+a^2)(s+b^2)(s+c^2)}} \tag{20}$$

For depolarization factors N_b and N_c, interchange b and a, and c and a in Eq. (20), respectively. The depolarization factors satisfy $N_a + N_b + N_c = 1$ for any ellipsoid, and for a sphere, these are equal: $N_a = N_b = N_c = 1/3$. The other two special cases are a disk (depolarization factors 1, 0, 0) and a needle (0, 1/2, 1/2). Closed-form expressions can be written for ellipsoids of revolution [17]. Osborn and Stoner have given tabulated values for the depolarization factors of general ellipsoids [18,19].

The depolarization dyadic is

$$\bar{\bar{L}} = N_a \bar{u}_a \bar{u}_a + N_b \bar{u}_b \bar{u}_b + N_c \bar{u}_c \bar{u}_c \tag{21}$$

with $\bar{u}_a, \bar{u}_b, \bar{u}_c$ being the unit vectors along the axes.

Using the same reasoning as for bi-isotropic spheres, the polarizability components of the bi-isotropic ellipsoid can be calculated. The internal fields are constant, and Equations (7) read in the ellipsoidal case as

$$\begin{pmatrix} \bar{E}_i \\ \bar{H}_i \end{pmatrix} = \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix} - \bar{\bar{L}} \cdot \begin{pmatrix} \bar{P}_e/\epsilon_0 \\ \bar{P}_m/\mu_0 \end{pmatrix} \tag{22}$$

Therefore the dipole moments can be written as functions of the incident fields, and the polarizability matrix components are now dyadics [21]:

$$\begin{pmatrix} \bar{p}_e \\ \bar{p}_m \end{pmatrix} = \begin{pmatrix} \bar{\bar{\alpha}}_{ee} & \bar{\bar{\alpha}}_{em} \\ \bar{\bar{\alpha}}_{me} & \bar{\bar{\alpha}}_{mm} \end{pmatrix} \cdot \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix} \tag{23}$$

Written explicitly, the components of the polarizability dyadics $\bar{\bar{\alpha}}_{rs} = \sum_{i=1}^{3} \alpha_{rs,i} \bar{u}_i \bar{u}_i$ are

$$\alpha_{ee,i} = \frac{\epsilon_0 V}{\Delta_i} \left\{ (\epsilon - \epsilon_0)[N_i \mu + (1 - N_i)\mu_0] - N_i(\chi^2 + \kappa^2)\mu_0 \epsilon_0 \right\} \tag{24}$$

$$\alpha_{mm,i} = \frac{\mu_0 V}{\Delta_i} \left\{ (\mu - \mu_0)[N_i \epsilon + (1 - N_i)\epsilon_0] - N_i(\chi^2 + \kappa^2)\mu_0 \epsilon_0 \right\} \quad (25)$$

$$\alpha_{em,i} = \frac{\mu_0 \epsilon_0 V}{\Delta_i}(\chi - j\kappa)\sqrt{\mu_0 \epsilon_0}, \qquad \alpha_{me,i} = \frac{\mu_0 \epsilon_0 V}{\Delta_i}(\chi + j\kappa)\sqrt{\mu_0 \epsilon_0} \quad (26)$$

with

$$\Delta_i = [N_i \mu + (1 - N_i)\mu_0][N_i \epsilon + (1 - N_i)\epsilon_0] - N_i^2(\chi^2 + \kappa^2)\mu_0 \epsilon_0 \quad (27)$$

and $V = 4\pi abc/3$ being the volume of the ellipsoid. These expressions reduce to Eqs. (13) $\cdots$ (16) as the ellipsoids degenerate into spheres: $N_i \to 1/3$.

3.3 Bi-isotropic Background Material

The previous expressions about sphere and ellipsoid polarizabilities assumed that the background was isotropic in which the inclusion was embedded. If the background is also bi-isotropic, the steps in the analysis and the complexity of matrix algebra remain exactly the same as in the previous subsections. Now the electric and magnetic polarizations have to include also the magnetoelectric coupling in the background medium. This has been done in [20].

Be the material parameters of the background $\epsilon_1, \mu_1, \chi_1, \kappa_1$ and those of the spherical inclusions $\epsilon_2, \mu_2, \chi_2, \kappa_2$. Then the polarizabilities (generalizations of (13)–(16)) read:

$$\begin{aligned} &\alpha_{ee} \\ &= 3V \frac{\epsilon_1(\epsilon_2 - \epsilon_1)(\mu_2 + 2\mu_1) - \epsilon_1[(\chi_2 - \chi_1)^2 + (\kappa_2 - \kappa_1)^2]\mu_0 \epsilon_0}{(\mu_2 + 2\mu_1)(\epsilon_2 + 2\epsilon_1) - [(\chi_2 + 2\chi_1)^2 + (\kappa_2 + 2\kappa_1)^2]\mu_0 \epsilon_0} \\ &+ 3V \frac{-3(\epsilon_2 - \epsilon_1)(\chi_1^2 + \kappa_1^2)\mu_0 \epsilon_0}{(\mu_2 + 2\mu_1)(\epsilon_2 + 2\epsilon_1) - [(\chi_2 + 2\chi_1)^2 + (\kappa_2 + 2\kappa_1)^2]\mu_0 \epsilon_0} \end{aligned} \quad (28)$$

$$\alpha_{em} = 3V\sqrt{\mu_0\epsilon_0}\frac{3\mu_1\epsilon_1[(\chi_2-\chi_1)-j(\kappa_2-\kappa_1)]+A}{(\mu_2+2\mu_1)(\epsilon_2+2\epsilon_1)-[(\chi_2+2\chi_1)^2+(\kappa_2+2\kappa_1)^2]\mu_0\epsilon_0} \tag{29}$$

$$\alpha_{me} = 3V\sqrt{\mu_0\epsilon_0}\frac{3\mu_1\epsilon_1[(\chi_2-\chi_1)+j(\kappa_2-\kappa_1)]+A^*}{(\mu_2+2\mu_1)(\epsilon_2+2\epsilon_1)-[(\chi_2+2\chi_1)^2+(\kappa_2+2\kappa_1)^2]\mu_0\epsilon_0} \tag{30}$$

$$\begin{aligned}\alpha_{mm} =& 3V\frac{\mu_1(\mu_2-\mu_1)(\epsilon_2+2\epsilon_1)-\mu_1[(\chi_2-\chi_1)^2+(\kappa_2-\kappa_1)^2]\mu_0\epsilon_0}{(\mu_2+2\mu_1)(\epsilon_2+2\epsilon_1)-[(\chi_2+2\chi_1)^2+(\kappa_2+2\kappa_1)^2]\mu_0\epsilon_0}\\ &+3V\frac{-3(\mu_2-\mu_1)(\chi_1^2+\kappa_1^2)\mu_0\epsilon_0}{(\mu_2+2\mu_1)(\epsilon_2+2\epsilon_1)-[(\chi_2+2\chi_1)^2+(\kappa_2+2\kappa_1)^2]\mu_0\epsilon_0}\end{aligned} \tag{31}$$

where the background-dependent correction term A is

$$\begin{aligned}A =&(\chi_1-j\kappa_1)\{(\mu_2-\mu_1)(\epsilon_2-\epsilon_1)-[(\chi_2-\chi_1)-j(\kappa_2-\kappa_1)]\\ &[(\chi_2+2\chi_1)+j(\kappa_2+2\kappa_1)]\mu_0\epsilon_0\}\end{aligned} \tag{32}$$

$$\begin{aligned}A^* =&(\chi_1+j\kappa_1)\{(\mu_2-\mu_1)(\epsilon_2-\epsilon_1)-[(\chi_2-\chi_1)+j(\kappa_2-\kappa_1)]\\ &[(\chi_2+2\chi_1)-j(\kappa_2+2\kappa_1)]\mu_0\epsilon_0\}\end{aligned} \tag{33}$$

3.4 Discussion on the Polarizabilities

A look at the polarizability expressions reveals several facts. The polarizabilities of bi-isotropic inclusions are dependent on the four material parameters of the inclusion material, and trivially[6] on the volume of the sphere. In addition, the ellipsoidal inclusions display the shape effect as in the isotropic dielectric case: the smaller the depolarization factor, the larger the polarizabilities. In the following let us concentrate on the manner how the magnetoelectric parameters affect the sphere polarizabilities.

Often it happens that the chirality and nonreciprocity parameters are small compared with the relative permittivity and permeability parameters. Equations (13)–(14) show that because the copolarizabilities α_{ee}, α_{mm} are even functions of both χ and κ, which could have values of the order of 0.1 or less, the effect of these is rather small. The main behavior of the copolarizabilities is that an increase in ϵ increases α_{ee} (similarly happens for α_{mm} affected by μ). The effect of increasing κ or χ is that both copolarizabilities decrease. In addition, the introduction of a nonzero κ or χ leads to the crossdependence of α_{ee} on μ and also α_{mm} on ϵ.

The crosspolarizabilities (15), (16) increase for increasing χ and κ, and decrease for increasing ϵ and μ. For illustrations on these behaviors, see [22].

One extremely interesting effect can be seen in Fig. 1 where the magnetic copolarizability α_{mm} is plotted as a function of the inclusion permeability. The monotonic increase of α_{mm} with μ is evident but for sufficiently small values of μ, the magnetic copolarizability can be negative. In other words, for a bi-isotropic sphere, the magnetic polarization response can be opposite to the incident magnetic field, although the material permeability is larger than that of the vacuum[7]. Figure 1 shows also that only in the case of nonchiral and reciprocal sphere, α_{mm} is always nonnegative.

To paraphrase once more, a paramagnetic sphere can display diamagnetic behavior if it is chiral and/or nonreciprocal. The effect is more pronounced if the chirality or nonreciprocity is increased. The figure also shows the effect of ϵ on this phenomenon: making the per-

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[6] linear proportionality

[7] This means that the depolarization field — which is opposite to the internal field — has an amplitude larger than the incident field.

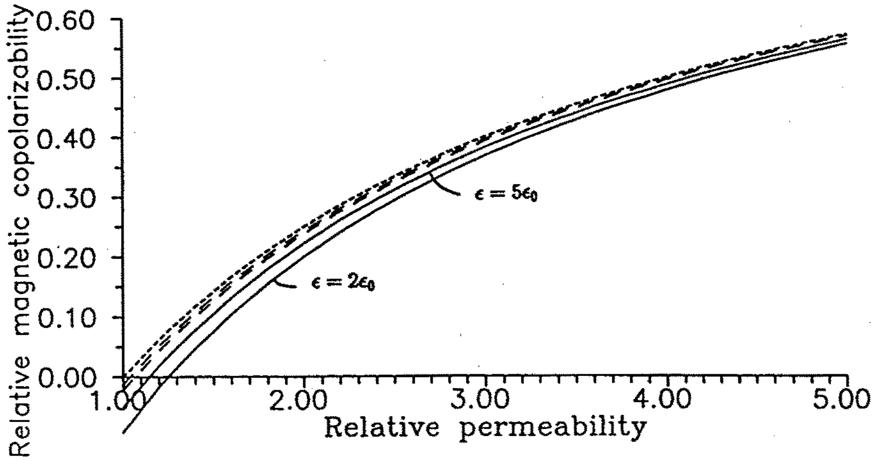

Figure 1. **Normalized magnetic copolarizability of a chiral sphere $\alpha_{mm}/(3\mu_0 V)$ in equation (15) as a function of the relative permeability of the sphere μ/μ_0 for two permittivity values $\epsilon = 2\epsilon_0, \epsilon = 5\epsilon_0$. Solid line: $\kappa = 1$; dashed line: $\kappa = 0.5$; dotted line: $\kappa = 0$. Note the diamagnetic behavior at sufficiently small values of μ.**

mittivity larger increases the effect. In other words, as has been noted elsewhere [23]: even though chiral (or even bi-isotropic) inclusions be nonmagnetic to begin with, the composite medium will have magnetic properties.

Also, due to the duality, or the fact that equations (13) and (16) remain the same if the electric and magnetic quantities are interchanged, a similar phenomenon exists for the electric copolarizability: a "diadielectric" material[8] can be manufactured from "paradielectric" bi-isotropic spheres that have permittivity sufficiently close to that of vacuum, in such a way which corresponds to a state below the zero-crossing point of Fig. 1.

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[8] I call a medium "diadielectric" if its electric susceptibility is negative.

4. Effective Parameters of Bi-isotropic Mixtures

Once the polarizability matrices of bi-isotropic inclusions are known, it is possible to characterize electromagnetically mixtures that are composed of these inclusions embedded in a host medium. The macroscopic parameters depend, in addition to the polarizabilities, also on the fractional volumes of the components making the mixture. Furthermore, in the case of ellipsoidal inclusions, their orientation distribution may have a strong effect on the macroscopic effective parameters of the mixture. As for the position distribution, the situation may be more complicated if the inclusions form clusters or aggregates, leading to complex interaction effects. The following discussion will not touch problems associated with the pair-distribution functions.

The simplest classical mixing rule is Maxwell-Garnett formula [25] which gives the effective permittivity $\epsilon_{\rm eff}$ of a dielectric mixture where spheres of permittivity ϵ occupy a volume fraction f in a host medium of permittivity ϵ_0:

$$\epsilon_{\rm eff} = \epsilon_0 + 3f\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0 - f(\epsilon - \epsilon_0)} \tag{34}$$

or, in the form labeled as Rayleigh formula

$$\frac{\epsilon_{\rm eff} - \epsilon_0}{\epsilon_{\rm eff} + 2\epsilon_0} = f \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \tag{35}$$

These formulas have been generalized to chiral, nonreciprocal, and general bi-isotropic mixtures [14,21,22]. Even bianisotropic mixtures have been treated [24]. Naturally the quasistatic assumption restricts the range of these formulas, too: the scatterer sizes have to be small compared with the wavelength[9]. In this section, a short presentation of bi-isotropic mixing rules is given first, followed by conclusions about the predictions and results of these equations.

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[9] As a matter of fact, it may be not correct to call the inclusions as "scatterers" in the quasistatic regime, where these only have effect through the average polarization they cause with their electric and magnetic dipole moments, higher multipoles being neglected.

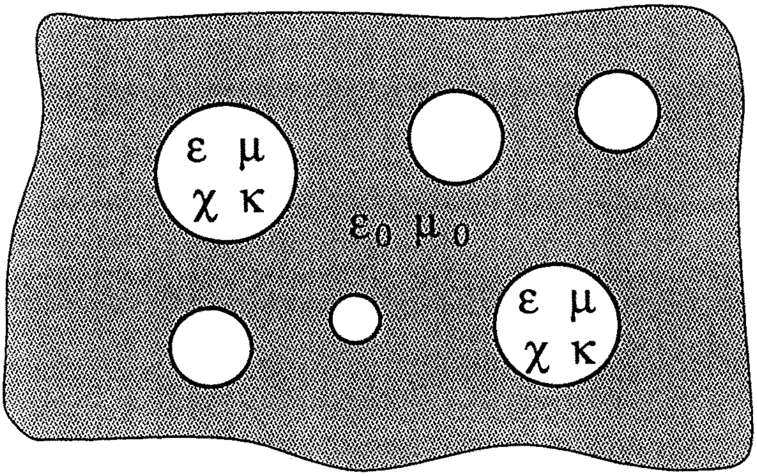

Figure 2. A mixture of bi-isotropic spherical inclusions in isotropic host material.

4.1 Mixtures with Spherical Inclusions

Consider the geometry shown in Fig. 2 where spherical bi-isotropic inclusions occupy a volume fraction f in the mixture. The host medium is isotropic (permittivity is ϵ_0 and permeability μ_0). Let the material parameters of the inclusions be $\epsilon, \mu, \chi, \kappa$ and polarizability components $\alpha_{ee}, \alpha_{em}, \alpha_{me}, \alpha_{mm}$. The effective isotropic parameters of the mixture $\epsilon_{\text{eff}}, \mu_{\text{eff}}, \chi_{\text{eff}}, \kappa_{\text{eff}}$ are defined as the relations between the average flux densities and the incident fields:

$$\begin{pmatrix} < \bar{D} > \\ < \bar{B} > \end{pmatrix} = \begin{pmatrix} \epsilon_{\text{eff}} & (\chi_{\text{eff}} - j\kappa_{\text{eff}})\sqrt{\mu_0\epsilon_0} \\ (\chi_{\text{eff}} + j\kappa_{\text{eff}})\sqrt{\mu_0\epsilon_0} & \mu_{\text{eff}} \end{pmatrix} \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix} \tag{36}$$

The flux densities can be calculated from the electric and magnetic polarizations $< \bar{P}_e >$ and $< \bar{P}_m >$ that are caused collectively by all the dipole moments of the scatterers in the material:

$$\begin{pmatrix} <\bar{D}> \\ <\bar{B}> \end{pmatrix} = \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix} \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix} + \begin{pmatrix} <\bar{P}_e> \\ <\bar{P}_m> \end{pmatrix} \tag{37}$$

The average polarization is the dipole moment density:

$$\begin{pmatrix} <\bar{P}_e> \\ <\bar{P}_m> \end{pmatrix} = n \begin{pmatrix} \bar{p}_e \\ \bar{p}_m \end{pmatrix} = \begin{pmatrix} n\alpha_{ee} & n\alpha_{em} \\ n\alpha_{me} & n\alpha_{mm} \end{pmatrix} \begin{pmatrix} \bar{E}_L \\ \bar{H}_L \end{pmatrix} \tag{38}$$

where n is the number density of the scatterers. Here, the exciting fields $\bar{E}_L$ and $\bar{H}_L$ are not the same as the average fields $\bar{E}, \bar{H}$ but rather the Lorentzian fields [26]. These are larger than the incident fields because they include the contribution from the surrounding polarization. This effect is

$$\begin{pmatrix} \bar{E}_L \\ \bar{H}_L \end{pmatrix} = \begin{pmatrix} \bar{E} \\ \bar{H} \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 1/\epsilon_0 & 0 \\ 0 & 1/\mu_0 \end{pmatrix} \begin{pmatrix} <\bar{P}_e> \\ <\bar{P}_m> \end{pmatrix} \tag{39}$$

From these matrix equations, the average polarizations can be solved, whence, using the polarizability expressions (13)–(16), the final result for bi-isotropic Maxwell-Garnett formulae reads

$$\epsilon_{\text{eff}} = \epsilon_0 + 3f\epsilon_0 \frac{(\epsilon-\epsilon_0)[\mu+2\mu_0-f(\mu-\mu_0)] - (\chi^2+\kappa^2)\mu_0\epsilon_0(1-f)}{[\mu+2\mu_0-f(\mu-\mu_0)][\epsilon+2\epsilon_0-f(\epsilon-\epsilon_0)] - (\chi^2+\kappa^2)\mu_0\epsilon_0(1-f)^2} \tag{40}$$

$$\mu_{\text{eff}} = \mu_0 + 3f\mu_0 \frac{(\mu-\mu_0)[\epsilon+2\epsilon_0-f(\epsilon-\epsilon_0)] - (\chi^2+\kappa^2)\mu_0\epsilon_0(1-f)}{[\mu+2\mu_0-f(\mu-\mu_0)][\epsilon+2\epsilon_0-f(\epsilon-\epsilon_0)] - (\chi^2+\kappa^2)\mu_0\epsilon_0(1-f)^2} \tag{41}$$

$$\chi_{\text{eff}} = \frac{9f\chi\mu_0\epsilon_0}{[\mu+2\mu_0-f(\mu-\mu_0)][\epsilon+2\epsilon_0-f(\epsilon-\epsilon_0)] - (\chi^2+\kappa^2)\mu_0\epsilon_0(1-f)^2} \tag{42}$$

$$\kappa_{\text{eff}} = \frac{9f\kappa\mu_0\epsilon_0}{[\mu+2\mu_0-f(\mu-\mu_0)][\epsilon+2\epsilon_0-f(\epsilon-\epsilon_0)]-(\chi^2+\kappa^2)\mu_0\epsilon_0(1-f)^2} \tag{43}$$

with $f = nV$ being the fractional volume of the bi-isotropic inclusion phase in the mixture.

It requires substantial bookkeeping to prove that these equations are algebrally equal to the following ones:

$$\frac{(\epsilon_{\text{eff}}-\epsilon_0)(\mu_{\text{eff}}+2\mu_0)-(\chi_{\text{eff}}^2+\kappa_{\text{eff}}^2)\mu_0\epsilon_0}{(\mu_{\text{eff}}+2\mu_0)(\epsilon_{\text{eff}}+2\epsilon_0)-(\chi_{\text{eff}}^2+\kappa_{\text{eff}}^2)\mu_0\epsilon_0} = f\frac{(\epsilon-\epsilon_0)(\mu+2\mu_0)-(\chi^2+\kappa^2)\mu_0\epsilon_0}{(\epsilon+2\epsilon_0)(\mu+2\mu_0)-(\chi^2+\kappa^2)\mu_0\epsilon_0} \tag{44}$$

$$\frac{(\mu_{\text{eff}}-\mu_0)(\epsilon_{\text{eff}}+2\epsilon_0)-(\chi_{\text{eff}}^2+\kappa_{\text{eff}}^2)\mu_0\epsilon_0}{(\mu_{\text{eff}}+2\mu_0)(\epsilon_{\text{eff}}+2\epsilon_0)-(\chi_{\text{eff}}^2+\kappa_{\text{eff}}^2)\mu_0\epsilon_0} = f\frac{(\mu-\mu_0)(\epsilon+2\epsilon_0)-(\chi^2+\kappa^2)\mu_0\epsilon_0}{(\mu+2\mu_0)(\epsilon+2\epsilon_0)-(\chi^2+\kappa^2)\mu_0\epsilon_0} \tag{45}$$

$$\frac{\chi_{\text{eff}}}{(\mu_{\text{eff}}+2\mu_0)(\epsilon_{\text{eff}}+2\epsilon_0)-(\chi_{\text{eff}}^2+\kappa_{\text{eff}}^2)\mu_0\epsilon_0} = f\frac{\chi}{(\mu+2\mu_0)(\epsilon+2\epsilon_0)-(\chi^2+\kappa^2)\mu_0\epsilon_0} \tag{46}$$

$$\frac{\kappa_{\text{eff}}}{(\mu_{\text{eff}}+2\mu_0)(\epsilon_{\text{eff}}+2\epsilon_0)-(\chi_{\text{eff}}^2+\kappa_{\text{eff}}^2)\mu_0\epsilon_0} = f\frac{\kappa}{(\mu+2\mu_0)(\epsilon+2\epsilon_0)-(\chi^2+\kappa^2)\mu_0\epsilon_0} \tag{47}$$

These are the generalization of the Rayleigh mixing formula (35). The decoupling into separate electric and magnetic Rayleigh formulae is easily seen for $\kappa \to 0, \chi \to 0$. The appearance of the generalized Maxwell–Garnett and Rayleigh formulas is slightly different if other constitutive relations are followed; for explicit expressions, see [21].

4.2 Mixtures with Ellipsoidal Inclusions

If the bi-isotropic inclusions in the mixture have ellipsoidal form, the polarizability dyadics (24)–(26) have to be used. The orientation distribution of the inclusions affects the macroscopic parameters: if all ellipsoids are randomly oriented, there is no preferred global direction in the macroscopic picture, and the effective parameters are multiples of unit dyadic, equivalent to scalars. The mixture is bi-isotropic in this case.

The other extreme is the case when all ellipsoids have the same alignment of their axes within the mixture. The consequence is that the medium is also anisotropic, or rather, bianisotropic. The average polarization $\bar{P}_e, \bar{P}_m$ can be calculated along the lines of the case of spheres. There is a difference with the Lorentzian fields $\bar{E}_L; \bar{H}_L$ due to the shape effect. The depolarization dyadic (21) needs to be taken into account:

$$\begin{pmatrix}\bar{E}_L\\ \bar{H}_L\end{pmatrix} = \begin{pmatrix}\bar{E}\\ \bar{H}\end{pmatrix} + \bar{\bar{L}}\cdot\begin{pmatrix}<\bar{P}_e>/\epsilon_0\\ <\bar{P}_m>/\mu_0\end{pmatrix} \qquad (48)$$

The effective parameters are now dyadic with the principal coordinate system spanned by the axes $\bar{u}_a, \bar{u}_b, \bar{u}_c$ of the single inclusion ellipsoid:

$$\bar{\bar{\vartheta}}_{\text{eff}} = \sum_{i=a,b,c} \vartheta_{\text{eff},i}\bar{u}_i\bar{u}_i \qquad (49)$$

where ϑ stands for $\epsilon, \mu, \chi, \kappa$. The results for the components are

$$\begin{aligned}\epsilon_{\text{eff,i}} =& \epsilon_0 + \frac{f\epsilon_0}{d_i}\{(\epsilon-\epsilon_0)[\mu_0 + (\mu-\mu_0)N_i(1-f)] \\ & -(\chi^2+\kappa^2)\mu_0\epsilon_0 N_i(1-f)\}\end{aligned} \qquad (50)$$

$$\begin{aligned}\mu_{\text{eff,i}} =& \mu_0 + \frac{f\mu_0}{d_i}\{(\mu-\mu_0)[\epsilon_0 + (\epsilon-\epsilon_0)N_i(1-f)] \\ & -(\chi^2+\kappa^2)\mu_0\epsilon_0 N_i(1-f)\}\end{aligned} \qquad (51)$$

$$\chi_{\text{eff,i}} = \frac{f\chi\mu_0\epsilon_0}{d_i} \tag{52}$$

$$\kappa_{\text{eff,i}} = \frac{f\kappa\mu_0\epsilon_0}{d_i} \tag{53}$$

where

$$\begin{aligned} d_i =& [\mu_0 + (\mu - \mu_0)N_i(1-f)][\epsilon_0 + (\epsilon - \epsilon_0)N_i(1-f)] \\ &- (\chi^2 + \kappa^2)\mu_0\epsilon_0 N_i^2(1-f)^2 \end{aligned} \tag{54}$$

with $f = nV$ being the fractional volume of the bi-isotropic inclusion phase in the mixture.

As the ellipsoids degenerate into spheres ($N_i \rightarrow 1/3$), it can be seen that all three axial components become equal, and the spherical Maxwell-Garnett formulas (40)–(43) are recovered.

For ellipsoids with an orientation distribution within the mixture, the dipole moments have to be averaged with this distribution function as the polarization is integrated. In this case the final formulas cannot be represented in such compact form as the spheres or aligned ellipsoids. However, a closed-form solution still exists [21].

The treatments and results above only allowed one type of inclusions, such that the spheres (or ellipsoids) had to have the same bi-isotropic material parameters. This restriction can be relaxed. If there are different guest phases in the host medium, their average polarization contributions can be added together, the other parts of the analysis remaining as before.

4.3 Discussion on the Effective Material Parameters

It may be difficult to see the effect of a given mixture parameter in the mixture rules (40)–(43), or (50)–(53). This is because of the coupled nature of the macroscopic equations for the effective parameters: for example, the effective permittivity depends not only on the permittivities of the constituent phases but also on their permeabilities, and the guest chirality and nonreciprocity.

4.3.1 Duality

A clear fact is, however, that there exist wonderful dualities in the macroscopic material formula expressions. For example, the way the *permeability* of the inclusion μ affects the effective permeability μ_{eff}, chirality κ_{eff}, nonreciprocity χ_{eff}, and permittivity ϵ_{eff}, is the same as the way the *permittivity* of the inclusion ϵ affects the effective permittivity, chirality, nonreciprocity, and permeability. Also, the functional dependence of the macroscopic permittivity[10] on the inclusion chirality is exactly the same as on the inclusion nonreciprocity. Put into a more nonredundant form: $\epsilon_{\text{eff}}(\epsilon, \mu, \kappa, \chi; \epsilon_0, \mu_0) = \epsilon_{\text{eff}}(\epsilon, \mu, \chi, \kappa; \epsilon_0, \mu_0)$. Further; the macroscopic chirality depends on inclusion nonreciprocity identically with macroscopic nonreciprocity dependence on inclusion chirality.

An essential observation is the fact that the effective (=macroscopic) permittivity and permeability of a mixture are even functions of both the chirality and nonreciprocity of the component material. Hence, firstly, the sign of handedness, *i.e.* whether left or right handed, should not have effect on these parameters, and they are true scalars, invariant of spatial inversion. This is obvious: samples of media that are mirror images of one another should have the same permittivity and the same permeability; Nature should not prefer left to right. Although in the atomic level, in the weak interaction process, this asymmetry has been predicted by Lee and Yang in 1956 [27], and experimentally observed in 1957 [28–30], one would expect this not to happen at the macroscopic level where different racemization processes produce right and left hands in equal proportions.

But similar, and even identical, is also the dependence of ϵ_{eff} and μ_{eff} on the inclusion nonreciprocity parameter: changing the sign of χ does not affect the macroscopic copolarizability measures ϵ_{eff}, μ_{eff}. Intuitively, this phenomenon is not as clear as in the case of handedness and chirality. One way to gain a physical picture of naturally nonreciprocal materials is to model the microstructure by rigid units that consist of bound electric and magnetic dipoles according to Figure 3. The polar mechanism is as follows: As the electric field exerts a torque on the electric dipole, it also produces cophasal magnetic polarization. If all these hybrid units have the same "state", *i.e.* in all elements, the permanent electric and magnetic dipole moment vectors

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[10] The same applies for the macroscopic permeability.

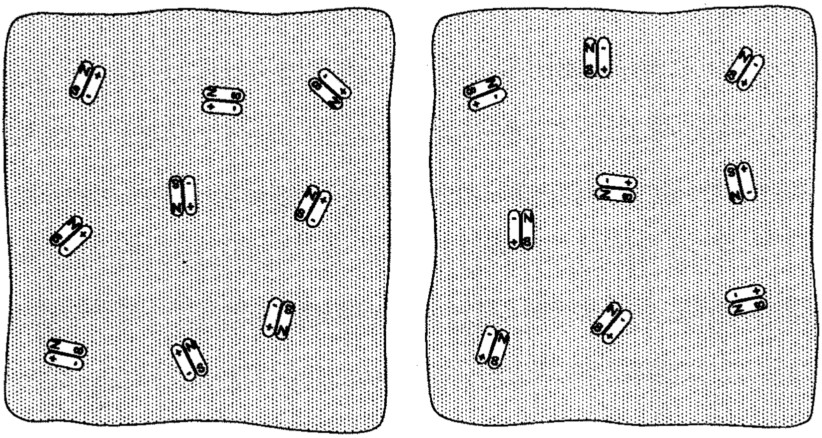

Figure 3. A phenomenological model for isotropic nonreciprocal mixture. The elements consist of an electric and a magnetic dipole that are coupled together by a nonelectromagnetic interaction. The relative direction of these dipole moment vectors within an element (the "state" of the element) is important. Note that in order to have a nonreciprocal mixture, there has to be nonsymmetry in the concentration of the element states. The two samples shown here represent media with the same magnitude of the nonreciprocity parameter χ, but of opposite sign.

are of the parallel direction (or antiparallel), but not mixed, the nonreciprocity effect manifests itself. Changing the sign of χ corresponds to reversing the electric (or magnetic) dipole moment vector direction in all units[11]. In this view it is a vaguely similar operation as taking the mirror image of a material.

From Figure 3, the evenness of ϵ_{eff} and μ_{eff} on χ can be understood. The change in the sign of χ does not change the overall copolarizability characteristics of the medium, and hence also the macroscopic permittivity and permeability should remain intact.

It is easy to accept the conclusion that the effective chirality κ_{eff} is an odd function of the chirality of the inclusion material (and a pseudoscalar): changing the handedness of the component changes the handedness of the mixture. κ_{eff} is even function of the inclusion nonre-

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[11] This means taking the other sample of Figure 3.

ciprocity, which consolidates the idea of handedness and nonreciprocity being separate physico–geometrical properties of the material structure. And again, similar conclusion holds for the effective nonreciprocity: χ_{eff} is odd function of χ and even of κ.

4.3.2 Perturbation expansions

Perturbation expansions of the mixing formulas give information about the first-order effects of magnetoelectric parameters. It is easy to see that the full parameter expressions converge uniformly to the classical dielectric (and magnetic) mixing relations as the chirality and nonreciprocity vanish. Hence, for values of χ and κ much less than 1, the dominant effect is the dielectric polarization in ϵ_{eff} and magnetic polarization in μ_{eff}. This fact, and also the decoupling of electric and magnetic polarizations, is evident from the perturbation expansions of (40)–(43):

$$\begin{aligned}\epsilon_{\text{eff}} \simeq & \,\epsilon_0 + 3f\epsilon_0 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0 - f(\epsilon - \epsilon_0)} \\ & - (\chi^2 + \kappa^2)\mu_0\epsilon_0 \frac{9f(1-f)\epsilon_0^2}{[\epsilon + 2\epsilon_0 - f(\epsilon - \epsilon_0)]^2[\mu + 2\mu_0 - f(\mu - \mu_0)]}\end{aligned} \tag{55}$$

$$\begin{aligned}\mu_{\text{eff}} \simeq & \,\mu_0 + 3f\mu_0 \frac{\mu - \mu_0}{\mu + 2\mu_0 - f(\mu - \mu_0)} \\ & - (\chi^2 + \kappa^2)\mu_0\epsilon_0 \frac{9f(1-f)\mu_0^2}{[\mu + 2\mu_0 - f(\mu - \mu_0)]^2[\epsilon + 2\epsilon_0 - f(\epsilon - \epsilon_0)]}\end{aligned} \tag{56}$$

$$\kappa_{\text{eff}} \simeq \kappa \frac{9f\mu_0\epsilon_0}{[\mu + 2\mu_0 - f(\mu - \mu_0)][\epsilon + 2\epsilon_0 - f(\epsilon - \epsilon_0)]} \tag{57}$$

$$\chi_{\text{eff}} \simeq \chi \frac{9f\mu_0\epsilon_0}{[\mu + 2\mu_0 - f(\mu - \mu_0)][\epsilon + 2\epsilon_0 - f(\epsilon - \epsilon_0)]} \tag{58}$$

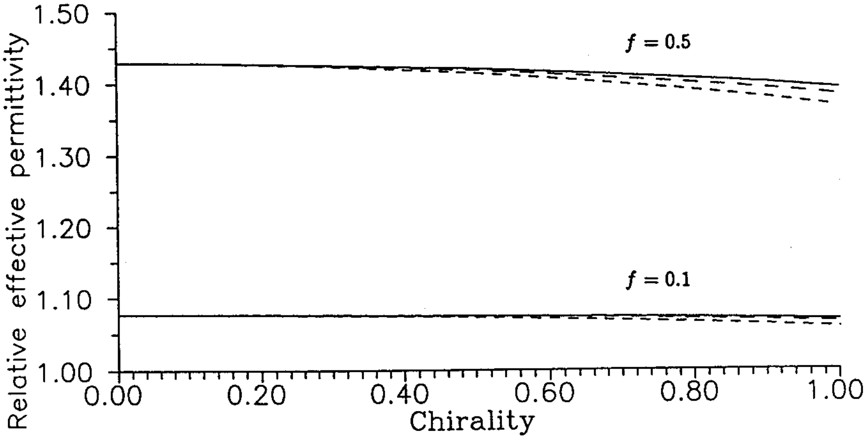

Figure 4. Relative effective permittivity $\epsilon_{\rm eff}/\epsilon_0$ of a mixture with relative inclusion permittivity of $\epsilon/\epsilon_0 = 2$ as a function of the chirality parameter of the inclusions for two volume fraction values; dilute mixture: inclusion volume fraction $f = 0.1$; dense mixture: $f = 0.5$. Solid line: permeability of the inclusion is $\mu = 5\mu_0$; dashed line: $\mu = 3\mu_0$; dotted line: $\mu = \mu_0$. Equation (40).

In other words, the chirality of the inclusion phase has little effect on the macroscopic permittivity and permeability, at least for high permittivity and permeability contrasts between the inclusion and background phases. On the other hand, naturally the chirality of the inclusion is the dominant parameter defining the effective chirality of the mixture. Finally, the effective chirality of a mixture is decreased by high permittivity and/or permeability of the inclusion phase.

Figure 4 illustrates the permittivity behavior. The effective permittivity of a chiral mixture ($\chi = 0$)[12] is shown as a function of the inclusion chirality for two volume fractions: $f = 0.1$ (dilute mixture), and $f = 0.5$ (dense mixture). The conclusions made above about small-chirality effects are supported by the figure: for $\kappa \ll 1$, there is little effect of κ on $\epsilon_{\rm eff}$, and also the inclusion permeability has

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[12] In the effective permittivity behavior, χ and κ have only collective, joint effect. Therefore this reciprocity assumption means no restriction on generality.

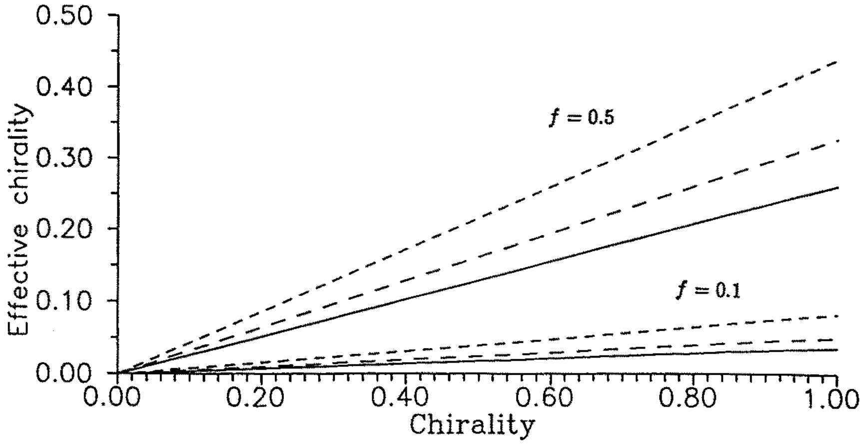

Figure 5. **Effective chirality κ_{eff} of a mixture with relative inclusion permittivity of $\epsilon/\epsilon_0 = 2$ as a function of the chirality parameter of the inclusions for two volume fraction values; dilute mixture: inclusion volume fraction $f = 0.1$; dense mixture: $f = 0.5$. Solid line: permeability of the inclusion is $\mu = 5\mu_0$; dashed line: $\mu = 3\mu_0$; dotted line: $\mu = \mu_0$. Equation (43).**

extremely small effect on ϵ_{eff} in this regime[13]. For larger values of κ, the inclusion chirality decreases ϵ_{eff}, but the inclusion permeability increases ϵ_{eff}.

Figure 5 focuses on the same chiral mixture as Figure 4 but shows the effective chirality behavior $\kappa_{\text{eff}}(\kappa)$. The mostly linear behavior of κ_{eff} on κ is clear. Note also that increasing the inclusion permeability (or permittivity, for that matter) decreases the macroscopic chirality.

4.3.3 Ellipsoidal mixtures

How, then, about inclusions of other shapes? The only other forms of inclusions whose polarizabilities can be solved in closed form are ellipsoids. By using chiral inclusions of ellipsoidal shapes, a further range of mixture parameters can be tailored. The effects vary: by us-

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[13] This is a consequence of the decoupling of electric and magnetic polarization phenomena.

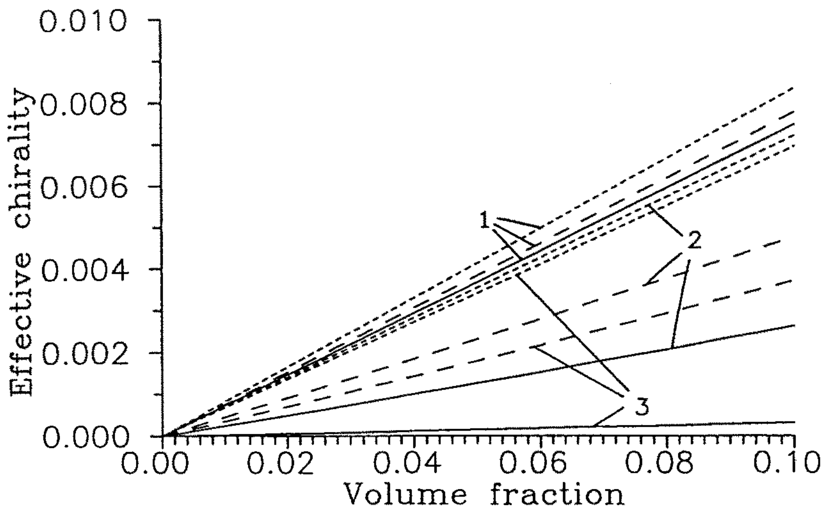

Figure 6. A mixture with isotropic background medium and reciprocal chiral bi-isotropic ($\chi = 0$) inclusion phase whose volume fraction is given as the abscissa. Shown is the effective chirality of the mixture relative to background permittivity. Line type denotes inclusion shape: solid – spheres; dashed – needles; dotted – discs. Inclusion permeability relative to background is 1.1 and inclusion chirality is $\kappa = 0.1$. The numbers refer to inclusion permittivity; $\epsilon = 2\epsilon_0$ (1); $\epsilon = 10\epsilon_0$ (2); $\epsilon = 100\epsilon_0$ (3).

ing needle-shaped or disk-shaped inclusions, larger effective parameters than with spheres can be achieved, although the shape effect depends on the dielectric and magnetic contrast between the inclusion and host phases. One observation is, however, always valid: spherical inclusions produce minimum effects in the macroscopic properties, and each deviation from this extremum shape increases the value achieved by the spherical geometry.

As an example how the shape effect can dominate the mixture parameters, Figure 6 shows the effective chirality of a mixture as a function of the volume fraction of the chiral inclusions. The dependence on κ_{eff} on κ is clear, again similar as in Figure 5, but the large shape effect (spheres, needles, disks) of the inclusions can only be compensated by extremely large variation in the inclusion permittivity ϵ.

5. Temporal Dispersion Effects in Bi-isotropic Media

All the previous analysis was done keeping the material parameters constant with respect to the frequency. The constitutive relations (5)–(6) themselves underlined the fixed-frequency nature of the approach: the time-harmonic dependence was assumed. However, the materials in nature are unavoidably dispersive. Their polarization responses depend on the frequency, i.e. on the rates of temporal change of the electric and magnetic fields that act on the medium.

In the frequency domain, this shows up in the frequency dependence of the material parameters $\epsilon, \mu, \chi, \kappa$. In the time-domain representation, the bi-isotropic constitutive relations will show that the electric polarization response is proportional to the time derivative of the magnetic field strength, and vice versa.

It is the advantage of time-domain representation, that the basic physical limitations and requirements — causality, independence of time shifts, etc. — of the material coefficients take can be formulated naturally. In frequency domain, these limitations take involved forms: for example causality confines the frequency dependencies of the real and imaginary parts of the material parameter functions to follow complicated-looking Kramers–Kronig integral relations [31]. Therefore, in the frequency-domain modeling of bi-isotropic media, we are not totally free to choose the frequency dependencies of the functions. The dispersion and explicit time-dependent formulations of chiral media have attracted recently the interest of several investigators [32–35].

In this section those phenomena are examined that emerge if we accept both the dispersion of bi-isotropic media and the heterogeneity of materials. In other words, the focus is on the question: How does the mixing process affect the magnetoelectric dispersion in composites? Special treatment in the following is given for chiral properties.

5.1 Dispersion Model for the Chirality

To analyze the dispersion of the chirality of the composite, we need to know the frequency behavior of the homogeneous chiral bulk material.

The earliest molecular theories that tried to explain the optical properties of materials have reached already an age of a century. The resonance models by Lorentz, Drude, and others have been quite successful in explaining and predicting several optical and electromagnetic phenomena. The models concentrated on permittivity.

Basing on these types of theories for the electric polarizability of molecules, Condon [36] proposed a model for the frequency dependence of the optical activity — chirality, in today's microwave parlance — in materials. In the following analysis this Condon's model is applied for mixtures[14].

If there is one dominant resonance that lies far away from other molecular transitions, the Condon model gives the following frequency behavior for the chirality parameter $\kappa(\omega)$

$$\kappa(\omega) \propto \frac{\omega R}{\omega_0^2 - \omega^2 + j\omega\Gamma} \tag{59}$$

where R is the rotational strength of the molecular transition, ω_0 is the resonant frequency, and Γ measures the damping associated with the transition. Nondimensionalizing the expression, we get

$$\kappa(\omega) = \frac{\tau\omega_0 x}{1 - x^2 + jdx} \tag{60}$$

where $x = \omega/\omega_0$ is the relative frequency and $d = \Gamma/\omega_0$ is again a measure for the damping. τ is a characteristic time constant describing the magnitude of the chirality. This model clearly shows the absence of handed effects in statics: κ disappears for DC $(x = 0)$.

Because of the absorption term, the chirality is now (in the frequency domain) a complex number, indicating real and imaginary parts. Using terms of classical optics, κ' is responsible for optical rotatory power and κ'' produces circular dichroism. The real and imaginary parts can be written explicitly:

$$\kappa(\omega) = \kappa'(\omega) - j\kappa''(\omega) \tag{61}$$

with

$$\frac{\kappa'(\omega)}{\tau\omega_0} = \frac{x(1 - x^2)}{1 - (2 - d^2)x^2 + x^4} \tag{62}$$

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[14] Condon used different constitutive relations for optically active media in his publication from the 1930's, but it can be translated to the notation followed here. For a closer look of the interdependencies, see [35].

$$\frac{\kappa''(\omega)}{\tau\omega_0} = \frac{dx^2}{1-(2-d^2)x^2+x^4} \tag{63}$$

It is clear that the “resonant frequency” ω_0 is the center for dispersion. At this frequency the imaginary part attains its maximum value (κ'' always is positive). At the same frequency, the real part vanishes, and it behaves symmetrically above and below this frequency if it is drawn with the logarithmic scale. The effect of the increasing damping term d is to broaden the resonance peak and to make it lower at the same time. The imaginary part of the chirality parameter in this Condon model (60) has the maximum value

$$\kappa''(\omega_0) = \frac{\tau\omega_0}{d} \tag{64}$$

at the resonant frequency ω_0.

5.2 Dispersion Effects in the Composite Chirality

How does the mixing process affect the frequency behavior of chirality which in the bulk medium is contained in equations (62)–(63)? To appreciate the most important effect in this dispersion transformation, let us keep all other variables constant in the mixing process. Therefore, consider a mixture with a small amount of chiral guest material in a nonchiral background medium, and let all the frequency behavior be contained in the chirality of the inclusion phase. Therefore the other material parameters will be kept nondispersive in the following analysis, i.e. the dielectric and magnetic dispersions are assumed to take place at other frequency bands; also, for simplicity, ϵ and μ are supposed to be real, since the imaginary parts of these quantities are connected to the dispersion of the real part through the Kramers–Kronig relations[15].

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[15] This approach suffers from the following weakness: The wave numbers of the two eigenwaves in a bulk chiral medium, $k_\pm = k_0(n \pm \kappa)$ with $k_0 = \omega\sqrt{\mu_0\epsilon_0}$ and $n = \sqrt{\mu\epsilon/\mu_0\epsilon_0}$ will become imaginary for dispersive media due to the imaginary part of the chirality κ. If the dispersion in permittivity and permeability are neglected, ϵ and μ are real constant quantities. Consequently, n will be real, and hence one of the eigenwaves will have positive, and the other negative imaginary part of the wave number. This means gain in one of the propagating

Due to the dual character of the chirality and nonreciprocity in material effects, it is sufficient to study the dispersion in chirality. From this, one is able to glean information about the effect of nonreciprocity.

To proceed, let us consider a two-phase mixture where the background is nonchiral of permittivity ϵ_0 and permeability μ_0, and spherical chiral ($\chi = 0$) inclusion particles of material parameters ϵ, μ, κ occupy a fraction f of the total volume. The assumption is $f \ll 1$. From (43), the macroscopic chirality of the mixture follows the formula (where the small volume fraction of the inclusion phase has been exploited) [16]

$$\kappa_{\text{eff}} = \frac{9f\kappa}{(\epsilon_r + 2)(\mu_r + 2) - \kappa^2} \tag{65}$$

where $\epsilon_r = \epsilon/\epsilon_0, \mu_r = \mu/\mu_0$ are the permittivity and permeability of the inclusion relative to the background. Assumed the model (60) for the chirality dispersion, the macroscopic chirality can be calculated. The following frequency dependence is the result:

$$\kappa_{\text{eff}} = \kappa'_{\text{eff}} - j\kappa''_{\text{eff}} \tag{66}$$

with

$$\kappa'_{\text{eff}} = B\frac{x(1-x^2)[1-(2-d^2+A)x^2+x^4]}{[1-(2+d^2+A)x^2+x^4]^2+4d^2x^2(1-x^2)^2} \tag{67}$$

$$\kappa''_{\text{eff}} = B\frac{dx^2[1-(2-d^2-A)x^2+x^4]}{[1-(2+d^2+A)x^2+x^4]^2+4d^2x^2(1-x^2)^2} \tag{68}$$

Here, the mixture parameters that have effect on the frequency behavior, are

$$d = \frac{\Gamma_{ba}}{\omega_0}, \quad A = \frac{(\tau\omega_0)^2}{(\mu_r + 2)(\epsilon_r + 2)}, \quad B = \frac{9f\tau\omega_0}{(\mu_r + 2)(\epsilon_r + 2)} \tag{69}$$

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modes, which contradicts with an assumption of a non-active material. Therefore, in a real physical situation where optical rotatory dispersion and circular dichroism are present, dispersion has to appear also in ϵ and μ, not only in κ.

[16] Note that (57) is valid for small chirality, $\kappa \ll 1$, and general volume fraction, therefore being different than what is needed here.

5.3 Conclusions about the Chiral Dispersion

Why would one expect interesting effects on the dispersion caused by the mixing process? A positive answer may not be evident. However, in the dielectric mixing studies, it is well known that the relaxation-type phenomena occur at drastically different frequency ranges for bulk and composite media. Relaxation is a frequency-dependent polarization phenomenon that occurs in liquids that consist of polar molecules. Water is one example, and the model used for the permittivity is called Debye model [37,38]. Mixtures with spherical inclusions change their relaxation behavior most strongly. It is worth noting that the shape of the inclusion particles is extremely crucial in the magnitude of the shift in relaxation frequency: in [39] the effect of the nonsphericity has been studied, and there the conclusion was that for ellipsoids deviating from the spherical shape, the relaxation frequency always was lower than for sphere-mixtures. For the case of needle- and disc-shaped inclusions, the relaxation frequency would come very close to the relaxation frequency of bulk water.

But chiral dispersion is different from the dispersion in permittivity and permeability. This is because chirality is embedded in nonchiral background, and if the guest vanishes, there is no handedness anymore. On the other hand, permittivity is floating in the background medium, which possesses certain host permittivity, at least that of vacuum. Therefore the dispersion in permittivity is modified in interaction with the host permittivity which does not vanish in any case. The chiral dispersion transformation is hence radically different from permittivity dispersion.

In the dispersion of the real part of composite chirality (Equation (67)), which is responsible for the optical (or electromagnetic) activity, the main effect in the composite dispersion is that the frequency range is broader than the bulk chirality dispersion. The dispersion surrounds the center frequency ω_0. Also it has more structure in its shape. For low damping factors $d = 0.01 \cdots 0.1$, four extrema in the $\kappa_{\text{eff}}(\omega)$ curve appear (instead of only two in the case of bulk chiral material): there appears one maximum and one minimum below ω_0, and also one maximum and one minimum above. The value for κ'_{eff} at the resonant frequency ω_0, however, always remains at 0, and this point in the curves seems like a pivot around which the curves twist as the parameters change.

How is the mixing process manifest in the dispersion of the imagi-

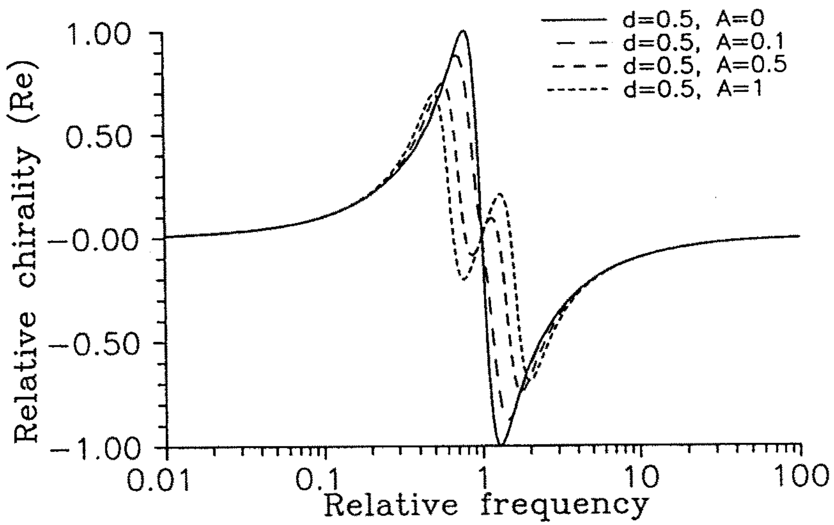

Figure 7. The frequency dependence of the real part of the chirality parameter of a mixture where chiral inclusions obeying the Condon model (62), (63) are embedded in a nonchiral background medium. The mixture is dilute, i.e. the volume fraction of the chiral phase is small. The two parameters affecting the dispersion are the damping factor d (being 0.5 in this figure) and the normalized chirality amplitude A. Effective chirality is shown relative to the value B defined in Equation (69).

nary part of the composite chirality κ''_{eff}? First of all, the imaginary part does not change sign: it is negative all the way throughout the frequency range (the positive values in the figures are due to the convention in Equation (66))[17]. Also, like in the case for the real part of κ_{eff}, the structure of the composite chirality frequency function is more detailed: there appear new maxima and minima.

Figures 7 and 8 show an example of the composite chirality behavior as functions of frequency. Although all possible quantities have been normalized, there remain two parameters (cf. definition (69)) affecting

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[17] This is similar convention in the discussion of dielectric properties of materials in the frequency domain, as the time-harmonic notation $\exp(j\omega t)$ is adopted: the complex permittivity is often written as $\epsilon = \epsilon' - j\epsilon''$ because ϵ'' stays positive for passive (dissipative) materials.

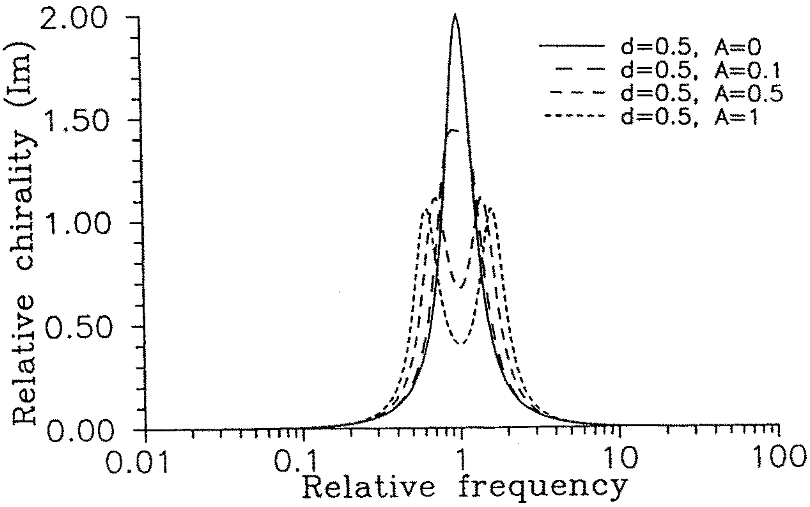

Figure 8. The same as Figure 6, for the imaginary part of chirality parameter.

the behavior of the curves: d is the relative damping of the resonance transition, and A is the measure of the chirality in the inclusion phase. The illustrated effective chirality is normalized with the value of B in equation (69).

Also more complicated mixture geometries can be analyzed with the Condon model. Using the mixing formula (53), the Condon model for chiral dispersion can be used to see the effect of ellipsoidal mixing on the chirality dispersion, as was done above for spherical mixing. The qualitative changes in the dispersive behavior become more detailed, because now there is increased structure in the mixture: the shape of the inclusions have lost part of their symmetry. The curve features become functions of the shape parameters, of which there are now two more than in the spherical case.

6. Analogies of Bi-isotropic Media with Anisotropic Dielectrics and Ferrites

There exist two main avenues in moving towards generalization from the domain of isotropic media. Of these two nonisotropic ideas, the first one is the class of bi-isotropic media; in accord with the topic of this article. This retains isotropy but allows magnetoelectric interactions. The second direction to more general media takes the opposite way: it does not recognize natural chirality or isotropic nonreciprocity but it admits that the polarization may depend on the direction of the electric or magnetic field. These materials are anisotropic.

Dielectric anisotropy has the effect that the polarization caused by electric field is generally not in the same direction as the field itself. Correspondingly, in anisotropic magnetic materials, the average magnetic dipole moment density is only in principal axes directions parallel to the magnetic field. In bi-isotropic media, on the other hand, there are no special axes. Therefore, it may seem strange that so different polarization mechanisms as in these two different classes of materials, there exist similar laws in the polarizability descriptions. This section will pinpoint these resemblancies.

6.1 Polarizability of Anisotropic Sphere

The constitutive relations of anisotropic media are formally simpler than bi-isotropic ones. For dielectrically anisotropic media, the permittivity is dyadic:

$$\bar{D} = \bar{\bar{\epsilon}} \cdot \bar{E} \tag{70}$$

and for magnetically anisotropic media, permeability is dyadic:

$$\bar{B} = \bar{\bar{\mu}} \cdot \bar{H} \tag{71}$$

Due to the noncoupling of electric and magnetic quantities, we may neglect one of these and concentrate on the other, say electrical field and polarization interaction. Using duality transformation, the solution for the magnetically anisotropic problem can be written directly.

There are different types of anisotropic media: uniaxial, biaxial, gyrotropic, etc. An important step in the distinction of anisotropic media comes through splitting the permittivity dyadic $\bar{\bar{\epsilon}}$ into two parts, symmetric and antisymmetric:

$$\overline{\overline{\epsilon}} = \overline{\overline{\epsilon}}_S + \overline{\overline{\epsilon}}_A \tag{72}$$

The antisymmetric part $\overline{\overline{\epsilon}}_A$ is nonzero for gyrotopic media, and it is this very antisymmetry that leads to analogies with bi-isotropic media. An example of dielectrically gyrotropic medium is magnetoplasma, e.g. in the ionosphere. Ferrite, on the other hand, has a gyrotropic permeability tensor, or dyadic. Both these media are nonreciprocal, but unlike bi-isotropic media, the origin for the nonreciprocity is not inside the material itself but it is the external (static) magnetic field. Therefore it is also intuitively clear that these media are anisotropic, because the external field breaks the spherical symmetry.

If the direction of the external magnetic field is denoted by the "gyration" unit vector $\bar{u}_g$, the antisymmetric part of the permittivity dyadic is $\epsilon_g \bar{u}_g \times \overline{\overline{I}}$ with $\overline{\overline{I}}$ as the unit dyadic. ϵ_g measures the amplitude of the gyrotropic coupling. Let the symmetric part of the permittivity dyadic have principal axis along the x, y, z directions[18] Therefore $\overline{\overline{\epsilon}}_S = \epsilon_x \bar{u}_x \bar{u}_x + \epsilon_y \bar{u}_y \bar{u}_y + \epsilon_z \bar{u}_z \bar{u}_z$.

Consider now the special case that the external field is along one of the principal axis: $\bar{u}_g = \bar{u}_z$. We then have the permittivity dyadic of the material

$$\overline{\overline{\epsilon}} = \epsilon_x \bar{u}_x \bar{u}_x + \epsilon_y \bar{u}_y \bar{u}_y + \epsilon_z \bar{u}_z \bar{u}_z + \epsilon_g \bar{u}_z \times \overline{\overline{I}} \tag{73}$$

What is the polarizability of a sphere made out of material that obeys the permittivity dyadic (73)?

The isotropic sphere of volume V and permittivity ϵ possesses polarizability

$$\alpha = 3\epsilon_0 V \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \tag{74}$$

It is known [40] that the polarizability of an anisotropic sphere is a (formally) straightforward generalization of the scalar case (74). The polarizability is naturally dyadic and obeys the expression

$$\overline{\overline{\alpha}} = 3\epsilon_0 V (\overline{\overline{\epsilon}} - \epsilon_0 \overline{\overline{I}}) \cdot (\overline{\overline{\epsilon}} + 2\epsilon_0 \overline{\overline{I}})^{-1} \tag{75}$$

To calculate the terms needed in (75), the dyadic inverse has to be enumerated. The inverse of a complete dyadic is [41]

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[18] This can be assumed without loss of generality for a real symmetric matrix.

$$\overline{\overline{A}}^{-1} = \frac{3(\overline{\overline{A}}\overset{\times}{\times}\overline{\overline{A}})^T}{\overline{\overline{A}}\overset{\times}{\times}\overline{\overline{A}} : \overline{\overline{A}}} \tag{76}$$

where the superscript T denotes transpose of a dyadic. Using rules of dyadic algebra [41] $(\overline{\overline{I}}\overset{\times}{\times}\overline{\overline{I}} = 2\overline{\overline{I}},\ \overline{\overline{I}} : \overline{\overline{I}} = 3,\ \bar{u}_g \times \overline{\overline{I}} \cdot \bar{u}_g \times \overline{\overline{I}} = \bar{u}_g\bar{u}_g - \overline{\overline{I}}, \cdots)$, the polarizability of the gyrotropic sphere can be calculated

$$\bar{\bar{\alpha}} = \sum_{i,j=x,y,z} \alpha_{ij}\bar{u}_i\bar{u}_j \tag{77}$$

with components

$$\begin{aligned}
\alpha_{xx} &= 3\epsilon_0 V \frac{(\epsilon_x - \epsilon_0)(\epsilon_y + 2\epsilon_0) + \epsilon_g^2}{(\epsilon_x + 2\epsilon_0)(\epsilon_y + 2\epsilon_0) + \epsilon_g^2} \\
\alpha_{yy} &= 3\epsilon_0 V \frac{(\epsilon_y - \epsilon_0)(\epsilon_x + 2\epsilon_0) + \epsilon_g^2}{(\epsilon_x + 2\epsilon_0)(\epsilon_y + 2\epsilon_0) + \epsilon_g^2} \\
\alpha_{zz} &= 3\epsilon_0 V \frac{\epsilon_z - \epsilon_0}{\epsilon_z + 2\epsilon_0} \\
\alpha_{xy} &= -\alpha_{yx} = 3\epsilon_0 V \frac{-3\epsilon_0\epsilon_g}{(\epsilon_x + 2\epsilon_0)(\epsilon_y + 2\epsilon_0) + \epsilon_g^2} \\
\alpha_{xz} &= \alpha_{zx} = \alpha_{yz} = \alpha_{zy} = 0
\end{aligned} \tag{78}$$

6.2 Comparison of Polarizabilities

There are striking similarities as one compares the gyrotropic polarizability components of (78) to the polarizability matrix of a bi-isotropic sphere (13)–(16).

In (78) the gyrotropy parameter ϵ_g affects the polarizability components. If it vanishes, the matrix becomes diagonal and the components become simple functions of the permittivities like in the perfect isotropic case. However, in the gyrotopic case, there is one component that is not affected by ϵ_g . This is the z-directed copolarizability α_{zz} which is the same as isotropic. It means that, for example in the case of a ferrite sphere, the gyrotropy has no effect on the copolarizability in the external magnetic field direction.

On the other hand, gyrotropy affects the transversal components α_{xx} and α_{yy} as also the off-diagonal components α_{xy} and α_{yx}. Here ϵ_g plays a similar role as the chirality parameter κ or nonreciprocity parameter χ in the case of bi-isotropic sphere. However, there is a change of sign: the denominator of the gyrotropic case is

$$(\epsilon_x + 2\epsilon_0)(\epsilon_y + 2\epsilon_0) + \epsilon_g^2$$

whereas the corresponding expression for the chiral (Pasteur) case is

$$(\epsilon + 2\epsilon_0)(\mu + 2\mu_0) - \kappa^2 \mu_0 \epsilon_0$$

and in the nonreciprocal (Tellegen) case

$$(\epsilon + 2\epsilon_0)(\mu + 2\mu_0) - \chi^2 \mu_0 \epsilon_0$$

and in the general bi-isotropic case

$$(\epsilon + 2\epsilon_0)(\mu + 2\mu_0) - (\chi^2 + \kappa^2) \mu_0 \epsilon_0$$

These quantitites, and also the polarizability expressions, agree more if the gyrotropy is imaginary: $\epsilon_g = jg$ where g is real. This is in fact the case in magnetoplasma [16,42] or in the case of the permeability of ferrites [43]:

$$\bar{\bar{\mu}} = \mu_a \bar{u}_z \bar{u}_z + \mu_t(\bar{\bar{I}} - \bar{u}_z \bar{u}_z) + jg\bar{u}_z \times \bar{\bar{I}} \tag{79}$$

The imaginary nature of the gyrotropy in the permittivity/permeability expression completes the analogy with respect to bi-isotropic media. We can write the following correspondence table:

Pasteur medium	Tellegen medium	Dielectrically gyrotropic	Magnetically gyrotropic
κ	χ	$j\epsilon_g$	$j\mu_g$
ϵ	ϵ	ϵ_x	μ_x
μ	μ	ϵ_y	μ_y
α_{ee}	α_{ee}	α_{xx}	α_{xx}
α_{mm}	α_{mm}	α_{yy}	α_{yy}
α_{em}	α_{em}	α_{xy}	α_{xy}
α_{me}	$-\alpha_{me}$	α_{yx}	α_{yx}

This correspondence table can serve as a tool to transfer conclusions from bi-isotropic media to anisotropic media. The present paper has brought forth several effects of chirality and nonreciprocity on material behavior, which resemble those found in anisotropic media. To study the effects in gyrotropic media, one needs to change the magnetoelectric quantities (κ, χ) to the gyrotropic ones according to the table above. Then one is left with quantities of anisotropic media that obey certain laws and functional dependencies, like, for example, the even and odd characteristics of material properties, along the lines discussed earlier in this paper.

7. Conclusion

It seems that the modeling theories of heterogeneous bi-isotropic materials have advanced considerably and have reached in recent years a level which can be used in the design of novel microwave materials, like chiral composites, for example. The classical approaches and strategies in the mixture theories maintain their applicability although more complicated inclusions have to be dealt with.

Mixture modeling starts with the problem of looking at one single inclusion exposed to the electromagnetic field. The question is to find out its response, *i.e.* the manner how the inclusion perturbates the field. This change in the field can be interpreted as scattering from a dipole. Once the dipole moment of this dipole is known, also the polarizability of the inclusion has been solved.

In classical mixing theories, it was sufficient to treat only the electric problem (in dielectric mixtures) or only the magnetic problem (in mixtures consisting of components with non-vacuous permeability). The extra complication in novel, complex media modeling is that the electric and magnetic problems are not anymore decoupled but they both need attention at the same time. The consequence is that there are crosspolarizability components in the polarizability matrix. Therefore along the modeling effort, inverses of matrices have to be accepted if one wants to keep the formal simplicity of the equations that describe the effective material parameters.

Anyway, the present article has hopefully proved that this approach is fertile. Polarizability modeling can be performed with conceptual simplicity and it can also be extended to ellipsoidally shaped inclusions. Having the expressions for the polarizability matrices, the effective bi-isotropic (even bianisotropic) parameters can be calculated for mixtures that contain inclusions of this type.

The effective parameters for heterogeneous bi-isotropic mixtures are not plain recipes for high-tech engineers wishing to design advanced composites but these formulas also display profound physics. The expressions have been interpreted in this article in many respects. The electromagnetic properties of electric and magnetic susceptibilities, chirality, and nonreciprocity, have acquired a clearer and more intuitive meaning than what can be gleaned by only looking at the constitutive relations of the material. Finally, there exist beautiful analogies between bi-isotropic and anisotropic materials, as was discussed in the last section of this article, where the similarity of different materials, like chiral samples and ferrites was underlined.

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