Progress In Electromagnetics Research, PIER 8, 173–298, 1994

THE DOUBLE DEFORMATION TECHNIQUE

M. J. Tsuk , S. Y. Poh, and J. A. Kong

1. Introduction to Double Deformation

a. Background

The problem of dipole radiation over media was first studied by Sommerfeld in 1909 for a vertical electric dipole in a two-halfspace configuration. The solutions apply only to the case where the dipole current varies harmonically in time. Corresponding problems have since been formulated and solved for the other fundamental cases of vertical magnetic, and horizontal electric and magnetic dipoles [1–2]. A comprehensive treatment and an extensive bibliography have been compiled by Baños [2]. Interest in this halfspace problem has remained, as evidenced by the abundant literature on the subject in recent years [3–6].

The problem of time-harmonic dipole radiation over stratified media has been studied by Ward [7] and Wait [8] in the context of geophysical probing. Wait solved the case of electric and magnetic dipoles radiating over a stratified isotropic medium [9–10]. The solutions of dipole radiation over a two-layer medium applied mainly to geophysical exploration have been investigated by several researchers [11–20]. Similar problems have been formulated and solved for the case of anisotropic media [21–25].

The problem of source radiation over media over a perfectly conducting ground plane is of interest in the analysis of isolated or coupled miniature transmission-lines or microstrip lines and elements [17, 26–28]. The advent of microwave integrated circuit technology has paved the way for the development of a new class of micro- and millimeter-wave networks using these microstrip elements.

The studies cited above are all concerned with the treatment of time-harmonic or single-frequency analysis. Research in the time-domain or transient solutions of the corresponding problems does not have as long a history but rather began only in the 1950's. However, electrical transient methods have been employed as early as the 1930's [29]. A major drawback to the development of time-domain electromagnetic research and applications lay in the practical difficulty encountered in taking transients measurements over very short times. With improving technology [30], interest in transient electromagnetics has been on the rise. In geophysical applications [31–32], a significant advantage of time-domain methods over frequency-domain methods is that with a transmitted wave having a broad spectrum of frequencies, a wide range of penetrations may be obtained simultaneously. In computer networks utilizing integrated-circuit technology, time-domain analysis of signals is essential in determining propagation effects, such as coupling or distortion, on transmission pulses.

In the case of non-time-harmonic excitations, the time-domain solutions for dipole radiation over stratified media may be obtained, in principle, through the evaluation of the Fourier inverse transform of the Sommerfeld time-harmonic solutions. Except for a few special configurations, the result is a double integral that is difficult to evaluate both analytically and numerically.

In 1951, Wait [33] treated the step response of electric and magnetic dipoles, a finite length of grounded wire and an infinite line source in an unbounded conducting medium. Expressions for the transient fields are derived through Laplace inversion of the time-harmonic solutions that have been simplified by neglecting the displacement current. The transient solutions thus obtained are valid only for times $t \gg \epsilon/\sigma$ where ϵ is the permittivity and σ the conductivity. Bhattacharyya [34–35] considered both negligible and significant displacement currents for the transient step response of an electric dipole in unbounded conducting medium. The approximation of negligible displacement current facilitates inversion of the frequency domain solutions but may not often be applicable due to high-frequency contents of excitation and potentially high permittivities of media.

Wait [9] studied the step response of a vertical magnetic dipole (VMD) over up to three layers of conductive stratification. Again, displacement currents are neglected allowing him to present simple expressions for surface fields and for the mutual impedance between the source and a current loop or a current element. The time-harmonic response of a VMD on a general halfspace evaluated also on the surface may be solved in closed form. This is attributed to Van der Pol [2]. As a consequence, the corresponding time-domain solution for lossless dielectrics may also be obtained in closed-form.

Poritsky [36] applied plane-wave decomposition of the time-harmonic solution and inversion by variable transformation to analyze the impulse response of both horizontal electric (HED) and vertical electric (VED) dipoles near the surface of a homogeneous, lossless earth. Closed-form solutions for the Hertzian potential for a VED on the halfspace with observation directions in the horizontal plane and along the dipole axis in the air were derived. In addition, a physical picture to explain the time-dependent solutions observed in the upper and lower halfspaces was presented. Van der Pol [37] evaluated the impulse response of a VED over a lossless halfspace employing operational calculus based on the two-sided Laplace transform. By directly inverting the Laplace integrals applied to the frequency domain solutions, he obtained closed-form field expressions for an unbounded medium as well as potential functions for the special configuration of dipole-observation point on the surface plane. In 1957, Pekeris and Alterman [38] also investigated the impulse response of a VED over a non-dissipative halfspace. As in the cases of [36–37], closed-form expressions for the Hertzian potential for special configurations were obtained. For more general configurations, the original double integral was modified to single finite-range integrals using a method developed by Cagniard [39] in connection with problems in seismic wave propagation.

It is interesting to note that no closed-form solution exists for the time-harmonic radiation of a VED over a lossless halfspace and yet the solution may be obtained in elementary form for the transient response, albeit for special source-observer positions only.

In [13, 40], Bhattacharyya investigated the transient fields due to a step-excited vertical magnetic dipole on and above a dissipative halfspace. Displacement currents were not neglected and approximate expressions for the limiting states of high and low frequencies were obtained. Wait [44] in 1960 gave a review of the basic theory used to describe the propagation of electromagnetic pulses in a homogeneous conducting earth.

In 1960 de Hoop and Frankena [42] examined the pulse radiation by a VED at a finite height above a plane halfspace. The original double integration was reduced to a finite single integral by deformation of one real-axis integration path into the complex plane and inverting directly. The method used was claimed to be an improved modification of the Cagniard method [39] for seismic wave propagation. Frankena [43] has applied the Cagniard-de Hoop method [41] to the study of pulse radiation by horizontal dipoles above a plane lossless halfspace. Bremmer [45] solved the step response for a VED on a two-dielectric interface directly from the partial differential equations for the potential entirely in the time-domain, avoiding the time-harmonic Sommerfeld solution.

In 1971, Hill [46] presented the exact closed-form solutions for the impulse responses of a VMD, VED, HED, and HMD (horizontal magnetic dipole) over lossless isotropic and uniaxial anisotropic halfspaces, each considered for special cases of source and observation point locations.

In a series of papers [47–50] Wait and his co-workers investigated the transient fields for a variety of combinations of source-observer configurations involving either a VMD or a finite-sized current loop in the presence of a conducting halfspace in the hope of applying the knowledge to mine-rescue operations. The results are obtained in the quasi-static regime and by neglecting displacement currents.

Fuller and Wait [51] attempted to simulate the actual earth environment by considering the permittivity and conductivity to be functions of frequency. The transient radiation of a VED in unbounded and halfspace medium are considered using direct numerical integration of the exact double infinite integrals with subtraction of asymptotic solutions to speed up computation.

In 1979, de Hoop [52] applied the Cagniard-de Hoop technique to derive closed-form expressions, valid everywhere and for all times, for the the transient fields of a line source over a lossless halfspace. A discussion on the extension of the technique to multiple layers was presented.

The impulse and step response of a VED on a conducting halfspace was evaluated through deformation in the complex wavenumber and frequency planes by Haddad and Chang [53]. The original double integral was shown to reduce to single integrals and physical interpretation of the component field expressions was attempted. The process of deforming in the complex frequency plane is similar to the singularities expansion method (SEM) [54] developed for problems involving isolated singularities in the complex frequency plane. More recently, Kuester [55] studied the step response of a pulsed line source over a conducting halfspace and arrived at an exact representation, valid for all times, in terms of a double integral over finite range.

As discussed previously, Wait [9] treated the case of the step response of a VMD over up to three conducting layers of stratification with an analysis that is simplified by the neglect of displacement currents. The shielding of transient dipole fields of a VMD and an HMD in air by a conductive sheet was also investigated by Wait [56]. Closed-form quasistatic solutions valid for very late times were derived by ignoring propagation effects in air.

Vanyan [57] considered transient fields of a pulsed HED in a layered conducting ground environment for geophysical exploration methods in the USSR. Early and late times solutions are provided through asymptotic analysis. The late time response for the magnetic field on the dipole axis of a VMD in a conducting bed of limiting thickness was obtained by Kaufman and Terent'yev [58].

Wait [59] treated electromagnetic transient coupling between two small ungrounded loops over a conducting halfspace and a two-layer conducting earth. Both VMD and HMD arrangements are analyzed using Laplace transform methods and by neglecting displacement current in the air. Electromagnetic coupling in both the frequency and time domains between grounded wires over a layered medium have been computed by Dey and Morrison [60]. The general approach employed involves brute-force numerical integration over frequency and the use of the Fast Fourier Transform (FFT) algorithm [61] for the integration over wavenumber. Results were obtained for up to two layers of stratification despite a multilayer formulation.

Kaufman [62] examined peculiarities in the behavior of transient, late-time, fields due to a VMD in a homogeneous conducting medium for applications to geophysical exploration. Lee and Lewis [63] analyzed the induced voltage in a large horizontal current loop, due to a step excitation in the loop, over layered conducting ground. The transient response of a multi-layered medium to an incident plane wave pulse of finite width has been discussed by Lytle and Lager [64]. In their paper, the natural frequencies, corresponding to pole singularities in the complex frequency plane of the layered structure determined from experiments are matched with those obtainable theoretically to deduce the electrical and physical parameters.

The early and very late time responses were evaluated for a pulsed VMD, HMD and infinite line current over a two-layer model for ground by Botros and Mahmoud [65]. The early time responses were deduced using geometrical ray theory while closed-form late-time responses were obtained by neglecting displacement current in air. The depth and conductivity of the bounded layer was shown to be deducible from the solutions. Kaufman [66] also investigated the late transient fields due to a step-excited horizontal current loop over a two-layer medium. The solution was expressed as a sum of terms proportional to inverse powers of time t. Mahmoud et al. [67] again discussed the transient electromagnetic fields of a VMD with step and pulsed current excitations over a two-layer earth model. Two methods, based on inverse Laplace transform and the natural frequencies concept [64, 54] respectively, were employed. Only displacement currents in the ground layers were neglected and no closed-form expressions were obtainable.

In 1981, Ezzeddine et al. [68] evaluated the time response of a VED over a two-layer nondispersive dielectric. The geometrical optics approach was employed for early arrivals, together with an explicit inversion scheme, analogous to the Cagniard-de Hoop method [41], that is valid for all times. The solution is expressible as single integrals. Ezzeddine et al. [69] continued the study of the pulse response of a VED over a two-layer medium by applying the double deformation technique [70] that involves complex plane deformation in both the wavenumber and frequency planes. The resulting solution, although numerical in nature, does not require excessive computation, and may be readily extended to consider dissipative and dispersive media. The results as presented in the paper, lacked a complete investigation for the early time solutions for both lossless and dissipative media.

The studies cited above have primarily dealt with lossless and dissipative media. Transient radiation in the presence of dispersive, stratified plasma medium has also been widely investigated, although more so for infinite plane waves [71–76] than for finite-source excitations [77–78]. The dispersive nature of the plasma, in general, makes the analysis more complex and the methods employed in the two classes of problems seldom overlap. For instance, the Cagniard-de Hoop approach, which gives elegant simplified solutions in a lossless environment, fails in the presence of dispersive media.

An understanding of time-domain dipole source radiation in the presence of media over a perfectly conducting ground plane is potentially useful in the analysis of transient signals propagation in transmission line systems. Time-domain analysis of transmission line systems have been investigated by Chang [79] and Agarwal [80]. Existing methods employ the TEM or quasi-TEM assumption for the wave propagation. In pulse propagation, a broad spectrum of frequencies exist and higher order modes propagation may not be neglected thus requiring more rigorous analyses.

We find that the problem of transient dipole radiation over layered medium is not new. However the general difficulty of evaluating the formal double integral solution based on the Fourier inversion of the Sommerfeld-type integral has continued to present a challenge to those seeking more efficient and general methods of solution. Aside from the few special cases where the field solutions exist in closed-form [46], the most elegant and well-known approach has been that attributed to Cagniard and de Hoop [41–43]. A weakness of this technique lies in its inherent inability to be applied to problems involving dispersive media. It is also most readily used in the evaluation of impulse or step responses thereby requiring an additional (convolution) integration in the case of more general current excitations. The geometrical optics theory approach, being a high frequency approximation, is valid only for very early time responses whereas neglect of displacement currents leads to solutions that are valid for late times. A significant time interval may exist where neither early nor late time approximations are valid [55].

There clearly remains a need to develop a reliable method general in the sense of being able to provide results that are valid for all times and to be applied to dispersive media. Insofar as this need is concerned, we choose to consider the double deformation technique, first suggested by Rosenbaum [81] in his study of elastic wave propagation, and applied by Ezzeddine et al. [69], Tsang and Kong [70] and Poh and Kong [82].

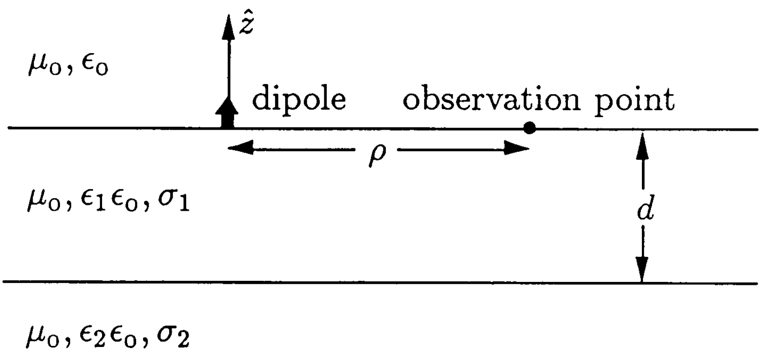

Figure 1.1 General configuration for double deformation solution.

b. Double Deformation

The starting point for the double deformation technique is to express the transient fields on the surface of a layered medium in terms of contour integrals in the complex frequency and transverse wavenumber planes. The general configuration studied, vertical electric (VED) and vertical magnetic (VMD) dipoles on the surface of a layered medium, is shown in Figure 1.1. In Sections 2 and 3, we will study dipoles on the surface of a single dielectric medium of infinite extent, which is equivalent to having d approach infinity in Figure 1.1, or letting $\epsilon_2 = \epsilon_1$. In Sections 4 and 5, we will let the conductivity of the bottom layer, medium 2, be infinity; we will refer to the resulting configuration as a "coated perfect conductor". Finally, in Sections 6 and 7, we will consider the full two-layer medium case; in Section 6, we will also let the bounded layer, medium 1, become slightly conductive.

From [83], we have the frequency-domain expression for the $\hat{z}$-directed magnetic field on the surface of a layered medium a distance ρ away from a VMD also on the surface of the medium:

$$H_z = -i\frac{IA}{8\pi}\int_{\text{SIP}} dk_\rho \frac{k_\rho^3}{k_z} H_0^{(1)}(k_\rho \rho)\left[1 + R^{TE}\right] \tag{1}$$

where $k_z = \sqrt{k_0^2 - k_\rho^2}$ and $k_0 = \omega\sqrt{\mu_0\epsilon_0}$. The Sommerfeld integration path (SIP) in the k_ρ plane is shown in Figure 1.2. Also, the response of the layered medium is entirely contained in R^{TE}; only this quantity

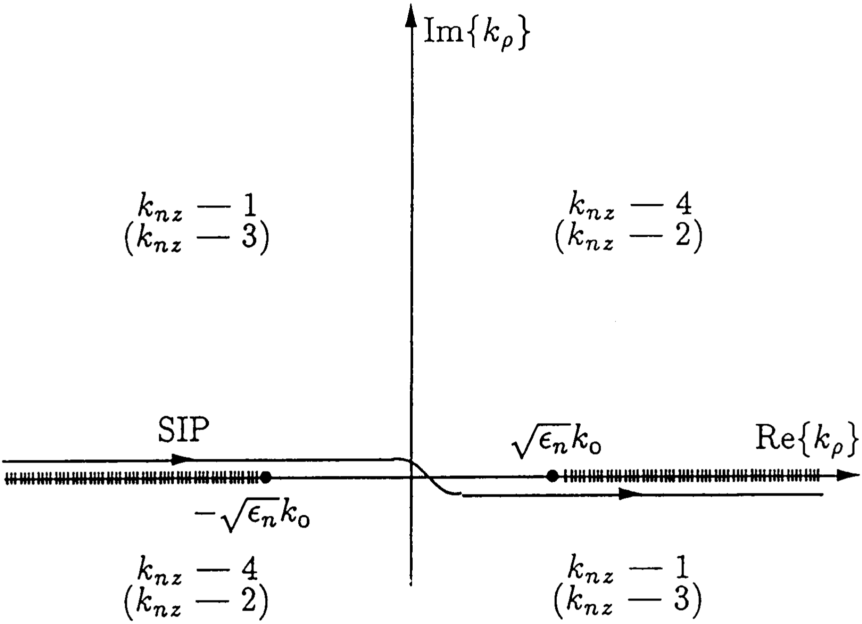

Figure 1.2 SIP, branch cuts and Riemann sheets.

will change as we consider different configurations. Using

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho E_\phi) = ik_0\eta_0 H_z, \tag{2}$$

where $\eta_0 = \sqrt{\mu_0/\epsilon_0}$, we obtain the frequency-domain E_ϕ :

$$E_\phi = \frac{IAk_0\eta_0}{8\pi}\int_{\text{SIP}} dk_\rho \, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho)\left[1 + R^{TE}\right] \tag{3}$$

We convert this to the time domain by means of the inverse Fourier transform, and since E_ϕ is a real function of time, it can be expressed in the following form:

$$E_\phi(\tau) = \frac{\eta_0}{8\pi}\text{Re}\left\{\int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\int_{\text{SIP}} dk_\rho\,\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\left[1 + R^{TE}\right]\right\} \tag{4}$$

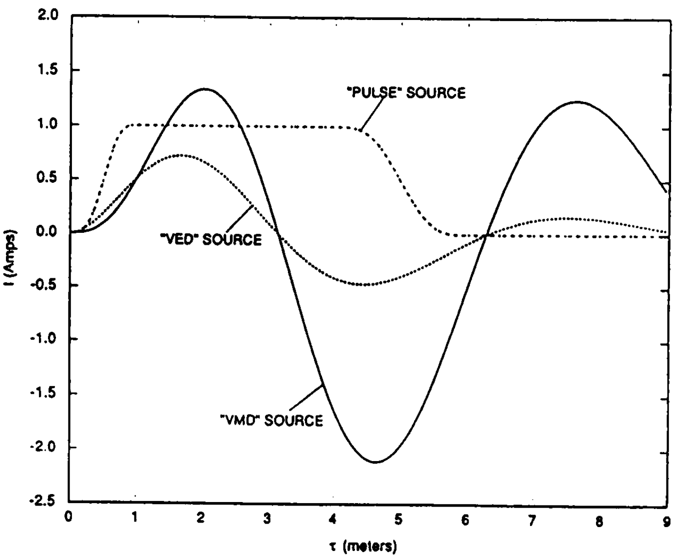

Figure 1.3 Current source waveforms, $\tau = ct, c = 3 \times 10^8$ m/s.

where $\tau = t/\sqrt{\mu_0\epsilon_0}$ and

$$\tilde{I}(k_0) = \int_{-\infty}^{\infty} d\tau' \, e^{ik_0\tau'} I(\tau') \tag{5}$$

We use the notation that $I(\tau')$ has the units of ampere-meter ^{2}, thus including the area of the current loop A. It is also important to note that, since we have normalized our time and frequency variables, $\tilde{I}(k_0)$ has the units of ampere-meter ^{3}. By duality, H_ϕ from a vertical electric dipole (VED) in the same configuration is

$$H_\phi(\tau) = \frac{1}{8\pi}\mathrm{Re}\left\{ i \int_0^\infty dk_0 \, e^{-ik_0\tau} \tilde{I}(k_0) \int_{\mathrm{SIP}} dk_\rho \, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho) \left[1 + R^{TM}\right] \right\} \tag{6}$$

In this case, $I(\tau')$ has the units of ampere-meter, and $\tilde{I}(k_0)$ the units of ampere-meter ^{2}.

In general, therefore, we can write the field quantities as

$$\mathrm{Re}\left\{ \int_0^\infty dk_0 \, f(k_0) \int_{\mathrm{SIP}} dk_\rho \, g(k_0, k_\rho) \right\} \tag{7}$$

We then deform the k_ρ integral to its steepest descent path. This gives us two contributions: one from the integral over the steepest descent path

$$\mathrm{Re}\left\{ \int_0^\infty dk_0\, f(k_0) \int_{\mathrm{SDP}} dk_{\rho_s}\, g(k_0, k_{\rho_s}) \right\} \tag{8}$$

where k_{ρ_s} is the value of k_ρ on the steepest descent path. The other contribution is from the residues of the poles in the k_ρ plane enclosed by the deformation,

$$\mathrm{Re}\left\{ \sum_n (\pm 2\pi i) \int_0^\infty dk_0\, f(k_0) \mathrm{Res}\left[g(k_0, k_\rho)\right]_{k_\rho = k_{\rho_n}} \right\}. \tag{9}$$

where k_{ρ_n} is the value of k_ρ at the nth pole. Next, we interchange the order of integration in (8) and deform the k_0 integral to its steepest descent path. Again, this gives two contributions: the first from the double steepest descent path:

$$\mathrm{Re}\left\{ \int_{\mathrm{SDP}} dk_{\rho_s} \int_{\mathrm{SDP}} dk_{0_s}\, f(k_{0_s}) g(k_{0_s}, k_{\rho_s}) \right\} \tag{10}$$

and the second from the enclosed poles in the k_0 plane:

$$\mathrm{Re}\left\{ \sum_n (\pm 2\pi i) \int_{\mathrm{SDP}} dk_{\rho_s}\, \mathrm{Res}\left[f(k_0) g(k_0, k_{\rho_s})\right]_{k_0 = k_{0_n}} \right\} \tag{11}$$

Thus, the highly oscillatory double integrals in the original expression have been converted to single integrals over pole residues, (9) and (11), and a much more rapidly convergent double integral (10). This reduces the overall computation time. The double deformation method has some analytical benefits as well. The enclosed poles in the complex wavenumber plane can be related to natural modes of the physical system, thus giving insight into the nature of the excitation. Also, the causality of the electromagnetic signal can be demonstrated analytically.

While the double deformation method is able to handle dispersive and lossy media, unlike the Caignard-de Hoop method, it has its limitations. In order for the deformations in the frequency plane to be possible, the Fourier transform of the source function must vanish at infinity in all directions in that plane. This limits the class of functions that can be used with double deformation to those whose Fourier transforms are purely algebraic — that is, those that have a source pole, and no exponential behavior.

In this chapter, a modification to the double deformation method will be presented. This modification is based on splitting the Fourier transform of the source current into "before" and "after" sections, the dividing point being the arrival time of the earliest electromagnetic signal at the observation point. The benefits of the modification are two-fold.

First, it permits a much wider range of source currents than are possible with the original double deformation. With the original technique, only those sources whose Fourier transforms vanished at infinity in all directions were acceptable. These functions have algebraic Fourier transforms in general, and thus have poles that have to be taken into account when deforming the frequency integral. With the modification, on the other hand, there are never any source poles if the source is zero before some initial time and remains finite thereafter. This is because the "before" part is in that case an integral of a finite function over a finite range, and must therefore be finite.

Secondly, the modification allows a stronger statement of causality than is possible with standard double deformation. It will be shown that only the "before" part of the current transform contributes to the response. This is intuitively appealing, since the "before" part is Fourier transform of all the current from times early enough so that the electromagnetic waves could have reached the observation point. In double deformation, the statement of causality is that the response is analytically identically zero before the first part of the signal can reach the observation point. With the modification, one can make a stronger statement: the response only depends on that part of the signal which could possibly have influence. If two signals are the same up to some time t_1, then their responses at a distance ρ away will be analytically identical up to time $t_1 + \rho/c$. With standard double deformation, one cannot make this claim; the response at any one time depends on the source at all times.

The modified method retains all of the advantages of standard double deformation: speed of calculation combined with the physical insight gained by examining the various modes. The insight gained from these modes is strengthened by the modification, since all modes are now continuous, without the step discontinuity at the first arrival common in the standard method. Thus, each mode represents a causal, continuous signal. Also, modified double deformation can treat lossy and/or dispersive media, as can the original technique.

In this article, we will consider the responses of various layered media to three current sources. The first two are decaying exponentials:

$$I(\tau) = I_0 \tau \sin \omega_0 \tau e^{-\alpha_0 \tau}, \tag{12}$$

which has the Fourier transform

$$\tilde{I}(k_0) = \frac{iI_0}{2}\left[\frac{1}{(k_0 + i\alpha_0 + \omega_0)^2} - \frac{1}{(k_0 + i\alpha_0 - \omega_0)^2}\right], \tag{13}$$

and

$$I(\tau) = I_0 \tau^2 \sin \omega_0 \tau e^{-\alpha_0 \tau}, \tag{14}$$

which has the Fourier transform

$$\tilde{I}(k_0) = I_0 \left[\frac{1}{(k_0 + i\alpha_0 - \omega_0)^3} - \frac{1}{(k_0 + i\alpha_0 + \omega_0)^3}\right]. \tag{15}$$

The first of these we will use in VED problems; the second in VMD problems, so that requirements of continuity of responses are satisfied. Since the Fourier transforms of these current sources contain only poles, they can be used in standard double deformation. These sources are shown in Figure 1.3 for $I_0 = 1$, $\omega_0 = 1$ and $\alpha_0 = 0.5$. We will also consider the following current source, which is a smoothed trapezoidal pulse:

$$I(\tau) = \begin{cases} 0 & x \leq 0 \text{ or } \\ & x \geq \tau_r + \tau_p + \tau_f \\ I_0(u+1)^4 & \\ \quad \cdot(16 - 29u + 20u^2 - 5u^3)/32 & 0 < x < \tau_r \\ I_0 & \tau_r < x < \tau_r + \tau_p \\ I_0[1 - (v+1)^4 & \\ \quad \cdot(16 - 29v + 20v^2 - 5v^3)/32] & \tau_r + \tau_p < x < \tau_r + \tau_p + \tau_f \end{cases} \tag{16}$$

where $u = (\tau - \tau_r/2)/(\tau_r/2)$ and $v = (\tau - \tau_r - \tau_p - \tau_f/2)/(\tau_f/2)$. This functional form was chosen so that the current would be sufficiently smooth, or in other words that its first, second, and third time derivatives would be continuous. The rise time is τ_r , the fall time is τ_f , and the pulse remains at its maximum value, I_0 for τ_p . The Fourier transform for this current is very complicated, but it can be expressed in closed form. It is not solely composed of poles, so this source can only be used with the modified double deformation technique. This current source is also shown in Figure 1.3, with $I_0 = 1$, $\tau_r = 1$, $\tau_p = 3$, and $\tau_f = 2$.

Finally, we must consider branch cuts and Riemann sheets. In all unbounded regions, $k_{nz} = \sqrt{\epsilon_n k_0^2 - k_\rho^2}$ will have branch points in the k_ρ plane at $k_\rho = \pm\sqrt{\epsilon_n}\, k_0$. (In bounded regions, all expressions are even in k_{nz} , so that the branch points are unimportant). We choose our branch cuts so that k_{nz} is purely imaginary along them; the upper Riemann sheet is defined where $\text{Re}\{k_{nz}\}$ is positive, the lower Riemann sheet where $\text{Re}\{k_{nz}\}$ is negative. With this definition, the Sommerfeld integration path in (4) and (6) lies entirely on the upper Riemann sheets. In Figure 1.2, for each quadrant in the k_ρ plane, we show the quadrants in which k_{nz} lies on both the upper and lower Riemann sheets (lower sheet in parentheses).

2. Modified Double Deformation for Vertical Magnetic Dipole over Lossless Halfspace

a. Single Deformation

The original expression for the electric field generated by a vertical magnetic dipole (VMD) on an infinite halfspace with dielectric constant ϵ_1 is:

$$E_\phi(\tau) = \frac{\eta_0}{8\pi^2}\text{Re}\left\{\int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\int_{\text{SIP}} dk_\rho\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho)\,\left[1 + R^{TE}\right]\right\} \tag{1}$$

where

$$R^{TE} = \frac{k_z - k_{1z}}{k_z + k_{1z}} \tag{2}$$

and

$$k_z = \sqrt{k_0^2 - k_\rho^2}, \quad k_{1z} = \sqrt{\epsilon_1 k_0^2 - k_\rho^2} \tag{3}$$

The first step in applying the double deformation technique to this problem is to deform the k_ρ integral to its steepest descent path (SDP). When we do this, we find that no pole singularities in the k_ρ plane are enclosed. We do have two integrals the integration paths of which loop around the branch points at $k_\rho = k_0$ and $k_\rho = \sqrt{\epsilon_1} k_0$, as shown in Figure 2.1. Our expression can thus be divided into two parts,

$$E_\phi = \mathrm{SDP}_0 + \mathrm{SDP}_1 \tag{4}$$

where

$$\mathrm{SDP}_0 = \frac{\eta_0}{2\pi^2(\epsilon_1 - 1)} \mathrm{Re}\left\{ i \int_{i\delta}^{\infty} dk_0 \, \frac{\tilde{I}(k_0)}{k_0} e^{-ik_0\tau} \int_0^{\infty} dq \, k_\rho^2 k_z H_1^{(1)}(k_\rho \rho) \right\} \tag{5}$$

with $k_\rho = k_0 + iq$, and

$$\mathrm{SDP}_1 = -\frac{\eta_0}{2\pi^2(\epsilon_1 - 1)} \mathrm{Re}\left\{ i \int_{i\delta}^{\infty} dk_0 \, \frac{\tilde{I}(k_0)}{k_0} e^{-ik_0\tau} \int_0^{\infty} dq \, k_\rho^2 k_{1z} H_1^{(1)}(k_\rho \rho) \right\} \tag{6}$$

with $k_\rho = \sqrt{\epsilon_1} k_0 + iq$. While the sum of these expressions is perfectly regular along the path of integration in the k_0 plane, individually they have single poles at the origin. In order to avoid this difficulty, we have changed our integration contour to start a small distance δ above the origin, make a quarter clockwise circle down to the real axis, and then go along the real axis to infinity. This slight deformation of the contour does not cross any poles of the integrand when both parts are together, so it makes no difference to the overall result.

b. Double Deformation

The evaluation of the two expressions is very similar; let us concentrate on SDP_0 (5). We now want to interchange the order of integration and deform the k_0 integral to its SDP. It is clear that this path will either be straight up ($k_0 = ip$) or straight down ($k_0 = -ip$). The decision depends on whether the integrand vanishes quickly enough in either the upper or lower half plane. The determining factor is the exponential behavior of the integrand. First of all, since we are now considering the behavior when $|k_0|$ gets large, we can replace $e^{-ik_0\tau} H_1^{(1)}(k_\rho \rho)$ with its asymptotic behavior, $-\sqrt{2i/(\pi(k_0 + iq)\rho)}\, e^{-q\rho} e^{-ik_0(\tau - \rho)}$.

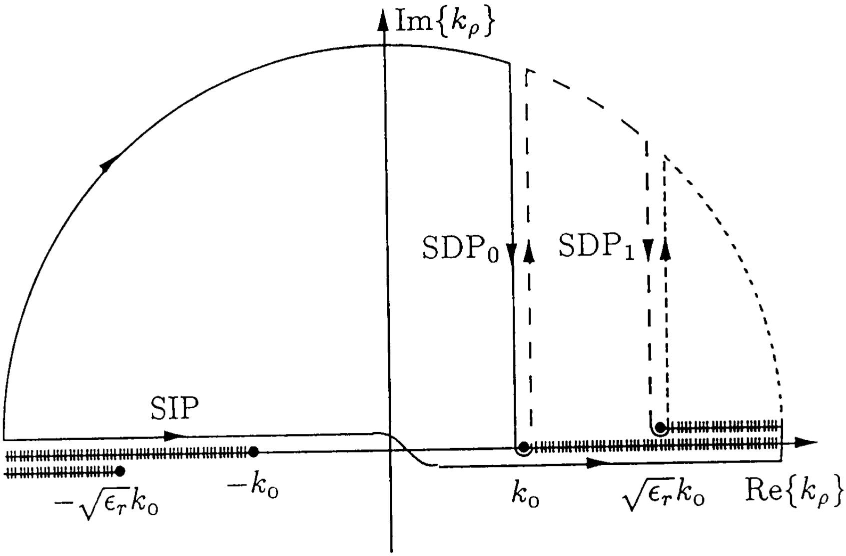

Figure 2.1 Deformation in k_ρ plane for Sommerfeld integration path (SIP); VMD on Lossless Halfspace.

Let us first consider a current source, such as

$$I(\tau) = I_0 \tau^2 \sin \omega_0 \tau e^{-\alpha_0 \tau}; \tag{7}$$

this source is shown in Figure 1.3 for $I_0 = 1$, $\omega_0 = 1$ and $\alpha_0 = 0.5$. The Fourier transform of this source is

$$\tilde{I}(k_0) = I_0 \left[\frac{1}{(k_0 + i\alpha_0 - \omega_0)^3} - \frac{1}{(k_0 + i\alpha_0 + \omega_0)^3} \right], \tag{8}$$

which is made up entirely of poles, and thus vanishes as $k_0 \to \infty$ in any direction in the complex plane. Therefore, in this case, the factor $e^{-ik_0(\tau-\rho)}$ is the only one with an exponential dependence on k_0, and we will deform upward for $\tau - \rho < 0$ and downward for $\tau - \rho > 0$. The k_0 plane for both of these cases is shown in Figure 2.2.

It is easy to show that when we deform upward, the contribution is identically zero. Since there are no singularities of the integrand in the upper half plane, SDP$_0$ becomes just the double steepest descent

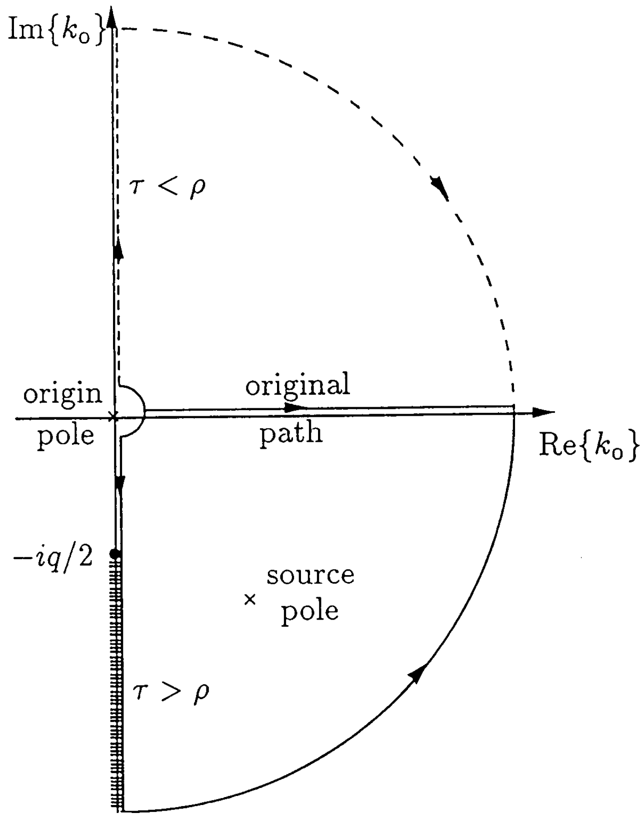

Figure 2.2 Deformation in k_0 plane for both SDP_0 and SDP_1.

path:

$$\mathrm{DSDP}_0 = \frac{\eta_0}{2\pi^2(\epsilon_1 - 1)} \mathrm{Re}\left\{ i \int_0^\infty dq \int_\delta^\infty dp \, \frac{\tilde{I}(ip)}{p} e^{p\tau} k_\rho^2 k_z H_1^{(1)}(k_\rho \rho) \right\}. \quad (9)$$

Since

$$\tilde{I}(ip) = \int_0^\infty d\tau' \, e^{-p\tau'} I(\tau') \quad (10)$$

is real, as is $k_z = \sqrt{q}\sqrt{q+2p}$, and since $k_\rho = i(p+q)$ is positive imaginary, we can use

$$H_n^{(1)}(ix) = \frac{2}{\pi}(-i)^{n+1}K_n(x) \tag{11}$$

to obtain

$$\mathrm{DSDP}_0 = \frac{\eta_0}{\pi^3(\epsilon_1 - 1)}\mathrm{Re}\left\{ i\int_0^\infty dq \int_\delta^\infty dp\, \frac{e^{p\tau}}{p}\left[\int_0^\infty d\tau'\, e^{-p\tau'}I(\tau')\right] (p+q)^2\sqrt{q}\sqrt{q+2p}K_1((p+q)\rho)\right\} = 0. \tag{12}$$

This result satisfies the causality requirement; since the source turns on at time $\tau = 0$, there can be no response a distance ρ away while $\tau - \rho < 0$.

What remains is to evaluate SDP$_0$ when $\tau - \rho > 0$. We wish to deform the k_0 integral in (5) to the negative imaginary axis. This will leave three parts: the residue due to the singularities of the source function, a residue due to the simple pole at the origin, and the integral along the negative imaginary k_0 axis. The latter is identically zero, as we will now show.

The integral along the axis is

$$\mathrm{DSDP}_0 = \frac{\eta_0}{2\pi^2(\epsilon_1 - 1)}\mathrm{Re}\left\{ i\int_\delta^\infty dp\, \frac{\tilde{I}(ip)}{p}e^{p\tau}\int_0^\infty dq\, k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)\right\}. \tag{13}$$

Now, $k_\rho = i(q-p)$ and $k_z = \sqrt{q}\sqrt{q-2p}$. The inner integral naturally divides into three regions:

Region 1: $q \le p$. Here, $k_z = -i\sqrt{q}\sqrt{2p-q}$ is negative imaginary, since we are on the upper Riemann sheet. $k_\rho = -i(p-q)$ is also negative imaginary. We have

$$H_1^{(1)}(-i(p-q)\rho) = -2iI_1((p-q)\rho) + \frac{2}{\pi}K_1((p-q)\rho). \tag{14}$$

We obtain

$$\mathrm{DSDP}_{01} = -\frac{\eta_0}{\pi^3(\epsilon_1 - 1)} \left\{ \int_\delta^\infty dp\, \frac{\tilde{I}(-ip)}{p}e^{-p\tau}\int_0^p dq\, (p-q)^2\sqrt{q}\sqrt{2p-q}K_1((p-q)\rho)\right\}. \tag{15}$$

Region 2: $p \le q \le 2p$. Here, k_z is still negative imaginary, but $k_\rho = i(q-p)$ is positive imaginary. Thus,

$$H_1^{(1)}(i(q-p)\rho) = -\frac{2}{\pi}K_1((q-p)\rho) \tag{16}$$

and we obtain

$$\mathrm{DSDP}_{02} = \frac{\eta_0}{\pi^3(\epsilon_1 - 1)}$$
$$\left\{ \int_\delta^\infty dp \, \frac{\tilde{I}(-ip)}{p} e^{-p\tau} \int_p^{2p} dq \, (q-p)^2 \sqrt{q}\sqrt{2p-q} K_1((q-p)\rho) \right\}. \tag{17}$$

Region 3: $q \ge 2p$. Here, $k_z = \sqrt{q}\sqrt{q-2p}$ is real, and k_ρ is still positive imaginary, so $H_1^{(1)}(k_\rho \rho)$ is real. Thus, the entire expression in brackets is imaginary, so the contribution is zero.

Now, in DSDP_{01}, let $q = p - u$. Then

$$\mathrm{DSDP}_{01} = -\frac{\eta_0}{\pi^3(\epsilon_1 - 1)}$$

$$\cdot \left\{ \int_\delta^\infty dp \, \frac{\tilde{I}(-ip)}{p} e^{-p\tau} \int_0^p du \, u^2 \sqrt{p-u}\sqrt{p+u} K_1(u\rho) \right\} \tag{18}$$

In DSDP_{02}, let $q = p + u$. Then

$$\mathrm{DSDP}_{02} = \frac{\eta_0}{\pi^3(\epsilon_1 - 1)}$$

$$\cdot \left\{ \int_\delta^\infty dp \, \frac{\tilde{I}(-ip)}{p} e^{-p\tau} \int_0^p du \, u^2 \sqrt{p+u}\sqrt{p-u} K_1(u\rho) \right\} \tag{19}$$

Since $\mathrm{DSDP}_{01} = -\mathrm{DSDP}_{02}$, the entire contribution to the total from this double steepest descent path is zero.

Next, we have the residues of all the pole singularities of the source function. For our example, this is just the triple pole at $k_0 = \omega_0 - i\alpha_0 \equiv$

k_s. This gives

$$\mathrm{SP}_0 = \frac{\eta_0}{2\pi(\epsilon_1 - 1)}$$

$$\cdot \operatorname{Re}\left\{ \int_0^\infty dq \, \frac{\partial^2}{\partial k_0^2} \left[(k_0 - k_s)^3 \frac{\tilde{I}(k_0)}{k_0} e^{-ik_0\tau} k_\rho^2 k_z H_1^{(1)}(k_\rho \rho) \right]_{k_0 = k_s} \right\} \tag{20}$$

Finally, we have the residue of the simple pole at the origin. We obtain $-\pi i$ times the residue, since we are detouring halfway around the pole clockwise:

$$\mathrm{DC}_0 = -\frac{\eta_0}{2\pi(\epsilon_1 - 1)} \operatorname{Re}\left\{ \tilde{I}(0) \int_0^\infty dq \, q^3 \left(-\frac{2}{\pi} K_1(q\rho) \right) \right\} \tag{21}$$

Using

$$\int_0^\infty dx \, x^n K_m(ax) = 2^{n-1} a^{-n-1} \Gamma\left(\frac{1+n+m}{2} \right) \Gamma\left(\frac{1+n-m}{2} \right) \tag{22}$$

we obtain

$$\mathrm{DC}_0 = \frac{3\eta_0}{2\pi(\epsilon_1 - 1)\rho^4} \operatorname{Re}\left\{ \tilde{I}(0) \right\} \tag{23}$$

The sum of SP_0 and DC_0 gives E_ϕ for times $\tau < \sqrt{\epsilon_1}\rho$. In order to obtain the rest of the response, we need to consider SDP_1 (6).

SDP_1 is very similar to SDP_0, except for a minus sign, and the fact that that $k_\rho = \sqrt{\epsilon_1}k_0 + iq$ and $k_{1z} = \sqrt{q^2 - 2i\sqrt{\epsilon_1}k_0 q}$. In this case, the deciding factor in the integrand is $e^{-ik_0(\tau - \sqrt{\epsilon_1}\rho)}$, so for $\tau < \sqrt{\epsilon_1}\rho$ we deform the k_0 integral upward, and for $\tau > \sqrt{\epsilon_1}\rho$ we deform it downward. When we deform upward, we enclose no poles, and the double SDP contribution again is identically zero.

The source pole contribution is

$$\mathrm{SP}_1 = -\frac{\eta_0}{2\pi(\epsilon_1 - 1)}$$

$$\operatorname{Re}\left\{ \int_0^\infty dq \, \frac{\partial^2}{\partial k_0^2} \left[(k_0 - k_s)^3 \frac{\tilde{I}(k_0)}{k_0} e^{-ik_0\tau} k_\rho^2 k_{1z} H_1^{(1)}(k_\rho \rho) \right]_{k_0 = k_s} \right\} \tag{24}$$

and the contribution from the pole at the origin is

$$\mathrm{DC}_1 = \frac{\eta_0}{2\pi(\epsilon_1 - 1)}\mathrm{Re}\left\{\tilde{I}(0)\int_0^\infty dq\, q^3\left(-\frac{2}{\pi}K_1(q\rho)\right)\right\} = -\frac{3\eta_0}{2\pi(\epsilon_1 - 1)\rho^4}\mathrm{Re}\left\{\tilde{I}(0)\right\} = \mathrm{DC}_0 \tag{25}$$

Thus, the total solution with the double deformation technique is

$$\begin{aligned}
E_\phi &= 0 \qquad \tau < \rho \\
&= \frac{\eta_0}{2\pi(\epsilon_1 - 1)} \cdot \mathrm{Re}\left\{\int_0^\infty dq\,\frac{\partial^2}{\partial k_0^2}\left[(k_0 - k_s)^3\frac{\tilde{I}(k_0)}{k_0}e^{-ik_0\tau}k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)\right]_{k_0=k_s}\right\} \\
&\quad - \frac{3\eta_0}{2\pi(\epsilon_1 - 1)\rho^4}\mathrm{Re}\left\{\tilde{I}(0)\right\} \qquad \rho < \tau < \sqrt{\epsilon_1}\rho \\
&= \frac{\eta_0}{2\pi(\epsilon_1 - 1)} \cdot \mathrm{Re}\left\{\int_0^\infty dq\,\frac{\partial^2}{\partial k_0^2}\left[(k_0 - k_s)^3\frac{\tilde{I}(k_0)}{k_0}e^{-ik_0\tau}k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)\right]_{k_0=k_s}\right\} \\
&\quad - \frac{\eta_0}{2\pi(\epsilon_1 - 1)} \cdot \mathrm{Re}\left\{\int_0^\infty dq\,\frac{\partial^2}{\partial k_0^2}\left[(k_0 - k_s)^3\frac{\tilde{I}(k_0)}{k_0}e^{-ik_0\tau}k_\rho^2 k_{1z} H_1^{(1)}(k_\rho\rho)\right]_{k_0=k_s}\right\} \\
&\qquad \tau > \sqrt{\epsilon_1}\rho
\end{aligned} \tag{26}$$

The total response is shown in Figure 2.3; the closed form solution, which will be discussed in the next section, overlays this result. The individual components of the solution are shown in Figures 2.4 and 2.5. We notice that the origin pole contributions are very small compared to the total result; this contrasts with the modified double deformation results, given below, which is entirely due to the origin pole.

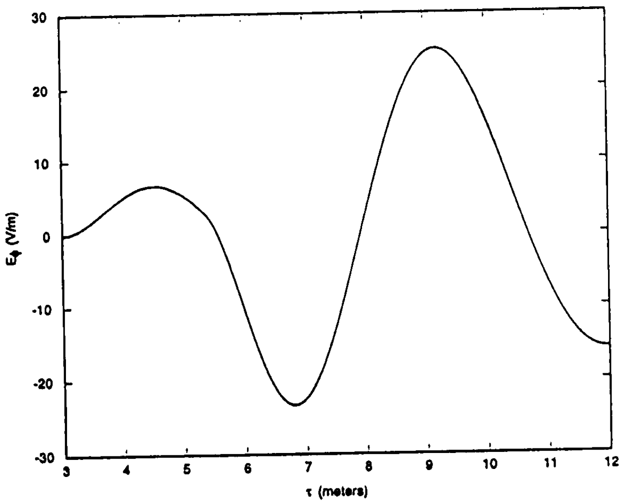

Figure 2.3 Total double deformation result and closed form solution; VMD on lossless halfspace; source, Eq. (14).

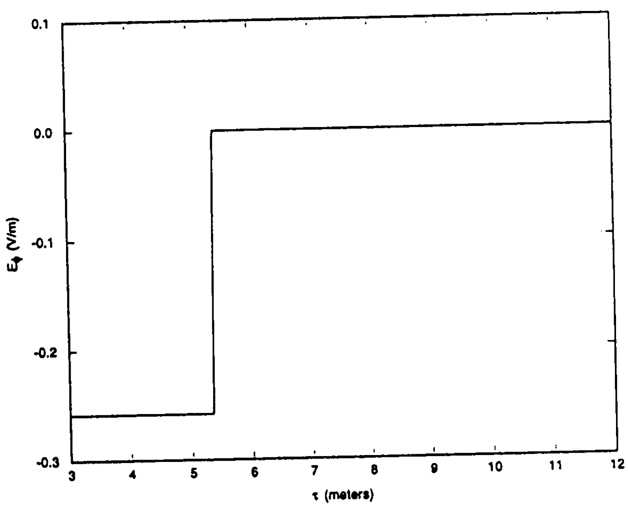

Figure 2.4 k_0-plane pole contribution to double deformation result; VMD on lossless halfspace; source, Eq. (14).

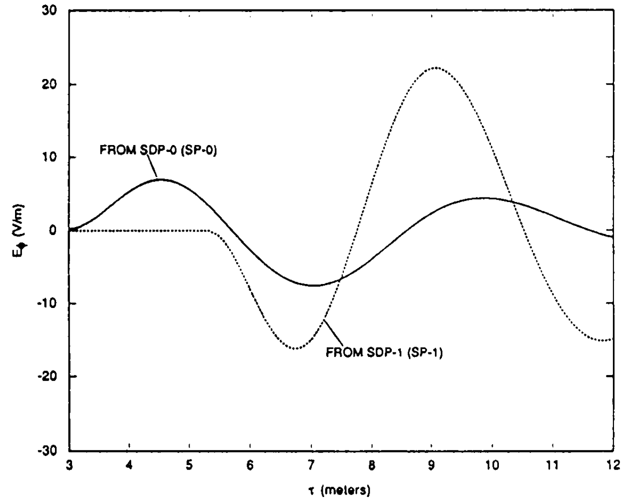

Figure 2.5 **k_0-plane source poles contribution; VMD on lossless halfspace; source, Eq. (14).**

c. Modified Double Deformation

In the previous section, we made use of the fact that the Fourier transform of the current, $\tilde{I}(k_0)$, was made up entirely of poles. This enabled us to use $e^{-ik_0(\tau-\rho)}$ as the deciding factor in whether to deform the k_0 integral up or down. However, only a special class of current functions have Fourier transforms that have no exponential k_0 behavior. In general, consider the two factors from asymptotic expansion of the integrand of SDP$_0$ (5),

$$e^{-ik_0(\tau-\rho)}\tilde{I}(k_0) \tag{27}$$

If we write $\tilde{I}(k_0)$ in its integral form and include $e^{-ik_0(\tau-\rho)}$ in the integrand, we obtain:

$$\int_{-\infty}^{\infty} d\tau' \, e^{-ik_0(\tau-\rho-\tau')} I(\tau') \tag{28}$$

If we now consider deforming the k_0 integral, we see that the deciding factor about which half-plane we can close the contour in is $\tau-\rho-\tau'$, which, in general, is positive for some values of τ' and negative for other values. Thus, we can neither deform upwards nor downwards and have the integrand vanish at infinity.

Since the problem comes about because of the wide range of τ' values, one possible solution is to split the Fourier transform of the current source into two halves, a "before" part, $\tilde{I}_b$, and an "after" part, $\tilde{I}_a$:

$$\tilde{I}(k_0) = \tilde{I}_b(k_0, \gamma) + \tilde{I}_a(k_0, \gamma) \tag{29}$$

where

$$\tilde{I}_b(k_0, \gamma) = \int_{-\infty}^{\gamma} d\tau' \, e^{ik_0\tau'} I(\tau') \tag{30}$$

and

$$\tilde{I}_a(k_0, \gamma) = \int_{\gamma}^{\infty} d\tau' \, e^{ik_0\tau'} I(\tau') \tag{31}$$

Combining these with $e^{-ik_0(\tau-\rho)}$ and setting $\gamma = \tau - \rho$ we obtain

$$\begin{aligned} \tilde{I}(k_0)e^{-ik_0(\tau-\rho)} &= \tilde{I}_b(k_0, \tau-\rho)e^{-ik_0(\tau-\rho)} + \tilde{I}_a(k_0, \tau-\rho)e^{-ik_0(\tau-\rho)} \\ &= \int_{-\infty}^{\tau-\rho} d\tau' \, e^{-ik_0(\tau-\rho-\tau')} I(\tau') \\ &\quad + \int_{\tau-\rho}^{\infty} d\tau' \, e^{-ik_0(\tau-\rho-\tau')} I(\tau'). \end{aligned} \tag{32}$$

Examining the above, we can see that, in the $\tilde{I}_b$ part, $\tau - \rho - \tau'$ is always positive, and so therefore we may deform it to the bottom half of the k_0 plane without it becoming singular. Conversely, in the $\tilde{I}_a$ part, $\tau - \rho - \tau'$ is always negative, so we may deform it to the top half of the k_0 plane.

However, there is one more difficulty that must be overcome; in order for the contribution from the section of the contour at infinity to vanish, the k_0 integrand must go to zero faster than $1/k_0$; this is Jordan's Lemma. Because of this, we require both $\tilde{I}_b$ and $\tilde{I}_a$ to vanish at infinity faster than $1/k_0^2$. Unfortunately, in general, the behavior of $\tilde{I}_b$ and $\tilde{I}_a$ is as $1/k_0$. As long as the time function of the current is sufficiently continuous, we can solve this by writing:

$$\tilde{I}(k_0) = \int_0^{\infty} d\tau' \, I(\tau')e^{ik_0\tau'} = -\frac{1}{k_0^2} \int_0^{\infty} d\tau' \, I''(\tau')e^{ik_0\tau'} \tag{33}$$

This assumes that the current is zero for $\tau < 0$, and that both the current and its derivative are zero at $\tau = 0$. We now split the Fourier transform of the current into two parts:

$$\tilde{I}(k_0) = -\frac{1}{k_0^2}\left[\int_0^{\tau-\rho} d\tau'\, I''(\tau')e^{ik_0\tau'} + \int_{\tau-\rho}^{\infty} d\tau'\, I''(\tau')e^{ik_0\tau'}\right] \tag{34}$$

$$\equiv -\frac{1}{k_0^2}\left[\tilde{I}_{2b}(k_0,\tau-\rho) + \tilde{I}_{2a}(k_0,\tau-\rho)\right] \tag{35}$$

We substitute this into our expression for SDP_0 (5) and deform the k_0 integral of the half containing $\tilde{I}_{2a}$ upward, the half containing $\tilde{I}_{2b}$ downward. Since $\tilde{I}_{2a}$ and $\tilde{I}_{2b}$ both vanish at infinity as $1/k_0$, Jordan's Lemma is satisfied.

It is easy to show that the part containing $\tilde{I}_{2a}$ is identically zero; the demonstration is very similar to that for standard double deformation for $\tau < \rho$ and is omitted here. It is important to note, however, that this demonstration of causality is stronger than that possible for standard double deformation. There, one can only show that the field is zero for $\tau < \rho$; the field for $\tau > \rho$ depends on the current for all time, through $\tilde{I}(k_0)$. With the modification, the field at point ρ at any time τ has no dependence at all on the current from the source later than time $\tau - \rho$.

What remains is to evaluate the part containing $\tilde{I}_{2b}$. We have

$$\mathrm{SDP}_0 = -\frac{\eta_0}{2\pi^2(\epsilon_1 - 1)}$$

$$\cdot\,\mathrm{Re}\left\{ i\int_0^{\infty} dq \int_{i\delta}^{\infty} dk_0\, \frac{\tilde{I}_{2b}(k_0,\tau-\rho)}{k_0^3} e^{-ik_0\tau} k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)\right\} \tag{36}$$

with $k_\rho = k_0 + iq$. We wish to deform the k_0 integral to the negative imaginary axis. This will leave only two parts: the residue due to the triple pole at the origin, and the integral along the negative imaginary k_0 axis, since $\tilde{I}_{2b}(k_0, \tau - \rho)$, being the integral of a finite integrand over a finite range (assuming the current is zero before a certain time), can have no pole singularities. The integral along the axis vanishes in the same way as it did for standard double deformation. The only contribution is therefore the residue of the triple pole at the origin of the k_0 plane. We obtain $-\pi i$ times the residue, since we are detouring halfway around the pole clockwise:

$$\mathrm{SDP}_0 = -\frac{\eta_0}{4\pi(\epsilon_1 - 1)}$$

$$\cdot \mathrm{Re}\left\{ \int_0^\infty dq \, \frac{\partial^2}{\partial k_0^2} \left[e^{-ik_0\tau} \tilde{I}_{2b}(k_0, \tau - \rho) k_\rho^2 k_z H_1^{(1)}(k_\rho \rho) \right]_{k_0=0} \right\} \tag{37}$$

Let

$$g(k_0) \equiv e^{-ik_0\tau} \tilde{I}_{2b}(k_0, \tau - \rho) \tag{38}$$

and

$$f(k_0, q) \equiv k_\rho^2 k_z H_1^{(1)}(k_\rho \rho). \tag{39}$$

where $k_\rho = k_0 + iq$. Then we can express the SDP_0 integral very simply as

$$\mathrm{SDP}_0 = -\frac{\eta_0}{4\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ g(0) \int_0^\infty dq \, f''(0, q) + 2g'(0) \int_0^\infty dq \, f'(0, q) + g''(0) \int_0^\infty dq \, f(0, q) \right\}. \tag{40}$$

Since

$$g(k_0) = \int_0^{\tau - \rho} d\tau' \, I''(\tau') e^{ik_0(\tau' - \tau)}, \tag{41}$$

we have

$$g(0) = I'(\tau - \rho) \tag{42}$$

$$\begin{aligned} g'(0) &= i \int_0^{\tau - \rho} d\tau' \, (\tau' - \tau) I''(\tau') \\ &= -i[\rho I'(\tau - \rho) + I(\tau - \rho)] \end{aligned} \tag{43}$$

and

$$\begin{aligned} g''(0) &= -\int_0^{\tau - \rho} d\tau' \, (\tau' - \tau)^2 I''(\tau') \\ &= -[\rho^2 I'(\tau - \rho) + 2\rho I(\tau - \rho) + 2 \int_0^{\tau - \rho} d\tau' \, I(\tau')] \end{aligned} \tag{44}$$

We must also keep in mind that $k_\rho = iq$ and $k_z = q$ when $k_0 = 0$. Now,

$$f(0,q) = -q^3 H_1^{(1)}(iq\rho) = \frac{2q^3}{\pi} K_1(q\rho), \tag{45}$$

so, using (22),

$$\int_0^\infty dq\, f(0,q) = \frac{8}{\pi\rho^4}\Gamma\left(\frac{5}{2}\right)\Gamma\left(\frac{3}{2}\right) = \frac{3}{\rho^4}. \tag{46}$$

Now,

$$f'(k_0,q) = k_z k_\rho H_1^{(1)}(k_\rho\rho) - \frac{iq}{k_z}k_\rho^2 H_1^{(1)}(k_\rho\rho) + k_\rho^2\rho k_z H_0^{(1)}(k_\rho\rho) \tag{47}$$

$$f'(0,q) = -\frac{4i}{\pi}q^2 K_1(q\rho) + \frac{2i}{\pi}\rho q^3 K_0(q\rho) \tag{48}$$

and

$$\int_0^\infty dq\, f'(0,q) = 0 \tag{49}$$

Finally,

$$f''(k_0,q) = \frac{q k_\rho^2 H_1^{(1)}(k_\rho\rho)}{k_z[q-2ik_0]} + 3k_z k_\rho \rho H_0^{(1)}(k_\rho\rho) - k_z k_\rho^2 \rho^2 H_1^{(1)}(k_\rho\rho)$$
$$- \frac{2iqk_\rho}{k_z}H_1^{(1)}(k_\rho\rho) - \frac{2iqk_\rho^2\rho}{k_z}H_0^{(1)}(k_\rho\rho) \tag{50}$$

$$f''(0,q) = \frac{2}{\pi}\left[5q^2\rho K_0(q\rho) - qK_1(q\rho) - q^3\rho^2 K_1(q\rho)\right] \tag{51}$$

and

$$\int_0^\infty dq\, f''(0,q) = \frac{1}{\rho^2} \tag{52}$$

Thus,

$$\mathrm{SDP}_0 = \frac{\eta_0}{2\pi(\epsilon_1 - 1)\rho^4}\left\{3\int_0^{\tau-\rho} d\tau'\, I(\tau') + 3\rho I(\tau-\rho) + \rho^2 I'(\tau-\rho)\right\}, \tag{53}$$

which is the closed form solution for times $\tau < \sqrt{\epsilon_1}\rho$. The other term, SDP_1, gives us the rest. It is very similar, except for a minus sign, and that $k_\rho = \sqrt{\epsilon_1}k_0 + iq$ and $k_{1z} = \sqrt{q^2 - 2i\sqrt{\epsilon_1}k_0 q}$. By following a similar process to that for SDP $_0$, we obtain

$$\mathrm{SDP}_1 = \frac{\eta_0}{4\pi(\epsilon_1 - 1)}\mathrm{Re}\left\{ g_1(0)\int_0^\infty dq\, f_1''(0,q) + 2g_1'(0)\int_0^\infty dq\, f_1'(0,q) + g_1''(0)\int_0^\infty dq\, f_1(0,q)\right\}. \quad (54)$$

where

$$g_1(k_0) \equiv e^{-ik_0\tau}\tilde{I}_{2b}(k_0, \tau - \sqrt{\epsilon_1}\rho) \quad (55)$$

and

$$f_1(k_0, q) \equiv k_\rho^2 k_{1z} H_1^{(1)}(k_\rho \rho) = f(\sqrt{\epsilon_1}k_0, q). \quad (56)$$

By applying these changes to our previous work, we find that

$$\mathrm{SDP}_1 = -\frac{\eta_0}{2\pi(\epsilon_1 - 1)\rho^4} \cdot \left\{ 3\int_0^{\tau - \sqrt{\epsilon_1}\rho} d\tau'\, I(\tau') + 3\sqrt{\epsilon_1}\rho I(\tau - \sqrt{\epsilon_1}\rho) + \epsilon_1\rho^2 I'(\tau - \sqrt{\epsilon_1}\rho)\right\} \quad (57)$$

which is the other half of the closed form solution.

The total solution for the electric field from a vertical magnetic dipole over a halfspace dielectric is therefore:

$$\begin{aligned}
E_\phi &= 0 && \tau < \rho \\
&= \frac{\eta_0}{2\pi(\epsilon_1 - 1)\rho^4}\left\{ 3\int_0^{\tau-\rho} d\tau'\, I(\tau') + 3\rho I(\tau - \rho) + \rho^2 I'(\tau - \rho)\right\} && \rho < \tau < \sqrt{\epsilon_1}\rho \\
&= \frac{\eta_0}{2\pi(\epsilon_1 - 1)\rho^4}\left\{ -3\int_{\tau-\rho}^{\tau - \sqrt{\epsilon_1}\rho} d\tau'\, I(\tau') + 3\rho I(\tau - \rho) - 3\sqrt{\epsilon_1}\rho I(\tau - \sqrt{\epsilon_1}\rho) \right. \\
&\qquad \left. + \rho^2 I'(\tau - \rho) - \epsilon_1\rho^2 I'(\tau - \sqrt{\epsilon_1}\rho)\right\} && \tau > \sqrt{\epsilon_1}\rho
\end{aligned} \quad (58)$$

We can see the advantages of the modified double deformation technique over the standard form. Not only are we able to consider a wider variety of sources, and have a stronger demonstration of causality, but modified double deformation gives us the *closed form* solution for the VMD over a dielectric halfspace.

3. Modified Double Deformation for Vertical Electric Dipole over Lossless Halfspace

a. Single Deformation

The original expression for the magnetic field from a vertical electric dipole (VED) on an infinite halfspace with dielectric constant ϵ_1 is:

$$H_\phi = \frac{1}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dk_0\, e^{-ik_0\tau}\tilde{I}(k_0)\int_{\mathrm{SIP}} dk_\rho\, \frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)[1+R^{TM}]\right\} \tag{1}$$

where

$$R^{TM} = R_{01} = \frac{\epsilon_1 k_z - k_{1z}}{\epsilon_1 k_z + k_{1z}} \tag{2}$$

The first step in applying the double deformation technique to this problem is to deform the k_ρ integral to its steepest descent path (SDP), as shown in Figure 3.1. When we do this, we find that we enclose no pole singularities in the k_ρ plane. The Sommerfeld poles, where $\epsilon_1 k_z + k_{1z} = 0$ and $R_{01}\to\infty$, occur at $k_\rho = \pm\sqrt{\epsilon_1/(\epsilon_1+1)}k_0$, but they require that k_z and k_{1z} be of opposite signs, and thus are not on the UU Riemann sheet. We do have two integrals which loop around the branch points at $k_\rho = k_0$ and $k_\rho = \sqrt{\epsilon_1}k_0$. Our expression can thus be divided into two parts,

$$H_\phi = \mathrm{SDP}_0 + \mathrm{SDP}_1 \tag{3}$$

where

$$\mathrm{SDP}_0 = \frac{\epsilon_1^2}{4\pi^2(\epsilon_1-1)}$$

$$\cdot\,\mathrm{Re}\left\{\int_0^\infty dk_0\,\tilde{I}(k_0)e^{-ik_0\tau}\int_0^\infty dq\,\frac{k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)}{\epsilon_1 k_0^2-(\epsilon_1+1)k_\rho^2}\right\} \tag{4}$$

$$k_\rho = k_0 + iq$$

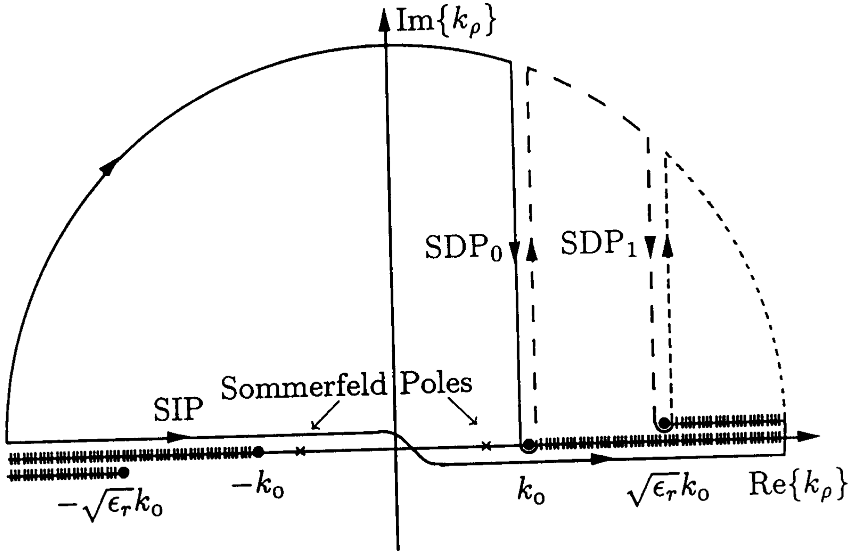

Figure 3.1 Deformation in k_ρ plane for Sommerfeld integration path (SIP) VED on lossless halfspace.

and

$$\mathrm{SDP}_1 = -\frac{\epsilon_1}{4\pi^2(\epsilon_1 - 1)}$$

$$\cdot \mathrm{Re}\left\{ \int_0^\infty dk_0\, \tilde{I}(k_0) e^{-ik_0\tau} \int_0^\infty dq\, \frac{k_\rho^2 k_{1z} H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2} \right\} \tag{5}$$

$$k_\rho = \sqrt{\epsilon_1} k_0 + iq$$

b. Double Deformation

The evaluation of the two expressions is very similar; let us first concentrate on SDP$_0$. We now want to interchange the order of inte-

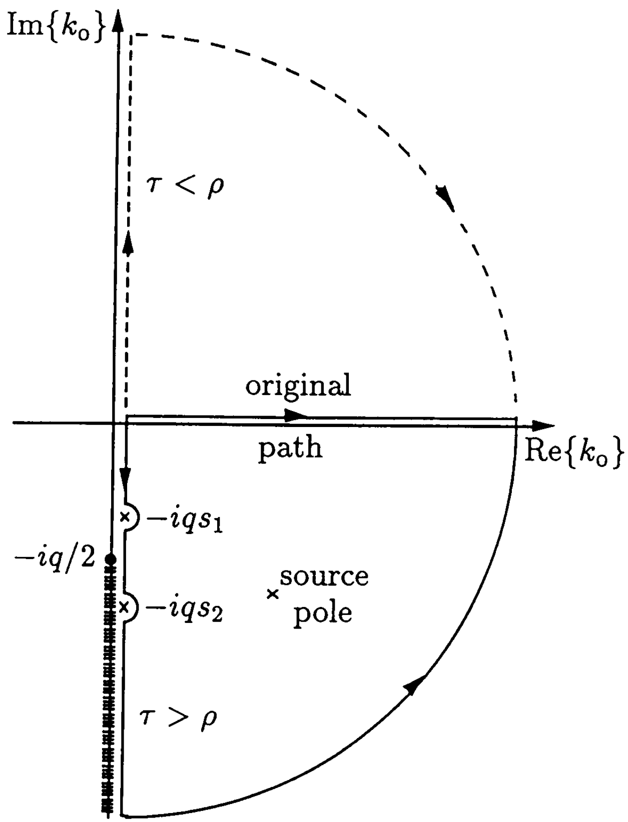

Figure 3.2 Deformation in k_0 plane for both SDP_0 and SDP_1 VED on lossless halfspace.

gration and deform the k_0 integral to its SDP. Let us first consider a current source, such as

$$I(\tau) = I_0 \tau \sin \omega_0 \tau e^{-\alpha_0 \tau}; \tag{6}$$

this source is shown in Figure 1.3 for $I_0 = 1$, $\omega_0 = 1$ and $\alpha_0 = 0.5$. The Fourier transform of this source is

$$\tilde{I}(k_0) = \frac{iI_0}{2} \left[\frac{1}{(k_0 + i\alpha_0 + \omega_0)^2} - \frac{1}{(k_0 + i\alpha_0 - \omega_0)^2} \right], \tag{7}$$

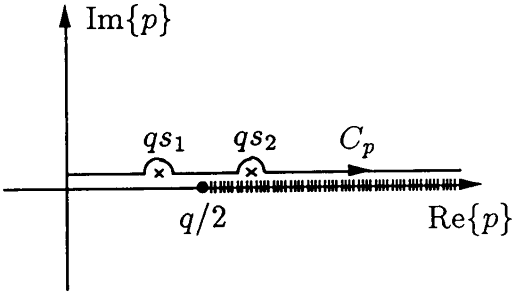

Figure 3.3 Integration contour C_p, Eq. (11) VED on lossless halfspace.

As was the case for the VMD on a halfspace medium, with this kind of source, the factor that determines whether the k_0 deformation is up or down is $e^{-ik_0(\tau-\rho)}$; for $\tau < \rho$ the deformation is up, for $\tau > \rho$, down. Both cases are shown in Figure 3.2. Examining the case with $\tau < \rho$ first, we deform the k_0 integral to the positive imaginary axis, letting $k_0 = ip$. Since both of the zeroes of the expression $\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2$ lie on the negative imaginary axis in the k_0 plane, we enclose no poles in this deformation. We are thus left with the double steepest descent path integral

$$\mathrm{DSDP}_0 = \frac{\epsilon_1^2}{2\pi^2(\epsilon_1 - 1)}\mathrm{Re}\Bigg\{ i\int_0^\infty dq \int_\delta^\infty dp\, \tilde{I}(ip)e^{p\tau} \frac{(p+q)^2\sqrt{q(q+2p)}H_1^{(1)}(i(p+q)\rho)}{\epsilon_1 p^2 - (\epsilon_1+1)(p+q)^2}\Bigg\} \tag{8}$$

Since both p and q are positive over the range of integration, $H_1^{(1)}(i(p+q)\rho) = -(2/\pi)K_1((p+q)\rho)$, which is purely real. Also,

$$\tilde{I}(ip) = \int_0^\infty d\tau'\, e^{-p\tau'} I(\tau') \tag{9}$$

is also real. Therefore, the entire contribution from (8) to the magnetic field is zero. This is required, by the principle of causality, since the electromagnetic field at time $\tau < \rho$ must be zero, since the waves travelling at the speed of light have not yet arrived at the observation point.

Considering now the situation when $\tau > \rho$, we wish to deform the k_0 integral in (4) down to its steepest descent path, on the negative imaginary axis. Since there are poles of the integrand on that axis, corresponding to the Sommerfeld poles in the k_ρ plane, we do not enclose them, but detour around them. If we let $k_0 = -ip$, our integration path C_p is shown in Figure 3.3. We also enclose the poles of the Fourier transform of the source function — for our example, the double pole at $k_0 = \omega_0 - i\alpha_0 \equiv k_s$. This gives

$$\mathrm{SP}_0 = \frac{\epsilon_1^2}{2\pi(\epsilon_1 - 1)}$$

$$\cdot \mathrm{Re}\left\{ \int_0^\infty dq \, \frac{\partial}{\partial k_0} \left[(k_0 - k_s)^2 \tilde{I}(k_0) e^{-ik_0\tau} \frac{k_\rho^2 k_z H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2} \right]_{k_0 = k_s} \right\} \tag{10}$$

From the integration along C_p, we obtain

$$\mathrm{DSDP}_0 = \frac{\epsilon_1^2}{2\pi^2(\epsilon_1 - 1)} \mathrm{Re}\left\{ \int_0^\infty dq \int_{C_p} dp \, \tilde{I}(-ip) e^{-p\tau} \frac{(q-p)^2 \sqrt{q(q-2p)} H_1^{(1)}(i(q-p)\rho)}{(p - s_1 q)(p - s_2 q)} \right\} \tag{11}$$

where

$$s_1 = \frac{1}{1 + (\sqrt{\epsilon_1}/\sqrt{\epsilon_1 + 1})} \tag{12}$$

and

$$s_2 = \frac{1}{1 - (\sqrt{\epsilon_1}/\sqrt{\epsilon_1 + 1})} \tag{13}$$

For $0 < \epsilon_1 < \infty$, $1/2 < s_1 < 1$ and $1 < s_2 < \infty$ so that there is never a double pole. In addition to the two poles, there are two other critical points along the integration path. First, for $p < q/2$, $k_z = \sqrt{q(q-2p)}$ is real and positive; for $p > q/2$, $k_z = -i\sqrt{q(2p-q)}$ is negative imaginary (since we are using k_z as on the top Riemann sheet). Second, for $p < q$, $H_1^{(1)}(i(q-p)\rho) = -(2/\pi)K_1((q-p)\rho)$ which is purely real; for $p > q$, $H_1^{(1)}(-i(p-q)\rho = (2/\pi)K_1((p - q)\rho) - 2iI_1((p-q)\rho)$ which is complex.

It can be shown that the total from the integrations along the axis, excluding the semicircles around the poles, is zero. To do this, we divide the integration interval into five regions: (1) $0 < p < q/2$, (2) $q/2 < p < s_1 q$, (3) $s_1 q < p < q$, (4) $q < p < s_2 q$, and (5) $s_2 < p$.

In region 1, both k_z and $H_1^{(1)}$ are real, so the total answer is zero. In region 2, k_z is negative imaginary but $H_1^{(1)}$ is still real. This gives

$$\mathrm{DSDP}_{02} = -\frac{\epsilon_1^2}{\pi^3(\epsilon_1 - 1)}\left\{\int_0^\infty dq \int_{q/2}^{s_1 q} dp\, \tilde{I}(-ip)e^{-p\tau} \right. \\ \left. \frac{(q-p)^2\sqrt{q(2p-q)}K_1((q-p)\rho)}{(p-s_1 q)(p-s_2 q)}\right\} \tag{14}$$

Region 3 is the same as region 2, except for a change in the limits of the p integral:

$$\mathrm{DSDP}_{03} = -\frac{\epsilon_1^2}{\pi^3(\epsilon_1 - 1)}\left\{\int_0^\infty dq \int_{s_1 q}^{q} dp\, \tilde{I}(-ip)e^{-p\tau} \right. \\ \left. \frac{(q-p)^2\sqrt{q(2p-q)}K_1((q-p)\rho)}{(p-s_1 q)(p-s_2 q)}\right\} \tag{15}$$

In region 4, k_z is still negative imaginary, but $H_1^{(1)}$ is now complex. Only the real part of $H_1^{(1)}$ will contribute:

$$\mathrm{DSDP}_{04} = \frac{\epsilon_1^2}{\pi^3(\epsilon_1 - 1)}\left\{\int_0^\infty dq \int_{q}^{s_2 q} dp\, \tilde{I}(-ip)e^{-p\tau} \right. \\ \left. \frac{(p-q)^2\sqrt{q(2p-q)}K_1((p-q)\rho)}{(p-s_1 q)(p-s_2 q)}\right\} \tag{16}$$

Region 5 is the same as region 4, except for a change in the limits of the p integral:

$$\mathrm{DSDP}_{05} = \frac{\epsilon_1^2}{\pi^3(\epsilon_1 - 1)}\left\{\int_0^\infty dq \int_{s_2 q}^{\infty} dp\, \tilde{I}(-ip)e^{-p\tau} \right. \\ \left. \frac{(p-q)^2\sqrt{q(2p-q)}K_1((p-q)\rho)}{(p-s_1 q)(p-s_2 q)}\right\} \tag{17}$$

Now, if we interchange the order of integration in DSDP_{02} and let $q = 2p - u$, we obtain

$$\mathrm{DSDP}_{02} = -\frac{\epsilon_1^2}{\pi^3(\epsilon_1 - 1)}\left\{\int_0^\infty dp \int_0^{p/s_2} du\, \tilde{I}(-ip)e^{-p\tau} \frac{(p-u)^2\sqrt{u(2p-u)}K_1((p-u)\rho)}{(p-s_1u)(p-s_2u)}\right\} \tag{18}$$

since $2 - 1/s_1 = 1/s_2$. Doing the same thing to DSDP_{03} we obtain

$$\mathrm{DSDP}_{03} = -\frac{\epsilon_1^2}{\pi^3(\epsilon_1 - 1)}\left\{\int_0^\infty dp \int_{p/s_2}^{p} du\, \tilde{I}(-ip)e^{-p\tau} \frac{(p-u)^2\sqrt{u(2p-u)}K_1((p-u)\rho)}{(p-s_1u)(p-s_2u)}\right\} \tag{19}$$

Interchanging the order of integration in DSDP_{04} and DSDP_{05}, without making a change of variable, gives us:

$$\mathrm{DSDP}_{04} = \frac{\epsilon_1^2}{\pi^3(\epsilon_1 - 1)}\left\{\int_0^\infty dp \int_{p/s_2}^{p} dq\, \tilde{I}(-ip)e^{-p\tau} \frac{(p-q)^2\sqrt{q(2p-q)}K_1((p-q)\rho)}{(p-s_1q)(p-s_2q)}\right\} \tag{20}$$

and

$$\mathrm{DSDP}_{05} = \frac{\epsilon_1^2}{\pi^3(\epsilon_1 - 1)}\left\{\int_0^\infty dp \int_0^{p/s_2} dq\, \tilde{I}(-ip)e^{-p\tau} \frac{(p-q)^2\sqrt{q(2p-q)}K_1((p-q)\rho)}{(p-s_1q)(p-s_2q)}\right\} \tag{21}$$

Thus, DSDP_{05} cancels DSDP_{02}, while DSDP_{04} cancels DSDP_{03}.

Now, all we are left with are the contributions from the detours around the two poles. Each of these we halfway enclose, going clockwise, so we obtain $-\pi i$ times the residue:

$$P_n = \frac{\epsilon_1^2}{2\pi(\epsilon_1 - 1)}\mathrm{Re}\left\{\int_0^\infty dq\, \mathrm{Res}\left[\tilde{I}(-ip)e^{-p\tau} \frac{(q-p)^2\sqrt{q(q-2p)}H_1^{(1)}(i(q-p)\rho)}{(p-s_1q)(p-s_2q)}\right]_{\text{pole } n}\right\} \tag{22}$$

The pole at $p = s_1 q$ contributes nothing, since, at that point, k_z is imaginary and $H_1^{(1)}$ is real. For the pole at $p = s_2 q$, only the imaginary part of $H_1^{(1)}$ contributes:

$$P_2 = -\frac{\epsilon_1^{5/2}(\sqrt{\epsilon_1+1}+\sqrt{\epsilon_1})^3}{2\pi(\epsilon_1-1)\sqrt{\epsilon_1+1}} \cdot \mathrm{Re}\left\{\int_0^\infty dq\, q^2 \tilde{I}(-is_2 q) e^{-s_2 q\tau} I_1((s_2-1)q\rho)\right\} \tag{23}$$

Now we consider the contribution from SDP$_1$ (5). Here, the dividing point is $\tau = \sqrt{\epsilon_1}\rho$; if $\tau < \sqrt{\epsilon_1}\rho$, we will deform upward, and obtain zero, just as we did for SDP$_0$. For $\tau > \sqrt{\epsilon_1}\rho$, we deform downward and let $k_0 = -ir/\sqrt{\epsilon_1}$ and $k_\rho = i(q-r)$. First, we consider the source pole:

$$\mathrm{SP}_1 = \frac{1}{2\pi(\epsilon_1-1)} \cdot \mathrm{Re}\left\{\int_0^\infty dq \frac{\partial}{\partial k_0}\left[(k_0-k_s)^2 \tilde{I}(k_0) e^{-ik_0\tau} \frac{k_\rho^2 k_{1z} H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 - (\epsilon_1+1)k_\rho^2}\right]_{k_0=k_s}\right\} \tag{24}$$

From the deformed integration, we obtain

$$\mathrm{DSDP}_1 = -\frac{1}{2\pi^2(\epsilon_1-1)}\mathrm{Re}\left\{\int_0^\infty dq \int_{C_r} dr\, \tilde{I}(-ir/\sqrt{\epsilon_1}) e^{-r\tau/\sqrt{\epsilon_1}} \frac{(q-r)^2\sqrt{q(q-2r)}H_1^{(1)}(i(q-r)\rho)}{(r-s_3 q)(r-s_4 q)}\right\} \tag{25}$$

where

$$s_3 = \frac{\sqrt{\epsilon_1+1}}{\sqrt{\epsilon_1+1}+1} \tag{26}$$

and

$$s_4 = \frac{\sqrt{\epsilon_1+1}}{\sqrt{\epsilon_1+1}-1} \tag{27}$$

and the contour C_r is just like C_p (Figure 3.3) except that the detours are around $r = s_3 q$ and $r = s_4 q$. For $0 < \epsilon_1 < \infty$, $1/2 < s_3 < 1$ and $1 < s_4 < \infty$ so that again there is never a double pole. In addition to the two poles, there are the same critical points as in SDP_0. Since $2 - 1/s_3 = 1/s_4$, we can apply the same reasoning as we did for SDP_0 to show that the total of all the on-axis integrations is zero. This leaves us with only the contributions from the detours around the poles:

$$P_n = -\frac{1}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ \int_0^\infty dq \, \mathrm{Res}\left[\tilde{I}(-ir/\sqrt{\epsilon_1}) e^{-r\tau/\sqrt{\epsilon_1}} \frac{(q-r)^2 \sqrt{q(q-2r)} H_1^{(1)}(i(q-r)\rho)}{(r - s_3 q)(r - s_4 q)} \right]_{\text{pole } n} \right\} \tag{28}$$

The pole at $r = s_3 q$ doesn't contribute, since the integrand is purely imaginary; the pole at $r = s_4 q$ gives:

$$P_4 = \frac{(1 + \sqrt{\epsilon_1 + 1})^3}{2\pi(\epsilon_1 - 1)\sqrt{\epsilon_1 + 1}\,\epsilon_1^{3/2}} \cdot \mathrm{Re}\left\{ \int_0^\infty dq \, q^2 \tilde{I}(-i s_4 q/\sqrt{\epsilon_1}) e^{-s_4 q\tau/\sqrt{\epsilon_1}} I_1((s_4 - 1)q\rho) \right\} \tag{29}$$

Therefore, the total field given by the double deformation technique is given by SP_0 (10), SP_1 (24), P_2 (23), and P_4 (29):

$$\begin{aligned} H_\phi &= 0 & \tau &< \rho \\ &= \mathrm{SP}_0 + P_2 & \rho < \tau &< \sqrt{\epsilon_1}\rho \\ &= \mathrm{SP}_0 + \mathrm{SP}_1 + P_2 + P_4 & \tau &> \sqrt{\epsilon_1}\rho \end{aligned} \tag{30}$$

For our specific current source, the total with $\epsilon_1 = 3.2$ is shown in Figure 3.4; this is a virtually perfect match with the closed form result. The various components of the solution are shown in Figures 3.5 and 3.6.

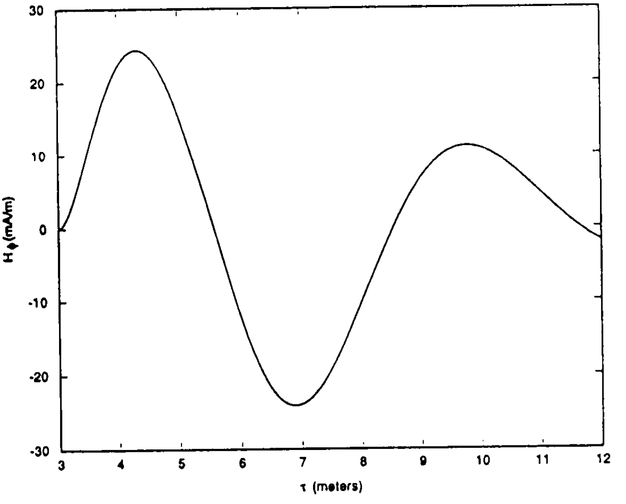

Figure 3.4 Total double deformation result and closed form solution; VED on lossless halfspace; source, Eq. (12).

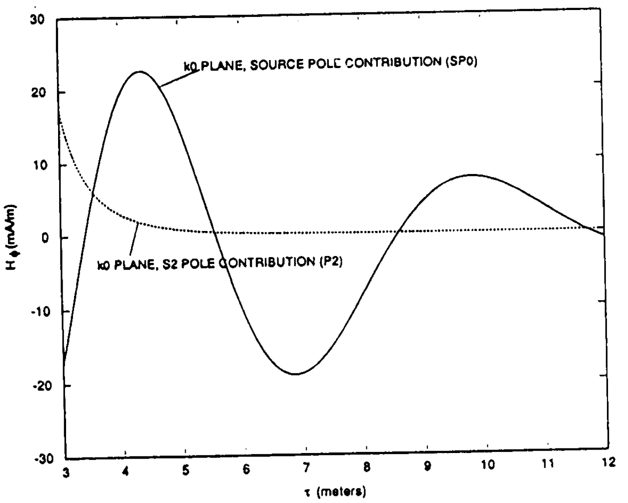

Figure 3.5 SDP_0 Contribution to double deformation solution; VED on lossless halfspace; source, Eq. (12).

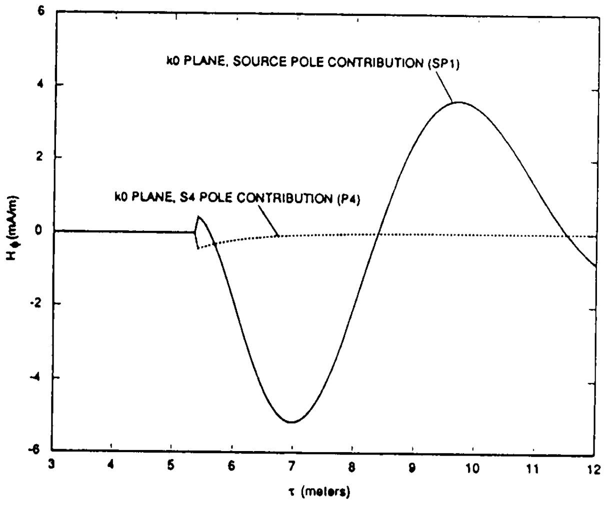

Figure 3.6 SDP$_1$ Contribution to double deformation solution; VED on lossless halfspace; source, Eq. (12).

c. Modified Double Deformation

We will need to split the Fourier transform of the current into two parts, one that allows us to deform in the upper half plane and one that allows us to deform in the lower half plane. Both parts also need to vanish sufficiently quickly in their respective half planes so that Jordan's Lemma will hold; in this case, we require better than $1/k_0$ asymptotic behavior as k_0 approaches infinity. As long as the time function of the current is sufficiently continuous, we can write:

$$\tilde{I}(k_0) = \int_0^\infty d\tau'\, I(\tau')e^{ik_0\tau'} = \frac{i}{k_0}\int_0^\infty d\tau'\, I'(\tau')e^{ik_0\tau'} \tag{31}$$

This assumes that the current is zero for $\tau \leq 0$. We now split the Fourier transform of the current into two parts. Due to the asymptotic nature of the Hankel function, the point of division will be $\tau - \rho$, in order for our integral expressions to vanish in the proper half planes. Thus,

$$\tilde{I}(k_0) = \frac{i}{k_0}\left[\int_0^{\tau-\rho} d\tau'\, I'(\tau')e^{ik_0\tau'} + \int_{\tau-\rho}^\infty d\tau'\, I'(\tau')e^{ik_0\tau'}\right]$$

$$\equiv \frac{i}{k_0}\left[\tilde{I}_{1b}(k_0, \tau - \rho) + \tilde{I}_{1a}(k_0, \tau - \rho)\right] \tag{32}$$

While the sum of these expressions is perfectly regular along the path of integration in the k_0 plane, individually they have single poles at the origin. In order to avoid this difficulty, we will change our integration contour to start a small distance δ above the origin, make a quarter clockwise circle down to the real axis, and then go along the real axis to infinity. There are no singularities encountered in this small deformation while the two halves of the current transform are together, and, when we split them apart, we avoid the singularity at the origin.

We therefore split the SDP expressions into two parts, one containing $\tilde{I}_a$, the other containing $\tilde{I}_b$.

$$\mathrm{SDP}_{0\,{}^a_b} = \frac{\epsilon_1^2}{2\pi^2(\epsilon_1 - 1)}\mathrm{Re}\left\{ i\int_{i\delta}^{\infty} dk_0\, \frac{\tilde{I}_{1\,{}^a_b}(k_0, \tau - \rho)}{k_0} e^{-ik_0\tau} \int_0^{\infty} dq\, \frac{k_\rho^2 k_z H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2} \right\} \tag{33}$$

and

$$\mathrm{SDP}_{1\,{}^a_b} = -\frac{\epsilon_1}{2\pi^2(\epsilon_1 - 1)}\mathrm{Re}\left\{ i\int_{i\delta}^{\infty} dk_0\, \frac{\tilde{I}_{1\,{}^a_b}(k_0, \tau - \sqrt{\epsilon_1}\rho)}{k_0} e^{-ik_0\tau} \int_0^{\infty} dq\, \frac{k_\rho^2 k_{1z} H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2} \right\} \tag{34}$$

Taking SDP_{0a} first, we interchange the order of integration, and deform the k_0 integral to the positive imaginary axis, letting $k_0 = ip$. Since both of the zeroes of the expression $\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2$ lie on the negative imaginary axis in the k_0 plane, we enclose no poles in this deformation. We are thus left with the double steepest descent path integral

$$\mathrm{DSDP}_{0a} = \frac{\epsilon_1^2}{2\pi^2(\epsilon_1 - 1)}\mathrm{Re}\left\{ i\int_0^{\infty} dq \int_{\delta}^{\infty} dp\, \frac{\tilde{I}_{1a}(ip, \tau - \rho)}{p} e^{p\tau} \frac{(p+q)^2\sqrt{q(q+2p)}H_1^{(1)}(i(p+q)\rho)}{\epsilon_1 p^2 - (\epsilon_1 + 1)(p+q)^2} \right\} \tag{35}$$

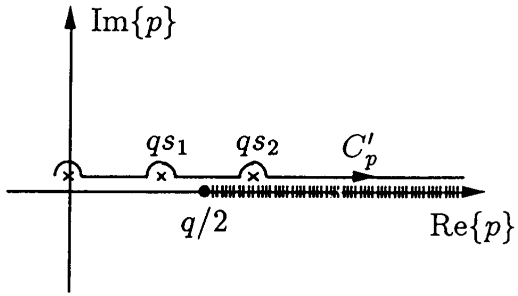

Figure 3.7 Integration contour C'_p for Eq. (36); VED on lossless halfspace.

which vanishes in the same way that (8) did for standard double deformation.

Turning now to SDP_{0b}, we wish to deform that k_0 integral down to its steepest descent path, on the negative imaginary axis. Since the only poles of the integral are on that axis, we do not enclose them, but detour around them. If we let $k_0 = -ip$, our integration path C'_p is shown in Figure 3.7. We therefore obtain

$$\mathrm{DSDP}_{0b} = -\frac{\epsilon_1^2}{2\pi^2(\epsilon_1 - 1)}\mathrm{Re}\left\{\int_0^\infty dq \int_{C_p} dp \frac{\tilde{I}_{1b}(-ip, \tau - \rho)}{p} e^{-p\tau} \frac{(q-p)^2\sqrt{q(q-2p)}H_1^{(1)}(i(q-p)\rho)}{(p - s_1 q)(p - s_2 q)}\right\} \tag{36}$$

where s_1 and s_2 are defined in (12) and (13), respectively. In addition to the three poles, there are the same two other critical points along the integration path. The total from the integrations along the axis, excluding the semicircles around the poles, is zero, just as before. We are left with are the contributions from the detours around the three poles. Each of these we halfway enclose, going clockwise, so we obtain $-\pi i$ times the residue:

$$P_n = -\frac{\epsilon_1^2}{2\pi(\epsilon_1 - 1)}\mathrm{Re}\left\{\int_0^\infty dq\, \mathrm{Res}\left[\frac{\tilde{I}_{1b}(-ip, \tau - \rho)}{p} e^{-p\tau} \frac{(q-p)^2\sqrt{q(q-2p)}H_1^{(1)}(i(q-p)\rho)}{(p - s_1 q)(p - s_2 q)}\right]_{\text{pole } n}\right\} \tag{37}$$

The pole at the origin gives

$$P_0 = -\frac{\epsilon_1^2}{2\pi(\epsilon_1 - 1)s_1 s_2} \mathrm{Re}\left\{ \int_0^\infty dq\, q \tilde{I}_{1b}(0, \tau - \rho) H_1^{(1)}(iq\rho) \right\} \quad (38)$$

$$= \frac{\epsilon_1^2}{\pi^2(\epsilon_1^2 - 1)} I(\tau - \rho) \int_0^\infty dq\, q K_1(q\rho) \quad (39)$$

since

$$\tilde{I}_{1b}(0, \tau - \rho) = \int_0^{\tau - \rho} d\tau' \, I'(\tau') = I(\tau - \rho) \quad (40)$$

Now, since

$$\int_0^\infty dx\, x^n K_m(ax) = 2^{n-1} a^{-n-1} \Gamma\left(\frac{1 + n + m}{2}\right) \Gamma\left(\frac{1 + n - m}{2}\right) \quad (41)$$

$$n \geq m$$

we obtain

$$\int_0^\infty dq\, q K_1(q\rho) = \frac{1}{\rho^2} \Gamma\left(\frac{3}{2}\right) \Gamma\left(\frac{1}{2}\right) = \frac{\pi}{2\rho^2} \quad (42)$$

which gives the contribution due to the pole at the origin as

$$P_0 = \frac{\epsilon_1^2}{2\pi(\epsilon_1^2 - 1)\rho^2} I(\tau - \rho) \quad (43)$$

The pole at $p = s_1 q$ contributes nothing, since, at that point, k_z is imaginary and $H_1^{(1)}$ is real. For the pole at $p = s_2 q$, only the imaginary part of $H_1^{(1)}$ contributes:

$$P_2 = \frac{\epsilon_1^2}{\pi(\epsilon_1 - 1)} \frac{(s_2 - 1)^2 \sqrt{2s_2 - 1}}{s_2(s_2 - s_1)} \cdot \int_0^\infty dq\, q \tilde{I}(-is_2 q, \tau - \rho) e^{-s_2 q \tau} I_1(q(s_2 - 1)\rho) \quad (44)$$

Remembering that

$$\tilde{I}(-is_2 q, \tau - \rho) = \int_0^{\tau - \rho} d\tau' \, e^{s_2 q \tau'} I'(\tau'), \quad (45)$$

we obtain

$$P_2 = \frac{\epsilon_1^{5/2}(\sqrt{\epsilon_1+1}+\sqrt{\epsilon_1})^2}{2\pi(\epsilon_1^2-1)} \int_0^{\tau-\rho} d\tau'\, I'(\tau') \int_0^{\infty} dq\, q e^{-s_2(\tau-\tau')q} I_1((s_2-1)q\rho) \tag{46}$$

We use the fact that

$$\int_0^{\infty} dx\, x^n e^{-\alpha x} I_m(\beta x) = \frac{1}{(\alpha^2-\beta^2)^{\frac{n+1}{2}}}\Gamma(n-m+1) P_n^m\left(\frac{\alpha}{\sqrt{\alpha^2-\beta^2}}\right) \tag{47}$$

where P_n^m is an associated Legendre function. For our case, where $m=1$ and $n=1$, we obtain

$$\int_0^{\infty} dx\, x e^{-\alpha x} I_1(\beta x) = \frac{\beta}{(\alpha^2-\beta^2)^{3/2}} \tag{48}$$

Applying this result to our expression for P_2, we obtain

$$P_2 = \frac{\epsilon_1^3\rho}{2\pi(\epsilon_1^2-1)} \int_0^{\tau-\rho} d\tau'\, I'(\tau') \frac{1}{[(\epsilon_1+1)(\tau-\tau')^2-\epsilon_1\rho^2]^{3/2}} \tag{49}$$

Using integration by parts to express everything in terms of $I(\tau)$, we can combine with P_0 to obtain the total contribution due to SDP$_0$:

$$\mathrm{SDP}_0 = \frac{1}{2\pi(\epsilon_1-1)\rho^2} \cdot \left\{ \epsilon_1^2 I(\tau-\rho) - 3\epsilon_1^3\rho^3 \int_\rho^\tau dx\, I(\tau-x) \frac{x}{[(\epsilon_1+1)x^2-\epsilon_1\rho^2]^{5/2}} \right\} \tag{50}$$

For $\tau < \rho$, this expression gives zero.

Now we consider the contribution from SDP$_1$ (34). The deformation of the term containing $\tilde{I}_{1a}$ will give zero, just as it did for SDP$_0$, so we are left with $\tilde{I}_{1b}$. Deforming downward, we let $k_0 = -ir/\sqrt{\epsilon_1}$ and $k_\rho = i(q-r)$. We obtain

$$\mathrm{DSDP}_{1b} = \frac{1}{2\pi^2(\epsilon_1-1)} \mathrm{Re}\left\{ \int_0^{\infty} dq \int_{C_r} dr\, \frac{\tilde{I}_{1b}(-ir/\sqrt{\epsilon_1}, \tau-\sqrt{\epsilon_1}\rho)}{r} \frac{e^{-r\tau/\sqrt{\epsilon_1}}(q-r)^2\sqrt{q(q-2r)}\,H_1^{(1)}(i(q-r)\rho)}{(r-s_3q)(r-s_4q)} \right\} \tag{51}$$

where s_3 and s_4 are given by (26) and (27), respectively. and the contour C_r' is just like C_p' (Figure 3.7) except that the detours are around $r = s_3 q$ and $r = s_4 q$. As before, only the contributions from the detours around the poles matter:

$$P_n = \frac{1}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ \int_0^\infty dq \, \mathrm{Res}\left[\frac{\tilde{I}_{1b}(-ir/\sqrt{\epsilon_1}, \tau - \sqrt{\epsilon_1}\rho)}{r} e^{-r\tau/\sqrt{\epsilon_1}} \frac{(q-r)^2\sqrt{q(q-2r)}H_1^{(1)}(i(q-r)\rho)}{(r - s_3 q)(r - s_4 q)} \right]_{\text{pole } n} \right\} \quad (52)$$

The pole at the origin gives

$$P_0 = -\frac{1}{\pi^2(\epsilon_1 - 1)s_3 s_4} \mathrm{Re}\left\{ \tilde{I}_{1b}(0, \tau - \sqrt{\epsilon_1}\rho) \int_0^\infty dq \, q K_1(q\rho) \right\} \quad (53)$$

$$= -\frac{\epsilon_1}{2\pi(\epsilon_1^2 - 1)\rho^2} I(\tau - \sqrt{\epsilon_1}\rho) \quad (54)$$

The pole at $r = s_3 q$ doesn't contribute, since the integrand is purely imaginary; the pole at $r = s_4 q$ gives:

$$P_4 = -\frac{1}{\pi(\epsilon_1 - 1)} \frac{(s_4 - 1)^2\sqrt{2s_4 - 1}}{s_4(s_4 - s_3)} \int_0^\infty dq \, q \tilde{I}_{1b}(-is_4 q/\sqrt{\epsilon_1}, \tau - \sqrt{\epsilon_1}\rho) e^{-s_4 q\tau/\sqrt{\epsilon_1}} I_1((s_4 - 1)q\rho) \quad (55)$$

Using (48) as above, we obtain:

$$P_4 = -\frac{\epsilon_1^3 \rho}{2\pi(\epsilon_1^2 - 1)} \int_0^{\tau - \sqrt{\epsilon_1}\rho} d\tau' \, I'(\tau') \frac{1}{[(\epsilon_1 + 1)(\tau - \tau')^2 - \epsilon_1 \rho^2]^{3/2}} \quad (56)$$

Using integration by parts, and combining with (54), we obtain

$$\mathrm{SDP}_1 = -\frac{1}{2\pi(\epsilon_1 - 1)\rho^2} \cdot \left\{ I(\tau - \sqrt{\epsilon_1}\rho) - 3\epsilon_1^3\rho^3 \int_{\sqrt{\epsilon_1}\rho}^{\tau} dx \, I(\tau - x) \frac{x}{[(\epsilon_1 + 1)x^2 - \epsilon_1\rho^2]^{5/2}} \right\} \quad (57)$$

This expression is zero for $\tau < \sqrt{\epsilon_1}\rho$.

Therefore, we obtain the complete modified double deformation result for a vertical electric dipole over a halfspace medium:

$$
\begin{aligned}
H_\phi &= 0 \qquad\qquad \tau < \rho \\
&= \frac{1}{2\pi(\epsilon_1 - 1)\rho^2}\Bigg\{\epsilon_1^2 I(\tau-\rho) \\
&\qquad - 3\epsilon_1^3\rho^3 \int_\rho^\tau dx\, I(\tau - x)\frac{x}{[(\epsilon_1+1)x^2 - \epsilon_1\rho^2]^{5/2}}\Bigg\} \\
&\qquad\qquad \rho < \tau < \sqrt{\epsilon_1}\rho \\
&= \frac{1}{2\pi(\epsilon_1 - 1)\rho^2}\Bigg\{\epsilon_1^2 I(\tau-\rho) - I(\tau - \sqrt{\epsilon_1}\rho) \\
&\qquad - 3\epsilon_1^3\rho^3 \int_\rho^{\sqrt{\epsilon_1}\rho} dx\, I(\tau - x)\frac{x}{[(\epsilon_1+1)x^2 - \epsilon_1\rho^2]^{5/2}}\Bigg\} \\
&\qquad\qquad \tau > \sqrt{\epsilon_1}\rho
\end{aligned}
\tag{58}
$$

Again, we see the power of the modified double deformation technique, since it provides for us the closed form solution for the VED over a halfspace medium.

4. Modified Double Deformation for Vertical Magnetic Dipole over Coated Perfect Conductor

a. Single Deformation

The electric field on the surface of a coated perfect conductor from a vertical magnetic dipole also on the surface a distance ρ away is

$$
E_\phi(\tau) = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{\int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\int_{\mathrm{SIP}} dk_\rho\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho)[1 + R^{TE}]\right\}
\tag{1}
$$

where

$$
R^{TE} = \frac{R_{01} - e^{2ik_{1z}d}}{1 - R_{01}e^{2ik_{1z}d}}, \qquad R_{01} = \frac{k_z - k_{1z}}{k_z + k_{1z}}
\tag{2}
$$

and

$$k_z = \sqrt{k_0^2 - k_\rho^2}, \quad k_{1z} = \sqrt{\epsilon_1 k_0^2 - k_\rho^2} \tag{3}$$

The dielectric coating is of thickness d and dielectric constant ϵ_1.

The principal difference between the coated perfect conductor case and the halfspace case studied earlier is the existence of pole singularities in both the k_ρ and k_0 planes due to resonances of the structure. We first deform the k_ρ integral in (1) to its steepest descent path (SDP). This path goes from $k_\rho = k_0 + i\infty$ on the top sheet, down to $k_\rho = k_0$, and back up on the bottom sheet, as shown in Figure 4.1; there is only one steepest descent path, since there is only one unbounded region. If we call height on the SDP q, then the steepest descent path contribution becomes

$$\text{SDP} = -\frac{\eta_0}{8\pi^2}\text{Re}\left\{ i \int_0^\infty dk_0\, k_0 e^{-ik_0\tau} \tilde{I}(k_0) \right.$$

$$\cdot \int_0^\infty dq \left[\frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)[1 + R^{TE}] \right]_{\text{top}}$$

$$\left. - \left[\frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)[1 + R^{TE}] \right]_{\text{bottom}} \right\} \tag{4}$$

which can be written as

$$\text{SDP} = -\frac{\eta_0}{8\pi^2}\text{Re}\left\{ i \int_0^\infty dk_0\, k_0 e^{-ik_0\tau} \tilde{I}(k_0) \right.$$

$$\left. \int_0^\infty dq\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1 + R_{01})^2 (1 - e^{2ik_{1z}d})^2}{(1 - R_{01} e^{2ik_{1z}d})(R_{01} - e^{2ik_{1z}d})} \right\} \tag{5}$$

where $k_\rho = k_0 + iq$ and k_z is evaluated on the top Riemann sheet. We must be careful, however, to consider any poles that we may have enclosed in deforming the k_ρ integration path. These poles are zeros of the denominator of (1),

$$1 - R_{01} e^{2ik_{1z}d} = 0 \tag{6}$$

This transcendental equation has an infinite number of solutions in the complex k_ρ plane. In order to find and examine them, we resort to a modal approach. We let

$$R_{01} = e^{i\phi} \tag{7}$$

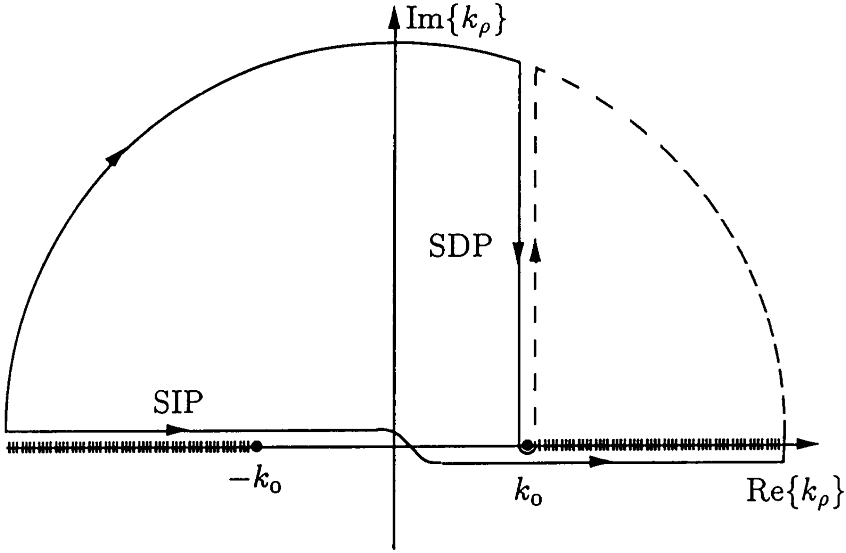

Figure 4.1 Deformation in k_ρ plane for Sommerfeld integration path; VMD on coated perfect conductor.

where ϕ is a complex number whose real part lies between 0 and 2π. We substitute this into (6) and, by taking the logarithm of both sides, arrive at the modal equation

$$\phi + 2k_{1z}d = 2(m+1)\pi, \qquad m = 0, 1, 2, \ldots \tag{8}$$

where the different m's distinguish the different modes.

Some typical loci of poles ($m = 0, 1, 2$) are shown in Figure 4.2. The solid lines refer to poles on the upper Riemann sheet; the dashed curves to poles on the lower Riemann sheet. Since (1) is even in k_ρ, the locus is symmetric; we will concentrate on the half with $\mathrm{Re}\{k_\rho\} > 0$. For k_0 very large, we have two poles, both near $k_\rho = \sqrt{\epsilon_1}\, k_0$ — one on the top Riemann sheet, and one on the bottom. As k_0 decreases, both poles move towards the point $k_\rho = k_0$, remaining on the real axis. The pole on the bottom sheet reaches that point at the "cutoff" frequency:

$$k_{cm} = \frac{(m + 1/2)\pi}{\sqrt{\epsilon_1 - 1}\, d} \tag{9}$$

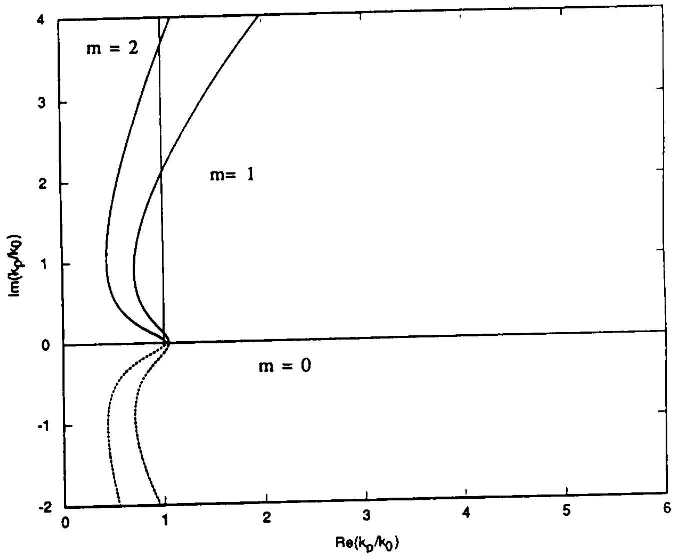

Figure 4.2 k_ρ **pole loci,** $m = 0, 1, 2$**; VMD on coated perfect conductor; vertical line at** $\mathrm{Re}(k_\rho/k_0) = 1$ **is steepest descent path.**

at which point it switches to the top sheet and approaches the other pole. They meet, and one moves away from the axis to the first quadrant on the top sheet, while the other moves to the fourth quadrant on the bottom sheet. As k_0 approaches zero, these loci are symmetric about the real k_ρ axis. The one in the first quadrant intersects the SDP at $k_0 = k_{bm}$ and $k_0 = k_{am}$ before moving out towards infinity. The locus for mode 0 is slightly different. One pole remains at $k_\rho = \sqrt{\epsilon_1}\, k_0$ on the top sheet for all values of k_0, while the other starts at $k_\rho = \sqrt{\epsilon_1}\, k_0$ on the bottom sheet for large k_0, moves toward $k_\rho = k_0$, crosses there to the top sheet, and moves back toward $k_\rho = \sqrt{\epsilon_1}\, k_0$ as k_0 approaches zero.

Thus, for each mode, only two parts of each locus are enclosed when the integration path is deformed. For a range of frequency between k_{am} and k_{bm}, poles on the upper Riemann sheet are enclosed. These poles, with $\mathrm{Re}\{k_\rho\} > 0$, $\mathrm{Im}\{k_\rho\} > 0$, $\mathrm{Re}\{k_z\} > 0$, and $\mathrm{Im}\{k_z\} < 0$, correspond to so-called "leaky" waves. For frequencies greater than k_{cm}, poles along the real k_ρ axis on the lower Riemann sheet are excited; these poles, with k_ρ and k_{1z} purely real and positive, and k_z purely imaginary and positive, correspond to "guided" waves. Mode 0 has only guided waves.

The contribution to the total field from these k_ρ poles is

$$\frac{\eta_0}{4\pi}\text{Re}\left\{ i \int dk_0\, k_0 e^{-ik_0\tau} \tilde{I}(k_0)\text{Res}\left[\frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)[1+R^{TE}]\right]\right\} \tag{10}$$

which becomes

$$\sum_{m=0}^{\infty} -\frac{\eta_0}{2\pi(\epsilon_1-1)}\text{Re}\left\{ i \int_{k_{cm}}^{\infty} dk_0\, \frac{\tilde{I}(k_0)}{k_0} e^{-ik_0\tau}\left[\frac{k_\rho k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{1/k_z - id}\right]\right\} \tag{11}$$

for the guided poles, where k_z is evaluated on the bottom Riemann sheet, and

$$\sum_{m=1}^{\infty} -\frac{\eta_0}{2\pi(\epsilon_1-1)}\text{Re}\left\{ i \int_{k_{am}}^{k_{bm}} dk_0\, \frac{\tilde{I}(k_0)}{k_0} e^{-ik_0\tau}\left[\frac{k_\rho k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{1/k_z - id}\right]\right\} \tag{12}$$

for the leaky poles, where k_z is evaluated on the top Riemann sheet. The total field is thus given by the sum of (5), (11), and (12). If we stop the analytical process here, and evaluate these three expressions numerically, the technique is called single deformation. While we have eliminated one of our highly oscillatory integrals, we still have the integral over k_0 in its original form.

b. Double Deformation

The double deformation method takes the single deformation result and deforms the k_0 integral in the SDP term, (5). To evaluate this, we want to interchange the order of the q and k_0 integrations, and deform the latter to its steepest descent path. The issues which decide whether the deformation in the k_0 plane will be up or down are exactly the same as in the halfspace case; for a source, such as

$$I(\tau) = I_0 \tau^2 \sin\omega_0\tau e^{-\alpha_0\tau}, \tag{13}$$

which has the Fourier transform

$$\tilde{I}(k_0) = I_n\left[\frac{1}{(k_0+i\alpha_0-\omega_0)^3} - \frac{1}{(k_0+i\alpha_0+\omega_0)^3}\right], \tag{14}$$

we deform upward to the positive imaginary k_0 axis for $\tau < \rho$ and downward to the negative imaginary k_0 axis for $\tau > \rho$.

It can be easily shown (Appendix A) that the total contribution for $\tau < \rho$ is identically zero, which satisfies causality. Therefore, we will concentrate on the contribution for $\tau > \rho$. For this case, we will deform the k_0 integral in the single SDP term (5) downward, to $k_0 = -ip$. In so doing, we obtain two more contributions: the double steepest descent path integral and the residues of the enclosed poles in the k_0 plane.

The double steepest descent path integral is not difficult. After manipulation, we obtain

$$\mathrm{DSDP} = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dp\, pe^{-p\tau}\tilde{I}(-ip) \int_0^\infty dq\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)\frac{(1+R_{01})^2(1-e^{2ik_{1z}d})^2}{(1-R_{01}e^{2ik_{1z}d})(R_{01}-e^{2ik_{1z}d})}\right\} \tag{15}$$

where $k_\rho = i(q-p)$, $k_z = \sqrt{q^2-2pq}$, and $k_{1z} = \sqrt{q^2-2pq-(\epsilon_1-1)p^2}$. First we notice that $\tilde{I}(-ip)$ is purely real. There will be three critical points on the q integration path, $q = p$, $q = 2p$, and $q = (1+\sqrt{\epsilon_1})p$, and thus four separate regions to consider.

Region 1: $0 \le q \le p$. Here, $k_\rho = -i(p-q)$, so

$$H_1^{(1)}(-i(p-q)\rho) = \frac{2}{\pi}K_1((p-q)\rho) - 2iI_1((p-q)\rho). \tag{16}$$

Also, both $k_z = -i\sqrt{2pq-q^2}$ and $k_{1z} = -i\sqrt{(\epsilon_1-1)p^2+2pq-q^2}$ are negative imaginary. Thus, both R_{01} and $e^{2ik_{1z}d}$ are real. This leaves us with

$$\mathrm{DSDP}_1 = \frac{\eta_0}{4\pi^3}\left\{\int_0^\infty dp\, pe^{-p\tau}\tilde{I}(-ip) \int_0^p dq\, \frac{(p-q)^2}{\sqrt{2pq-q^2}}K_1((p-q)\rho)\frac{(1+R_{01})^2(1-e^{2ik_{1z}d})^2}{(1-R_{01}e^{2ik_{1z}d})(R_{01}-e^{2ik_{1z}d})}\right\} \tag{17}$$

Region 2: $p \le q \le 2p$. Here, $k_\rho = i(q-p)$, so

$$H_1^{(1)}(i(q-p)\rho) = -\frac{2}{\pi}K_1((q-p)\rho). \tag{18}$$

However, both k_z and k_{1z} remain negative imaginary. This gives

$$\mathrm{DSDP}_2 = -\frac{\eta_0}{4\pi^3}\left\{\int_0^\infty dp\, pe^{-p\tau}\tilde{I}(-ip)\right.$$
$$\left.\int_p^{2p} dq\,\frac{(q-p)^2}{\sqrt{2pq-q^2}}K_1((q-p)\rho)\frac{(1+R_{01})^2(1-e^{2ik_{1z}d})^2}{(1-R_{01}e^{2ik_{1z}d})(R_{01}-e^{2ik_{1z}d})}\right\} \tag{19}$$

If we make the substitution $q = 2p - u$, we can write (19) as

$$\mathrm{DSDP}_2 = -\frac{\eta_0}{4\pi^3}\left\{\int_0^\infty dp\, pe^{-p\tau}\tilde{I}(-ip)\right.$$
$$\left.\int_0^{p} du\,\frac{(p-u)^2}{\sqrt{2pu-u^2}}K_1((p-u)\rho)\frac{(1+R_{01})^2(1-e^{2ik_{1z}d})^2}{(1-R_{01}e^{2ik_{1z}d})(R_{01}-e^{2ik_{1z}d})}\right\} \tag{20}$$

where, in terms of u, $k_z = -i\sqrt{2pu-u^2}$ and $k_{1z} = -i\sqrt{(\epsilon_1-1)p^2+2pu-u^2}$. It is clear, by comparison of (20) with (17), that the contributions from regions 1 and 2 exactly cancel each other.

Region 3: $2p \le q \le (1+\sqrt{\epsilon_1})p$. Here, k_ρ is positive imaginary, as before, so $H_1^{(1)}$ is purely real. The change is that k_z is real, while k_{1z} remains complex. This means that R_{01} is a complex number with a magnitude of unity, or that we can write $R_{01} = e^{2i\psi}$, where ψ is real. Let $e^{2ik_{1z}d} \equiv X$, where X is also real, and we obtain:

$$\frac{(1+R_{01})^2(1-e^{2ik_{1z}d})^2}{(1-R_{01}e^{2ik_{1z}d})(R_{01}-e^{2ik_{1z}d})} = \frac{(1+e^{2i\psi})^2(1-X)^2}{(1-e^{2i\psi}X)(e^{2i\psi}-X)} =$$
$$\frac{(e^{i\psi}-e^{-i\psi})^2(1-X)^2}{(e^{-i\psi}-e^{i\psi}X)(e^{i\psi}-e^{-i\psi}X)} \tag{21}$$

By inspection, we note that this expression is equal to its complex conjugate and is therefore real. Since k_z is real, we can see by examining (15) that the contribution from this range is zero.

Region 4: Here, $q \ge (1+\sqrt{\epsilon_1})p$. Now, both k_z and k_{1z} are real. This means that R_{01} is real, but that $|e^{2ik_{1z}d}| = 1$. We write

$$\frac{(1+R_{01})^2(1-e^{2ik_{1z}d})^2}{(1-R_{01}e^{2ik_{1z}d})(R_{01}-e^{2ik_{1z}d})} =$$

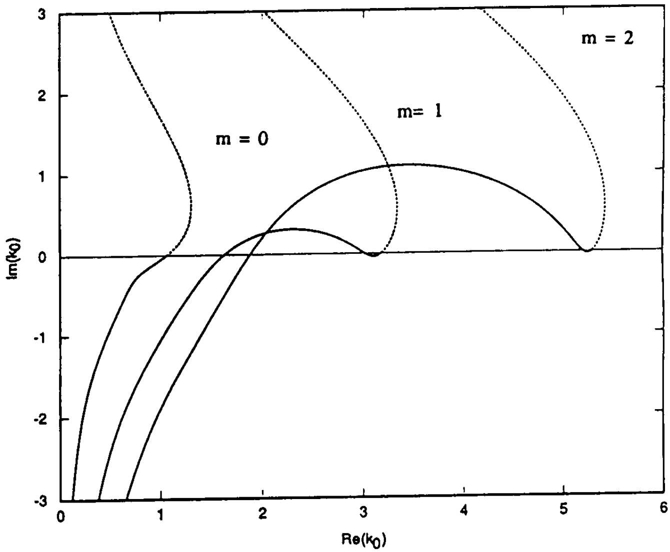

Figure 4.3 k_0 **pole loci,** $m = 0, 1, 2$**; VMD on coated perfect conductor.**

$$-\frac{(1+R_{01})^2(e^{ik_{1z}d}-e^{ik_{1z}d})^2}{(R_{01}e^{ik_{1z}d}-e^{-ik_{1z}d})(R_{01}e^{-ik_{1z}d}-e^{ik_{1z}d})} \tag{22}$$

Again, we can see that this expression equals its complex conjugate and is therefore real, and therefore the q integration for this range is zero. Thus, since the contribution from region 1 cancels that from region 2, and both region 3 and region 4 give zero, the entire double steepest descent path integral (15) is zero.

When we deform the k_0 integral to its SDP, we enclose poles of the integrand. There are two kinds of poles; the source pole, due to the singularities of $\tilde{I}(k_0)$, and the poles that arise from zeroes of the denominator (6). In $\tilde{I}(k_0)$, only the triple pole at $k_0 = \omega_0 - i\alpha_0$ is enclosed, so we obtain:

$$-\frac{\eta_0 I_0}{8\pi}\mathrm{Re}\left\{\int_0^\infty dq\,\frac{\partial^2}{\partial k_0^2}\left[k_0 e^{-ik_0\tau}\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\right.\right.$$

$$\left.\left.\frac{(1+R_{01})^2(1-e^{2ik_{1z}d})^2}{(1-R_{01}e^{2ik_{1z}d})(R_{01}-e^{2ik_{1z}d})}\right]_{k_0=\omega_0-i\alpha_0}\right\} \tag{23}$$

Turning now to the roots of (6), we look for solutions with complex k_0 and $k_\rho = k_0 + iq$ on both the top and bottom Riemann sheets, instead of solutions with real k_0 and complex k_ρ as we did in the k_ρ plane. We can use the same modal technique as previously, and discover loci such as those shown in Figure 4.3. The solid lines refer to poles on the upper Riemann sheet; the dashed curves to poles on the lower Riemann sheet. For all modes except $m = 0$, two sections of the locus are enclosed by the deformation of the integration path to the negative imaginary axis: the part from $k_0 = k_{cm}$ towards $k_0 = k_{bm}$, which we will call "k_0 finite", and from $k_0 = k_{am}$ to $k_0 \to -i\infty$, which we will call "k_0 infinite". For $m = 0$, since the locus in the k_ρ plane never crosses the SDP, the locus in the k_0 plane never crosses the k_0-real axis, so there is only a k_0 infinite locus. The integrals for these poles are

$$-\frac{\eta_0}{4\pi}\mathrm{Re}\left\{\int dq\,\mathrm{Res}\left[k_0 e^{-ik_0\tau}\tilde{I}(k_0)\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)[1+R^{TE}]\right]\right\} \tag{24}$$

which, after calculating the residues, gives the following expressions for the contributions:

$$\sum_{m=0}^{\infty}\frac{\eta_0}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{\int_{q_{am}}^{\infty}dq\,e^{-ik_0\tau}\tilde{I}(k_0)H_1^{(1)}(k_\rho\rho)\left[\frac{k_\rho^2 k_{1z}^2(q-2ik_0)}{k_\rho\left[k_0^2(\epsilon_1-1)d-ik_z\right]+k_0 d k_{1z}^2}\right]\right\} \tag{25}$$

for the k_0 infinite poles, and

$$\sum_{m=1}^{\infty}\frac{\eta_0}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{\int_{0}^{q_{bm}}dq\,e^{-ik_0\tau}\tilde{I}(k_0)H_1^{(1)}(k_\rho\rho)\left[\frac{k_\rho^2 k_{1z}^2(q-2ik_0)}{k_\rho\left[k_0^2(\epsilon_1-1)d-ik_z\right]+k_0 d k_{1z}^2}\right]\right\} \tag{26}$$

for the k_0 finite poles, where q_{am} and q_{bm} are the value of q where the locus in the k_ρ plane crosses the SDP; for $m = 0$, $q_{a0} = 0$.

The complete double deformation solution is thus made up of five parts: the k_ρ guided modes (11), the k_ρ leaky modes (12), the k_0

infinite modes (25), the k_0 finite modes (26), and the source pole contribution (23).

c. *Modified Double Deformation*

We can apply the modified double deformation technique to this problem, just as we did for the halfspace case. As in that case, to satisfy Jordan's Lemma in this case, we will have to do two integrations by parts:

$$\tilde{I}(k_0) = \int_0^\infty d\tau'\, I(\tau')e^{ik_0\tau'} = -\frac{1}{k_0^2}\int_0^\infty d\tau'\, I''(\tau')e^{ik_0\tau'} \qquad (27)$$

This assumes that the current is zero for $\tau < 0$, and that the current and its first derivative are zero at $\tau = 0$. We now split the Fourier transform of the current into two parts:

$$\tilde{I}(k_0) = -\frac{1}{k_0^2}\left[\int_0^{\tau-\rho} d\tau'\, I''(\tau')e^{ik_0\tau'} + \int_{\tau-\rho}^\infty d\tau'\, I''(\tau')e^{ik_0\tau'}\right] \qquad (28)$$

$$\equiv -\frac{1}{k_0^2}\left[\tilde{I}_{2b}(k_0,\tau-\rho) + \tilde{I}_{2a}(k_0,\tau-\rho)\right] \qquad (29)$$

We substitute this into our expression for SDP (5) and deform the k_0 integral of the half containing $\tilde{I}_{2a}$ upward, the half containing $\tilde{I}_{2b}$ downward. However, since we have introduced a pole at the origin, we must deform our integration path in the k_0 up to $i\delta$ before separating, as we did in the halfspace case. If we also split the integrals over the poles in the k_ρ plane into a part containing $\tilde{I}_{2a}$ and one containing $\tilde{I}_{2b}$, we can show (Appendix A) that the total for $\tilde{I}_{2a}$ is identically zero, which satisfies causality. Therefore, we will concentrate on the part containing $\tilde{I}_{2b}$:

$$\text{SDP} = \frac{\eta_0}{8\pi^2}\text{Re}\left\{ i\int_{i\delta}^\infty dk_0\, e^{-ik_0\tau}\frac{\tilde{I}_{2b}(k_0,\tau-\rho)}{k_0^2} \int_0^\infty dq\,\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1-e^{2ik_{1z}d})^2}{(1-R_{01}e^{2ik_{1z}d})(R_{01}-e^{2ik_{1z}d})}\right\} \qquad (30)$$

We will deform the k_0 integral downward, to $k_0 = -ip$. In so doing, we obtain three contributions: the double steepest descent path contribution, the residue from the pole at the origin, and the residues of the enclosed poles in the k_0 plane.

The double steepest descent path contribution is similar to that for standard double deformation, except that $-(1/k_0^2)\tilde{I}_{2b}(k_0, \tau - \rho)$ replaces $\tilde{I}(k_0)$. Examination of equations (15) through (22) shows that this change will not alter the conclusion that the total contribution is zero.

There is, however, a contribution from the pole at the origin. Since we half enclose this pole clockwise, we obtain:

$$Z = \frac{\eta_0}{8\pi}\mathrm{Re}\left\{ \int_0^\infty dq \left[\tilde{I}_{2b}(k_0, \tau - \rho)e^{-ik_0\tau} \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1 + R_{01})^2(1 - e^{2ik_{1z}d})^2}{(1 - R_{01}e^{2ik_{1z}d})(R_{01} - e^{2ik_{1z}d})} \right]_{k_0=0} \right\} \tag{31}$$

Using the fact that

$$\tilde{I}_{2b}(0, \tau - \rho) = I'(\tau - \rho) \tag{32}$$

we obtain

$$Z = \frac{\eta_0}{\pi^2} I'(\tau - \rho) \int_0^\infty dq\, q \sin^2(qd) K_1(q\rho) \tag{33}$$

which becomes

$$Z = \frac{\eta_0}{4\pi} I'(\tau - \rho) \left[\frac{1}{\rho^2} - \frac{\rho}{[\rho^2 + (2d)^2]^{3/2}} \right] \tag{34}$$

The contributions from the poles of (6) in the k_0 plane are very similar to the standard double deformation results, except that there is no source pole. We obtain

$$\sum_{m=0}^{\infty} -\frac{\eta_0}{2\pi(\epsilon_1 - 1)}\mathrm{Re}\left\{ \int_{q_{am}}^\infty dq\, e^{-ik_0\tau} \frac{\tilde{I}_{2b}(k_0, \tau - \rho)}{k_0^2} H_1^{(1)}(k_\rho \rho) \left[\frac{k_\rho^2 k_{1z}^2 (q - 2ik_0)}{k_\rho \left[k_0^2(\epsilon_1 - 1)d - ik_z\right] + k_0 d k_{1z}^2} \right] \right\} \tag{35}$$

for the k_0 infinite poles, and

$$\sum_{m=1}^{\infty} -\frac{\eta_0}{2\pi(\epsilon_1 - 1)}\mathrm{Re}\left\{ \int_0^{q_{bm}} dq\, e^{-ik_0\tau} \frac{\tilde{I}_{2b}(k_0, \tau - \rho)}{k_0^2} H_1^{(1)}(k_\rho \rho) \left[\frac{k_\rho^2 k_{1z}^2 (q - 2ik_0)}{k_\rho \left[k_0^2(\epsilon_1 - 1)d - ik_z\right] + k_0 d k_{1z}^2} \right] \right\} \tag{36}$$

for the k_0 finite poles.

We also have the contributions for the k_ρ poles, which are now

$$\sum_{m=0}^{\infty} \frac{\eta_0}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ i \int_{k_{cm}}^{\infty} dk_0 \, \frac{\tilde{I}_{2b}(k_0, \tau - \rho)}{k_0^3} e^{-ik_0\tau} \left[\frac{k_\rho k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{1/k_z - id} \right] \right\} \tag{37}$$

for the guided poles and

$$\sum_{m=1}^{\infty} \frac{\eta_0}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ i \int_{k_{am}}^{k_{bm}} dk_0 \, \frac{\tilde{I}_{2b}(k_0, \tau - \rho)}{k_0^3} e^{-ik_0\tau} \left[\frac{k_\rho k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{1/k_z - id} \right] \right\} \tag{38}$$

for the leaky poles. The total modified double deformation solution is therefore given by the sum of the k_ρ guided modes (37), the k_ρ leaky modes (38), the k_0 infinite modes (35) the k_0 finite modes (36), and the origin pole contribution (34).

d. Results

Results are shown for the configuration with $\rho = 3$, $d = 1$, and $\epsilon_1 = 3.2$. Two sources are used; the first is the damped sinusoid (13) with $I_0 = 1$, $\omega_0 = 1$ and $\alpha_0 = 0.5$. The second is the smoothed pulse, discussed in Section 1. In order to analyze the response we calculate, it is helpful to look at the situation from the viewpoint of rays. We can look at the waves sent out from the dipole source as being rays coming out at all angles from the dipole. We can trace the path of each ray through the model to see if it arrives at the receiver, and if so, when. These arrival times will tell us when to expect changes in the total response due to new waves contributing to the field.

The simplest rays which arrive at the observation point are those which travel directly from the source, one in free space and one in the dielectric. The travel times for these are:

$$\tau = \rho \qquad \text{in free space} \tag{39}$$

$$\tau = \sqrt{\epsilon_1}\,\rho \qquad \text{in dielectric} \tag{40}$$

These waves are the only ones which exist in the halfspace case.

The next simplest ray is that which reflects off the bottom boundary at $\rho/2$ to reach the observation point from the dielectric. This "reflected wave" is the simplest member of an infinite family, all of which reflect off the ground plane n times and the air-dielectric interface $n-1$ times. The travel times for these rays are:

$$\tau = \sqrt{\epsilon_1}\sqrt{\rho^2 + (2nd)^2} \qquad n = 1, 2, 3, \ldots \tag{41}$$

The remaining rays are the so-called "lateral waves". These start out from the line source propagating downward at the critical angle (the angle at which total internal reflection would occur between the dielectric and free space). They reflect off the ground plane, and the first member of the family is transmitted into the freespace medium with a transmission angle of 90°. The wave then travels straight to the observation point. The other members of this family reflect off the ground plane n times and the air-dielectric interface $n-1$ times. The number of members in this family is not infinite, because, since the point of emergence from the dielectric must be to the left of the observation point, there is a limit on the number of reflections possible.

The members of this family are those for which

$$n < \frac{\rho\sqrt{\epsilon_1 - 1}}{2d}. \tag{42}$$

The transit time for these waves is:

$$\tau = \rho + 2nd\sqrt{\epsilon_1 - 1}, \qquad n = 1, 2, 3, \ldots n_{max} \tag{43}$$

In the following table we list all the ray arrivals for this case before $\tau = 12$:

Wave	Arrival Time
Direct (air)	3.000
Direct (dielectric)	5.367
First Lateral	5.966
First Reflected	6.450
Second Lateral	8.932
Second Reflected	8.944
Third Lateral	11.900

It is clear from this table that, before $\tau = 5.966$, the response should be identical to what it would be if the ground plane didn't exist. Comparison with the closed-form halfspace solution thus provides a good check on the results.

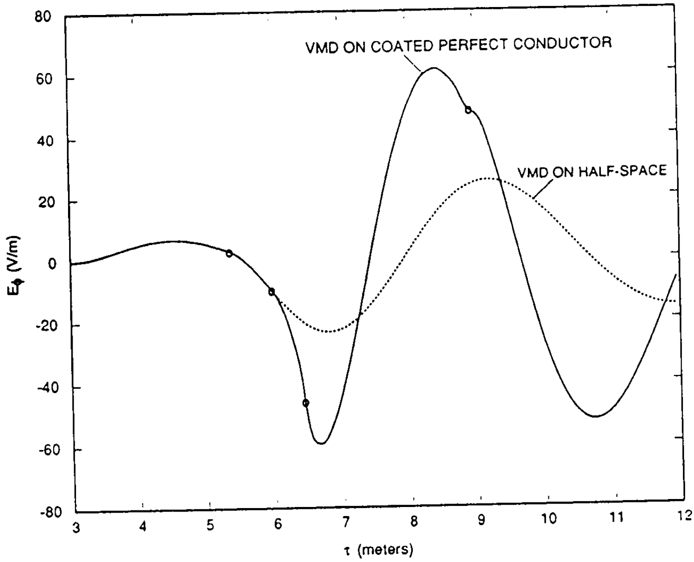

Figure 4.4 Total double deformation and halfspace solution; VMD on coated perfect conductor, source, Eq. (14); circles show various ray arrivals.

We present three sets of results. First, we use standard double deformation with the damped sinusoidal source. In Figure 4.4, we compare the total using eleven modes with the closed-form, halfspace result from Section 2. As can be seen, the fit is extremely good. One can also see the effects of the subsequent ray arrivals on the total waveform. In Figure 4.5, we show some of the components of the total; the source pole contribution and the first two modes.

Next, we do the same problem, but using modified double deformation. The total result would overlay the double deformation result in Figure 4.4. In Figure 4.6, we show the origin pole contribution and the first two modes. Comparing this with the standard double deformation result, we notice some interesting things. First, the agreement between the two techniques is very good. Also, in contrast with the standard double deformation method, all components in the modified results start smoothly from $\tau = \rho$, with no discontinuities of value or slope. This comes about from the stronger statement of causality possible with the modification (see Appendix A). Also, since the response at time τ only depends on the current for all times before $\tau - \rho$, in some sense the modes for modified double deformation are more

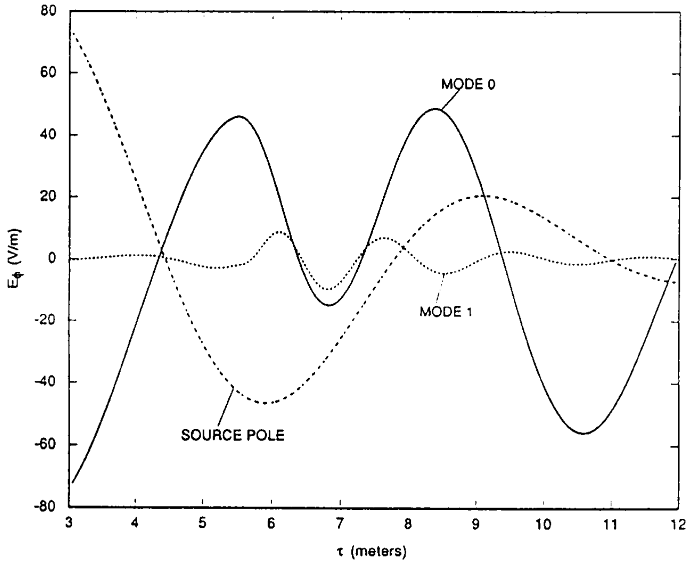

Figure 4.5 Modes 0 and 1, and source pole contribution; VMD on coated perfect conductor, source, Eq. (14).

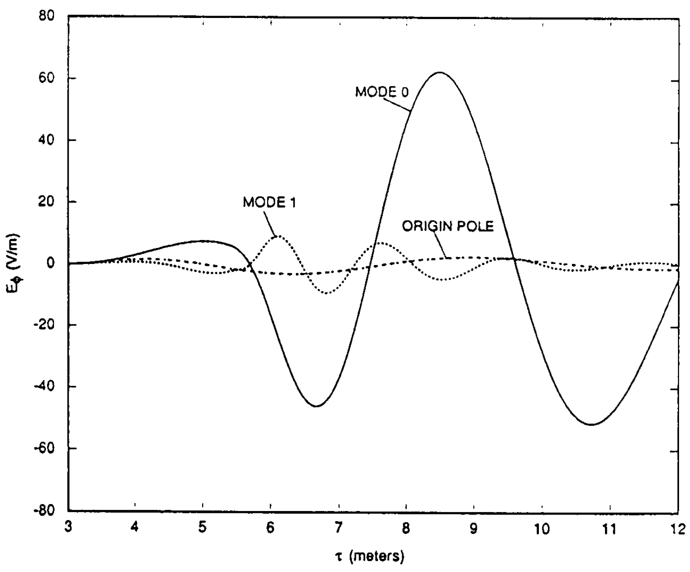

Figure 4.6 MDD: Modes 0 and 1, and origin pole contribution; VMD on coated perfect conductor, source, Eq. (14).

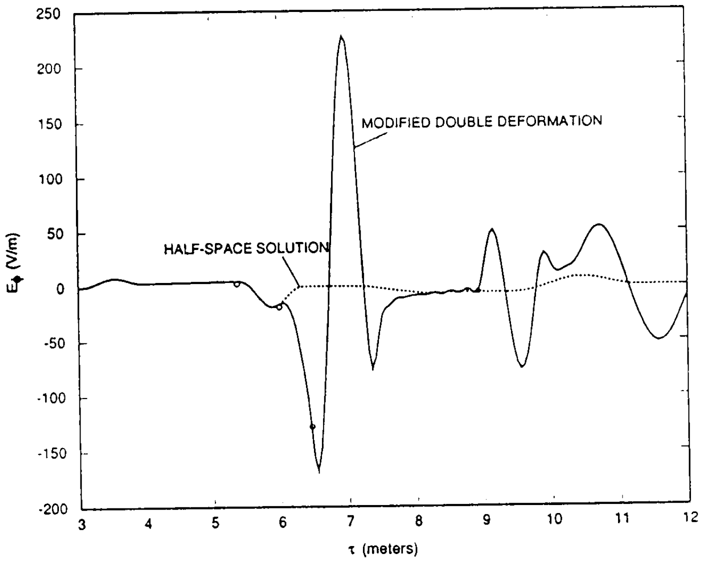

Figure 4.7 Modified double deformation result and halfspace solution; VMD on coated perfect conductor, source, Eq. (16); circles show various ray arrivals.

physical than those for the standard technique.

Finally, we use modified double deformation with the pulse source, to demonstrate the versatility of the new technique. In Figure 4.7, we compare the total with eleven modes to the halfspace result; in Figure 4.8, we show the origin pole contribution and the first two modes. While the character of the response is markedly different from the damped sinusoid, and the higher order modes relatively more prominent, we see that the total answer agrees with the halfspace solution almost exactly up to the arrival of the first lateral wave. Examination of later arrivals is complicated by the fact that we can expect changes in the response due not only to the start of the signal pulse, but also the points at $\tau = 1$, 4, and 6. Standard double deformation is incapable of considering this source, since its Fourier transform is not made up entirely of poles.

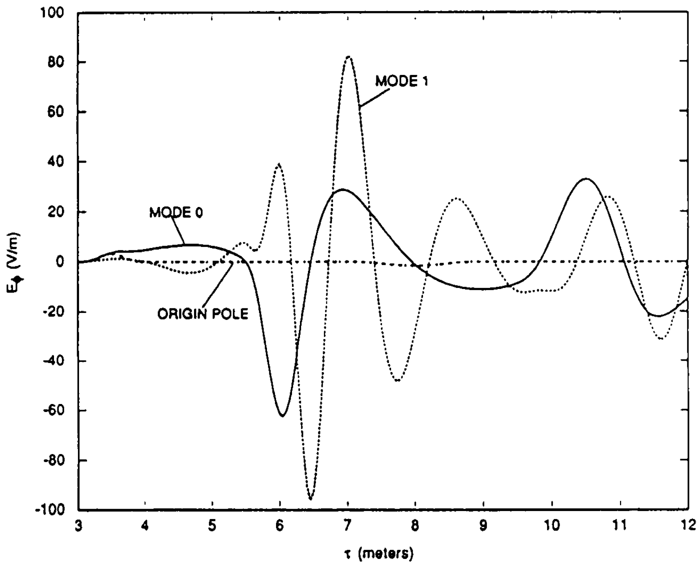

Figure 4.8 MDD: Modes 0 and 1, and origin pole contribution; VMD on coated perfect conductor, source, Eq. (16).

5. Modified Double Deformation for Vertical Electric Dipole over Coated Perfect Conductor

a. Single Deformation

The magnetic field on the surface of a coated perfect conductor from a vertical electric dipole also on the surface a distance ρ away is

$$H_\phi = \frac{1}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dk_0\, e^{-ik_0\tau}\tilde{I}(k_0)\int_{\mathrm{SIP}} dk_\rho\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho)[1+R^{TM}]\right\} \quad (1)$$

where

$$R^{TM} = \frac{R_{01}+e^{2ik_{1z}d}}{1+R_{01}e^{2ik_{1z}d}}, \qquad R_{01} = \frac{\epsilon_1 k_z - k_{1z}}{\epsilon_1 k_z + k_{1z}} \quad (2)$$

and

$$k_z = \sqrt{k_0^2 - k_\rho^2}, \quad k_{1z} = \sqrt{\epsilon_1 k_0^2 - k_\rho^2} \quad (3)$$

The coating is of thickness d and dielectric constant ϵ_1 .

The details of the double deformation technique for the VED are very similar to the VMD case, so we will concentrate on the differences. We first deform the k_ρ integral in (1) to the standard steepest descent path (SDP). Calling the height on the SDP q , we obtain

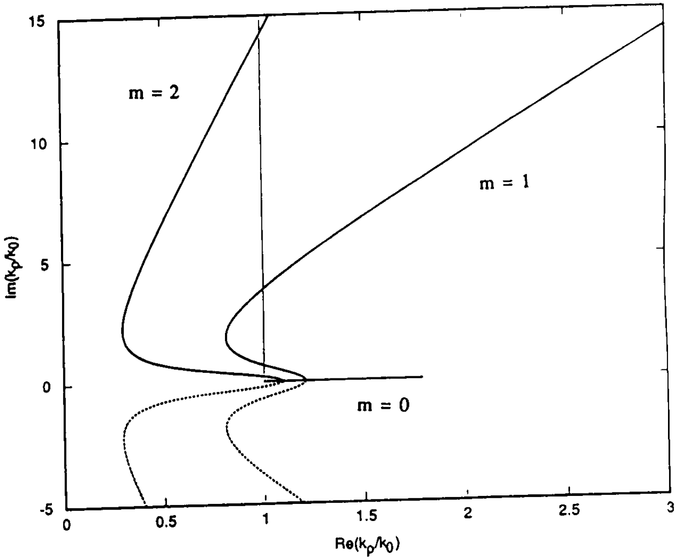

Figure 5.1 k_ρ **pole loci,** $m = 0, 1, 2$; **VED on coated perfect conductor; vertical line at** $\mathrm{Re}(k_\rho/k_0) = 1$ **is steepest descent path.**

$$\mathrm{SDP} = \frac{1}{8\pi^2}\mathrm{Re}\Bigg\{ \int_0^\infty dk_0\, e^{-ik_0\tau}\tilde{I}(k_0)$$
$$\int_0^\infty dq\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho) \frac{(1+R_{01})^2(1+e^{2ik_{1z}d})^2}{(1+R_{01}e^{2ik_{1z}d})(R_{01}+e^{2ik_{1z}d})} \Bigg\} \tag{4}$$

where $k_\rho = k_0 + iq$ and k_z is evaluated on the top Riemann sheet. The poles enclosed in the k_ρ plane are zeros of the denominator of (1),

$$1 + R_{01}e^{2ik_{1z}d} = 0 \tag{5}$$

We again use a modal approach to study these poles. We let

$$R_{01} = e^{i\phi} \tag{6}$$

where ϕ is a complex number whose real part lies between 0 and 2π. We substitute this into (5) and, by taking the logarithm of both sides, arrive at the modal equation

$$\phi + 2k_{1z}d = (2m+1)\pi, \qquad m = 0, 1, 2 \ldots \tag{7}$$

where the different m's distinguish the different modes.

Some typical loci of poles ($m = 0,1,2$) are shown in Figure 5.1. The solid lines refer to poles on the upper Riemann sheet; the dashed curves to poles on the lower Riemann sheet. The behavior is similar to that for the VMD over coated perfect conductor case, except with the cutoff frequency being defined by

$$k_{cm} = \frac{m\pi}{\sqrt{\epsilon_1 - 1}d} \tag{8}$$

We also have the intersection points of the locus with the SDP at $k_0 = k_{bm}$ and $k_0 = k_{am}$. This gives us the same guided and leaky wave poles; however, the locus for mode 0 is different. There is only one pole for this mode, which starts at $k_\rho = \sqrt{\epsilon_1}k_0$ on the bottom sheet for large k_0 and moves toward $k_\rho = k_0$, as k_0 approaches zero. Therefore, mode 0 has only guided waves.

The contribution to the total field from these k_ρ poles is

$$\sum_{m=0}^{\infty} \frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \operatorname{Re}\left\{ \int_{k_{cm}}^{\infty} dk_0 \, e^{-ik_0\tau} \tilde{I}(k_0) \left[\frac{k_\rho k_z k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 + ik_z d[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]} \right] \right\} \tag{9}$$

for the guided poles, where k_z is evaluated on the bottom Riemann sheet, and

$$\sum_{m=1}^{\infty} \frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \operatorname{Re}\left\{ \int_{k_{am}}^{k_{bm}} dk_0 \, e^{-ik_0\tau} \tilde{I}(k_0) \left[\frac{k_\rho k_z k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 + ik_z d[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]} \right] \right\} \tag{10}$$

for the leaky poles, where k_z is evaluated on the top Riemann sheet. The total field is thus given by the sum of (4), (9), and (10).

b. Double Deformation

Using our standard damped sinusoidal source,

$$I(\tau) = I_0 \tau \sin \omega_0 \tau e^{-\alpha_0 \tau} \tag{11}$$

with its Fourier transform

$$\tilde{I}(k_0) = \frac{iI_0}{2}\left[\frac{1}{(k_0 + i\alpha_0 + \omega_0)^2} - \frac{1}{(k_0 + i\alpha_0 - \omega_0)^2}\right] \tag{12}$$

we deform the k_0 integral in (4) upward to the positive imaginary k_0 axis for $\tau < \rho$ and downward to the negative imaginary k_0 axis for $\tau > \rho$. The total for $\tau < \rho$ is analytically zero; the details of this are almost identical to the VMD case and are therefore omitted. Deforming downward, then, our answer is in two parts: the double steepest descent path contribution, and the residues of the enclosed poles in the k_0 plane. The double steepest descent path contribution vanishes in the same way as it did for the VMD over coated perfect conductor.

The poles enclosed are due to singularities of $\tilde{I}(k_0)$ and zeros of (5) of (4). For this source, the source pole contribution is:

$$-\frac{I_0}{8\pi}\mathrm{Re}\left\{\int_0^\infty dq\,\frac{\partial}{\partial k_0}\left[e^{-ik_0\tau}\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\right.\right.$$
$$\left.\left.\frac{(1+R_{01})^2(1+e^{2ik_{1z}d})^2}{(1+R_{01}e^{2ik_{1z}d})(R_{01}+e^{2ik_{1z}d})}\right]_{k_0=\omega_0-i\alpha_0}\right\} \tag{13}$$

We now consider the contribution due to the zeros of (5). Using the same modal technique, we can find loci of these poles; loci for $m = 0$, 1, and 2 are shown in Figure 5.2. The solid lines refer to poles on the upper Riemann sheet; the dashed curves to poles on the lower Riemann sheet. For all modes except $m = 0$, two sections of the locus are enclosed by the deformation of the integration path to the negative imaginary axis: the part from $k_0 = k_{cm}$ towards $k_0 = k_{bm}$, which we call "k_0 finite", and from $k_0 = k_{am}$ to $k_0 \to -i\infty$, which we call "k_0 infinite". For $m = 0$, since the pole locus in the k_ρ plane never crosses the SDP, there are no finite poles. It turns out that the $m = 0$ locus in the k_0 plane lies along the negative imaginary axis; this means that these poles are only half-enclosed by the deformation. If we let $n(q)$ be 1 if the poles are fully enclosed, and $1/2$ if they are half enclosed, the integrals are

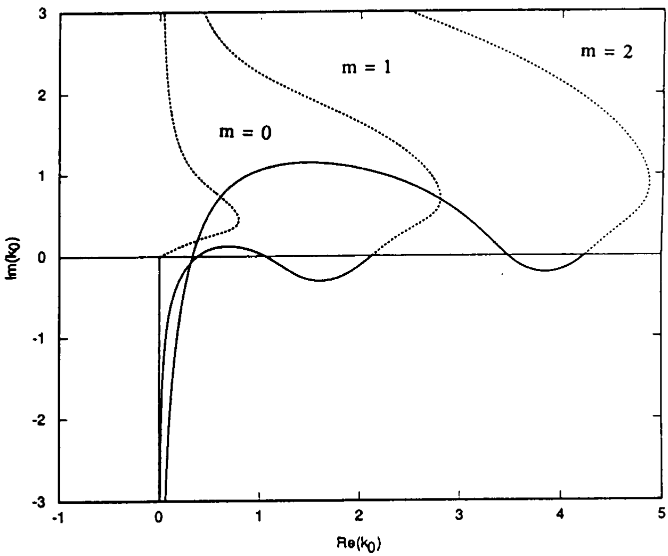

Figure 5.2 k_0 **pole loci,** $m = 0, 1, 2$**; VED on coated perfect conductor.**

$$\sum_{m=1}^{\infty} -\frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ \int_{q_{am}}^{\infty} dq\, n(q) e^{-ik_0\tau} \tilde{I}(k_0) \frac{k_\rho^2 k_{1z}^2 k_z H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0 k_\rho q + k_z d[(\epsilon_1 - 1)k_0 - iq][\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]} \right\} \quad (14)$$

for the k_0 infinite poles, and

$$\sum_{m=1}^{\infty} -\frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ \int_{0}^{q_{bm}} dq\, e^{-ik_0\tau} \tilde{I}(k_0) \frac{k_\rho^2 k_{1z}^2 k_z H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0 k_\rho q + k_z d[(\epsilon_1 - 1)k_0 - iq][\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]} \right\} \quad (15)$$

for the k_0 finite poles, where q_{am} and q_{bm} are the values of q where the locus in the k_ρ plane crosses the SDP. For $m = 0$, we define $q_{a0} = 0$.

The complete double deformation solution is thus made up of five parts: the k_ρ guided modes (9), the k_ρ leaky modes (10), the k_0

infinite modes, (14), the k_0 finite modes (15), and the source pole contribution (13).

c. Modified Double Deformation

Applying the modification as we did in the halfspace case, with one integration by parts necessary to satisfy Jordan's Lemma,

$$\tilde{I}(k_0) = \int_0^\infty d\tau' \, I(\tau') e^{ik_0\tau'} = \frac{i}{k_0} \int_0^\infty d\tau' \, I'(\tau') e^{ik_0\tau'} \tag{16}$$

We now split the Fourier transform of the current into two parts:

$$\tilde{I}(k_0) = \frac{i}{k_0} \left[\int_0^{\tau-\rho} d\tau' \, I'(\tau') e^{ik_0\tau'} + \int_{\tau-\rho}^\infty d\tau' \, I'(\tau') e^{ik_0\tau'} \right] \tag{17}$$

$$\equiv \frac{i}{k_0} \left[\tilde{I}_{1b}(k_0, \tau-\rho) + \tilde{I}_{1a}(k_0, \tau-\rho) \right] \tag{18}$$

We substitute this into our expression for SDP (4) and deform the k_0 integral of the half containing $\tilde{I}_{1a}$ upward, the half containing $\tilde{I}_{1b}$ downward. However, since we have introduced a pole at the origin, we must deform our integration path in the k_0 up to $i\delta$ before separating, as we did before. It can be shown that the total for $\tilde{I}_{1a}$ is identically zero, which satisfies causality; the details are similar to the case of VMD over coated perfect conductor. Therefore, we will concentrate on the part containing $\tilde{I}_{1b}$:

$$\mathrm{SDP} = \frac{1}{8\pi^2} \mathrm{Re} \left\{ i \int_{i\delta}^\infty dk_0 \, e^{-ik_0\tau} \frac{\tilde{I}_{1b}(k_0, \tau-\rho)}{k_0} \right.$$
$$\left. \int_0^\infty dq \, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1+R_{01})^2(1+e^{2ik_{1z}d})^2}{(1+R_{01}e^{2ik_{1z}d})(R_{01}+e^{2ik_{1z}d})} \right\} \tag{19}$$

We will deform the k_0 integral downward, to $k_0 = -ip$. In so doing, we obtain three contributions: the residue from the double steepest descent path contribution, the pole at the origin, and the residues of the enclosed poles in the k_0 plane. The double steepest descent path contribution is similar to that for standard double deformation, except that $(i/k_0)\tilde{I}_{1b}(k_0, \tau-\rho)$ replaces $\tilde{I}(k_0)$. This change does not alter the conclusion that the total contribution is zero.

There is, however, a contribution from the pole at the origin. Since we half enclose this pole clockwise, we can write this as follows:

$$Z = -\frac{1}{8\pi}\mathrm{Re}\left\{ \int_0^\infty dq \left[\tilde{I}_{1b}(k_0, \tau-\rho) e^{-ik_0\tau} \right.\right.$$
$$\left.\left. \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1+R_{01})^2(1+e^{2ik_{1z}d})^2}{(1+R_{01}e^{2ik_{1z}d})(R_{01}+e^{2ik_{1z}d})} \right]_{k_0=0} \right\} \tag{20}$$

Using the fact that $\tilde{I}_{1b}(0,\tau-\rho) = I(\tau-\rho)$, we can write this as

$$Z = \frac{1}{\pi^2} I(\tau-\rho) \int_0^\infty dq \frac{q\cos^2(qd)K_1(q\rho)}{\cos^2(qd)+\sin^2(qd)/\epsilon_1^2} \tag{21}$$

Unfortunately, unlike the VMD case, this integral cannot be evaluated in closed form.

The contributions from the poles of (5) in the k_0 plane are very similar to the standard double deformation results, except that there is no source pole. We obtain

$$\sum_{m=1}^\infty -\frac{\epsilon_1}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{ i\int_{q_{am}}^\infty dq\, n(q) \frac{\tilde{I}_{1b}(k_0,\tau-\rho)}{k_0} e^{-ik_0\tau} \right.$$
$$\left. \frac{k_\rho^2 k_{1z}^2 k_z H_1^{(1)}(k_\rho\rho)}{\epsilon_1 k_0 k_\rho q + k_z d[(\epsilon_1-1)k_0 - iq][\epsilon_1 k_0^2 - (\epsilon_1+1)k_\rho^2]} \right\} \tag{22}$$

for the k_0 infinite poles,

$$\sum_{m=1}^\infty -\frac{\epsilon_1}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{ i\int_0^{q_{bm}} dq \frac{\tilde{I}_{1b}(k_0,\tau-\rho)}{k_0} e^{-ik_0\tau} \right.$$
$$\left. \frac{k_\rho^2 k_{1z}^2 k_z H_1^{(1)}(k_\rho\rho)}{\epsilon_1 k_0 k_\rho q + k_z d[(\epsilon_1-1)k_0 - iq][\epsilon_1 k_0^2 - (\epsilon_1+1)k_\rho^2]} \right\} \tag{23}$$

and for the k_0 finite poles. We also have the contributions for the k_ρ poles, which are now

$$\sum_{m=0}^\infty \frac{\epsilon_1}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{ i\int_{k_{cm}}^\infty dk_0 \frac{\tilde{I}_{1b}(k_0,\tau-\rho)}{k_0} e^{-ik_0\tau} \right.$$
$$\left. \left[\frac{k_\rho k_z k_{1z}^2 H_1^{(1)}(k_\rho\rho)}{\epsilon_1 k_0^2 + ik_z d[\epsilon_1 k_0^2 - (\epsilon_1+1)k_\rho^2]} \right] \right\} \tag{24}$$

for the guided poles and

$$\sum_{m=1}^{\infty} \frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \text{Re}\left\{ \int_{k_{am}}^{k_{bm}} dk_0 \, \frac{\tilde{I}_{1b}(k_0, \tau - \rho)}{k_0} e^{-ik_0\tau} \left[\frac{k_\rho k_z k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 + ik_z d[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]} \right] \right\} \quad (25)$$

for the leaky poles. The total modified double deformation solution is therefore given by the sum of the k_ρ guided modes (24), the k_ρ leaky modes (25), the k_0 infinite modes, (22), the k_0 finite modes (23), and the origin pole contribution (21).

d. Results

Results are shown for the configuration with $\rho = 3$, $d = 1$, and $\epsilon_1 = 3.2$. Two sources are used; the first is the damped sinusoid (11) with $I_0 = 1$, $\omega_0 = 1$ and $\alpha_0 = 0.5$. The second is the smoothed pulse, discussed in Section 1. We have the same arrival times as in the VMD case:

Wave	Arrival Time
Direct (air)	3.000
Direct (dielectric)	5.367
First Lateral	5.966
First Reflected	6.450
Second Lateral	8.932
Second Reflected	8.944
Third Lateral	11.900

First, we solve the damped sinusoid with the standard double deformation technique. The total answer, with eleven modes, is shown in Figure 5.3, where it is compared to the closed-form halfspace result; the source pole and the first two modes are shown in Figure 5.4. As in the VMD case, all the arrivals can be clearly distinguished in the final result, but not in any of the separate components; they are features of the solution as a whole.

Next, we solve the same case, but using the modified double deformation technique. The total result overlays the double deformation result from Figure 5.3; the pole contribution and the first two modes are in Figure 5.5. As can be seen, the agreement between the modified and standard results is very good. Furthermore, in contrast to the standard

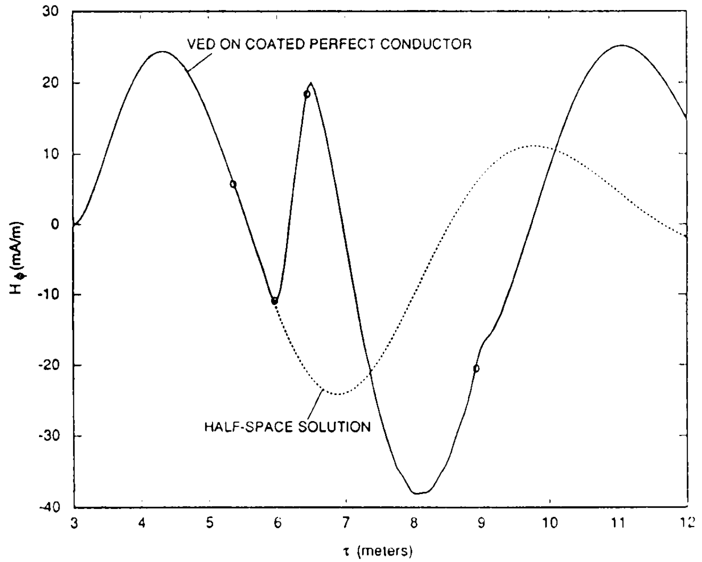

Figure 5.3 Total double deformation and halfspace solution; VED on coated perfect conductor, source, Eq. (12); circles show various ray arrivals.

case, the components of the modified result have no discontinuities of value or slope.

Finally, we use modified double deformation to solve the case with a pulse source. Again, the total response (eleven modes, Figure 5.6) agrees with the closed-form halfspace solution very well up until the arrival of the first lateral wave. The origin pole and the first two modes are shown in Figure 5.7. While the relative strength of the higher order modes is larger than in the damped sinusoid case, due to the high-frequency components of the pulse source, we still obtain a very good approximation to the total response with only eleven modes; all the ray arrivals can also be seen in this case.

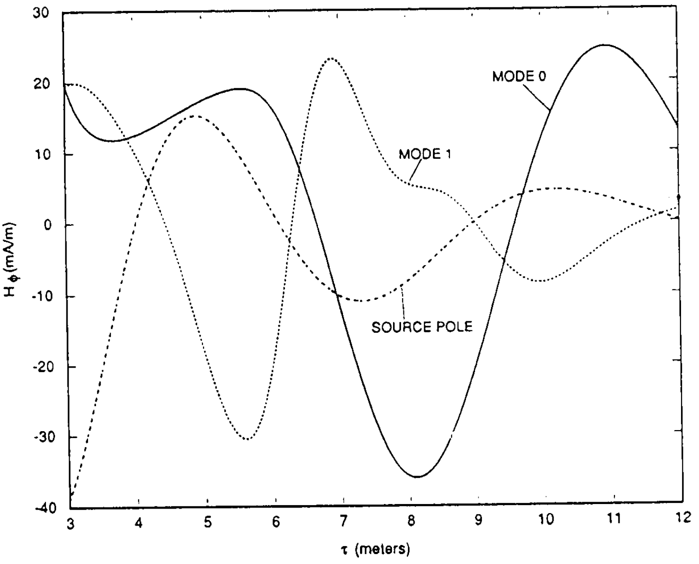

Figure 5.4 Modes 0 and 1, and source pole contribution; VED on coated perfect conductor, source, Eq. (12).

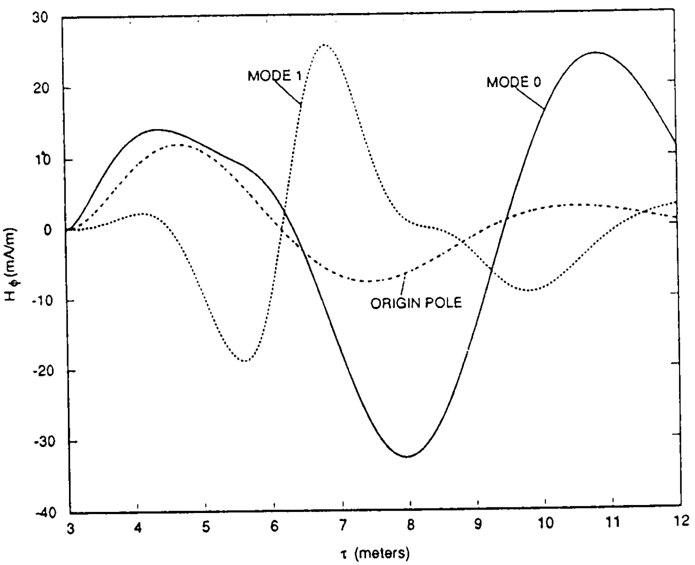

Figure 5.5 MDD: Modes 0 and 1, and origin pole contribution; VED on coated perfect conductor, source, Eq. (12).

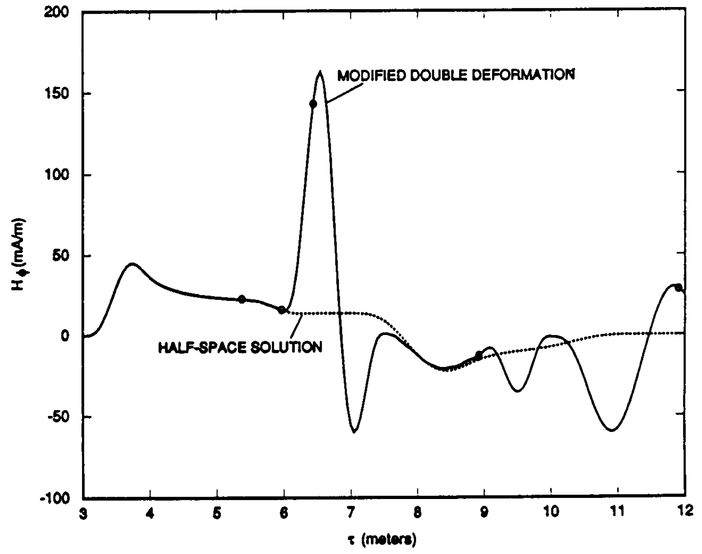

Figure 5.6 Modified double deformation result and halfspace solution; VED on coated perfect conductor, source, Eq. (16); circles show various ray arrivals.

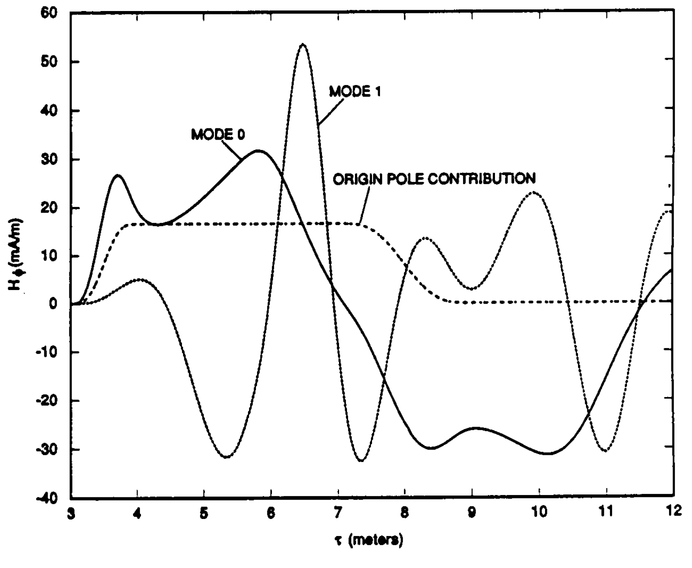

Figure 5.7 MDD: Modes 0 and 1, and origin pole contribution; VED on coated perfect conductor, source, Eq. (16).

6. Modified Double Deformation for Vertical Magnetic Dipole over Two-Layer Medium

a. Single Deformation

The electric field on the surface of a two-layer medium from a vertical magnetic dipole also on the surface a distance ρ away is

$$E_\phi(\tau) = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{\int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\int_{\mathrm{SIP}} dk_\rho \frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)[1+R^{TE}]\right\} \tag{1}$$

where

$$R^{TE} = \frac{R_{01}+R_{12}e^{2ik_{1z}d}}{1+R_{01}R_{12}e^{2ik_{1z}d}}, \qquad R_{01} = \frac{k_z-k_{1z}}{k_z+k_{1z}}, \qquad R_{12} = \frac{k_{1z}-k_{2z}}{k_{1z}+k_{2z}} \tag{2}$$

and

$$k_z = \sqrt{k_0^2-k_\rho^2}, \quad k_{1z} = \sqrt{\epsilon_1 k_0^2-k_\rho^2}, \quad k_{2z} = \sqrt{\epsilon_2 k_0^2-k_\rho^2} \tag{3}$$

The middle layer is of thickness d and dielectric constant ϵ_1; the bottom layer is of dielectric constant ϵ_2.

We first deform the k_ρ integral in (1) to its steepest descent path (SDP). This path goes from $k_\rho = k_0 + i\infty$ on the UU Riemann sheet, down to $k_\rho = k_0$, and back up on the LU sheet; we will call this part of the integration path SDP_0. It then descends from $k_\rho = \sqrt{\epsilon_2}\,k_0 + i\infty$, still on the LU sheet, to $k_\rho = \sqrt{\epsilon_2}\,k_0$ and back up on the LL sheet; we will call this part SDP_2. The convention for naming the Riemann sheets is that the first letter tells the sheet for k_z (U for $\mathrm{Re}\{k_z\} > 0$, L for $\mathrm{Re}\{k_z\} < 0$), and the second letter is for k_{2z} (U for $\mathrm{Re}\{k_{2z}\} > 0$, L for $\mathrm{Re}\{k_{2z}\} < 0$). Since layer 1 is bounded, there is no branch cut for k_{1z}, and therefore no Riemann sheets; indeed, (1) is even in k_{1z}. If we call the height on SDP_0 q and the height on SDP_2 r, then the steepest descent path contribution becomes

$$\mathrm{SDP}_0 = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{i\int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\right.$$
$$\left.\int_0^\infty dq\left[\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)[1+R^{TE}]\right]_{\mathrm{UU}} - \left[\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)[1+R^{TE}]\right]_{\mathrm{LU}}\right\} \tag{4}$$

where $k_\rho = k_0 + iq$, and

$$\mathrm{SDP}_2 = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0) \right.$$
$$\left. \int_0^\infty dr\left[\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)[1+R^{TE}]\right]_{\mathrm{LU}} - \left[\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)[1+R^{TE}]\right]_{\mathrm{LL}} \right\} \tag{5}$$

where $k_\rho = \sqrt{\epsilon_2}\, k_0 + ir$. These can be written as

$$\mathrm{SDP}_0 = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0) \right.$$
$$\left. \int_0^\infty dq\,\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \tag{6}$$

and

$$\mathrm{SDP}_2 = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0) \right.$$
$$\left. \cdot\int_0^\infty dr\,\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \tag{7}$$

where k_z and k_{2z} are both evaluated on the UU Riemann sheet. The poles enclosed in deforming the k_ρ integration path are zeros of the denominator of (1),

$$1 + R_{01}R_{12}e^{2ik_{1z}d} = 0 \tag{8}$$

Using our modal approach, we let

$$R_{01} = e^{i\phi} \tag{9}$$

and

$$R_{12} = e^{i\psi} \tag{10}$$

where ϕ and ψ are complex numbers whose real part lie between 0 and 2π. We thus obtain the modal equation

$$\phi + \psi + 2k_{1z}d = (2m+3)\pi, \qquad m = 0, 1, 2, \ldots \tag{11}$$

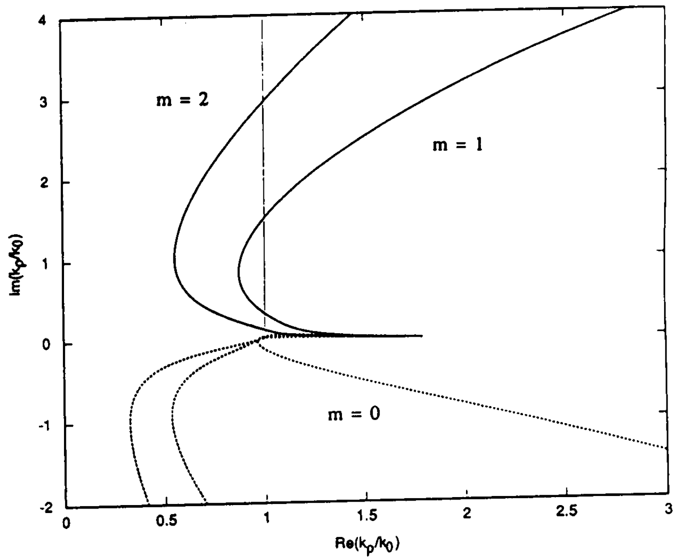

Figure 6.1 k_ρ **pole loci,** $m = 0, 1, 2$**; VMD on two-layer medium; vertical line at** $\text{Re}(k_\rho/k_0) = 1$ **is steepest descent path.**

where the different m's distinguish the different modes.

Some typical loci of poles ($m = 0, 1, 2$) are shown in Figure 6.1. The solid lines refer to poles on the UU Riemann sheet; the dashed curves to poles on the LU Riemann sheet. For the relatively large value of $\epsilon_2 = 80$ that we consider, the loci are not very different from that for the coated perfect conductor case, and the poles on the LL Riemann sheet do not contribute for times considered and can be ignored. The main difference is that the cutoff frequency, k_{cm}, cannot be expressed in closed form, and thus must be searched for numerically.

We have the same guided and leaky poles as we had previously:

$$\sum_{m=0}^{\infty} -\frac{\eta_0}{2\pi(\epsilon_1 - 1)} \text{Re}\left\{ i \int_{k_{cm}}^{\infty} dk_0 \, \frac{\tilde{I}(k_0)}{k_0} e^{-ik_0\tau} \left[\frac{k_\rho k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{1/k_z + 1/k_{2z} - id} \right] \right\} \quad (12)$$

for the guided poles, where k_z and k_{2z} are evaluated on the LU Riemann sheet, and

$$\sum_{m=1}^{\infty} -\frac{\eta_0}{2\pi(\epsilon_1 - 1)} \text{Re}\left\{ i \int_{k_{am}}^{k_{bm}} dk_0 \, \frac{\tilde{I}(k_0)}{k_0} e^{-ik_0\tau} \left[\frac{k_\rho k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{1/k_z + 1/k_{2z} - id} \right] \right\} \quad (13)$$

for the leaky poles, where k_z and k_{2z} are evaluated on the UU Riemann sheet. The total field is thus given by the sum of (6), (7), (12), and (13).

b. Double Deformation

The double deformation method takes the single deformation result and deforms the k_0 integrals in the SDP terms, (6) and (7). We use our standard damped sinusoidal source,

$$I(\tau) = I_0 \tau^2 \sin \omega_0 \tau e^{-\alpha_0 \tau}, \tag{14}$$

which has the Fourier transform

$$\tilde{I}(k_0) = I_0 \left[\frac{1}{(k_0 + i\alpha_0 - \omega_0)^3} - \frac{1}{(k_0 + i\alpha_0 + \omega_0)^3} \right], \tag{15}$$

There are three ranges of τ to consider. First, for $\tau < \rho$, we deform the k_0 integrals in both SDP_0 and SDP_2 upward to the positive imaginary k_0 axis. For the range $\rho < \tau < \sqrt{\epsilon_2}\,\rho$, we deform SDP_0 downward to the negative imaginary k_0 axis and SDP_2 upward. Finally, for $\tau > \sqrt{\epsilon_2}\,\rho$, we deform both integrals downward.

It can be easily shown that the total contribution for $\tau < \rho$ is identically zero, which satisfies causality. The cancelation of the pole residues in the k_ρ and k_0 planes is similar to the coated perfect conductor case, but the double steepest descent path contributions are more interesting, and are discussed in Appendix B. Also, we will not consider times for which $\tau > \sqrt{\epsilon_2}\,\rho$, and so will concentrate on the contribution for $\rho < \tau < \sqrt{\epsilon_2}\,\rho$. For this case, we will deform the k_0 integral in (6) downward, to $k_0 = -ip$, and the k_0 integral in (7) upward, to $k_0 = is$. In so doing, we obtain the two double steepest descent path contributions, and the residues of the enclosed poles in the k_0 plane.

The double steepest descent path contributions are more difficult than in the coated perfect conductor case, and we leave the details to Appendix B. For DSDP_0, we obtain

$$\mathrm{DSDP}_0 = \frac{\eta_0}{8\pi^2} \mathrm{Re} \Bigg\{ i \int_0^\infty dp\, p e^{-p\tau} \tilde{I}(-ip)$$

$$\int_{(\sqrt{\epsilon_2}+1)p}^\infty dq\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1 + R_{01})^2 (1 + R_{12} e^{2ik_{1z}d})^2}{(1 + R_{01} R_{12} e^{2ik_{1z}d})(R_{01} + R_{12} e^{2ik_{1z}d})} \Bigg\} \tag{16}$$

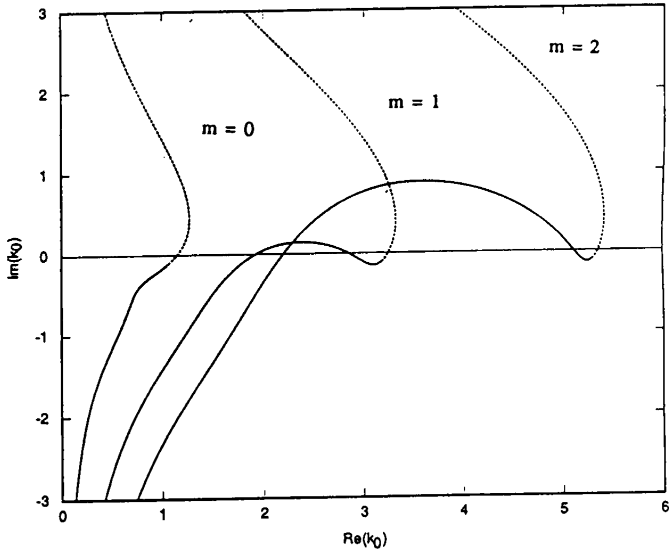

Figure 6.2 k_0 **pole loci,** $m = 0, 1, 2$; **VMD on two-layer medium.**

and for DSDP$_2$, we obtain

$$\mathrm{DSDP}_2 = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i \int_0^\infty ds\, se^{s\tau}\tilde{I}(is) \right.$$

$$\left. \int_{(\sqrt{\epsilon_2}-1)s}^\infty du\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho)\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \quad (17)$$

In the deformation, we enclose two kinds of poles; the source pole, due to the singularities of $\tilde{I}(k_0)$, and the poles that arise from zeroes of the denominator (8). Only the source pole at $k_0 = \omega_0 - i\alpha_0$ is enclosed, so we obtain:

$$-\frac{\eta_0 I_0}{8\pi}\mathrm{Re}\left\{ \int_0^\infty dq\, \frac{\partial^2}{\partial k_0^2}\left[k_0 e^{-ik_0\tau}\frac{k_\rho^2}{k_z} \right.\right.$$

$$\left.\left. H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right]_{k_0=\omega_0-i\alpha_0} \right\} \quad (18)$$

Turning now to the roots of (8), we look for solutions with complex k_0 and $k_\rho = k_0 + iq$ on both the UU and LU Riemann sheets, instead of solutions with real k_0 and complex k_ρ as we did in the k_ρ plane. We can use the same modal technique as previously, and discover loci such as the one shown in Figure 6.2. The solid lines refer to poles on the UU Riemann sheet; the dashed curves to poles on the LU Riemann sheet. For all modes except $m = 0$, two sections of the locus are enclosed by the deformation of the integration path to the negative imaginary axis: the part from $k_0 = k_{cm}$ towards $k_0 = k_{bm}$, which we will call "k_0 finite", and from $k_0 = k_{am}$ to $k_0 \to -i\infty$, which we will call "k_0 infinite". For $m = 0$, since the locus in the k_ρ plane never crosses the SDP, the locus in the k_0 plane never crosses the k_0-real axis, so there is only a k_0 infinite locus. The integrals for these poles are

$$-\frac{\eta_0}{4\pi}\mathrm{Re}\left\{\int dq\,\mathrm{Res}\left[k_0 e^{-ik_0\tau}\tilde{I}(k_0)\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)[1+R^{TE}]\right]\right\} \tag{19}$$

which, after calculating the residues of these poles, gives the following expressions for the contributions:

$$\frac{\eta_0}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{\int dq\,n(q)\tilde{I}(k_0)e^{-ik_0\tau}H_1^{(1)}(k_\rho\rho)\right.$$
$$\left.\left[\frac{k_\rho^2k_{1z}^2(q-2ik_0)}{k_\rho\left[k_0^2(\epsilon_1-1)d-ik_z(1+k_z/k_{2z})\right]+k_0dk_{1z}^2}\right]\right\} \tag{20}$$

which becomes

$$\sum_{m=0}^{\infty}\frac{\eta_0}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{\int_{q_{am}}^{\infty}dq\,n(q)e^{-ik_0\tau}\tilde{I}(k_0)H_1^{(1)}(k_\rho\rho)\right.$$
$$\left.\left[\frac{k_\rho^2k_{1z}^2(q-2ik_0)}{k_\rho\left[k_0^2(\epsilon_1-1)d-ik_z(1+k_z/k_{2z})\right]+k_0dk_{1z}^2}\right]\right\} \tag{21}$$

for the k_0 infinite poles, and

$$\sum_{m=1}^{\infty}\frac{\eta_0}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{\int_{q_{cm}}^{q_{bm}}dq\,n(q)e^{-ik_0\tau}\tilde{I}(k_0)H_1^{(1)}(k_\rho\rho)\right.$$
$$\left.\left[\frac{k_\rho^2k_{1z}^2(q-2ik_0)}{k_\rho\left[k_0^2(\epsilon_1-1)d-ik_z(1+k_z/k_{2z})\right]+k_0dk_{1z}^2}\right]\right\} \tag{22}$$

for the k_0 finite poles, where q_{am}, q_{bm} and q_{cm} are the values of q where the locus in the k_ρ plane crosses the SDP; for $m = 0$, $q_{a0} \equiv q_{c0}$. The function $n(q) = 1$ when the pole in question is on the UU Riemann sheet, and $n(q) = -1$ for LU.

The complete double deformation solution is thus made up of seven parts: the k_ρ guided modes (12), the k_ρ leaky modes (13), the k_0 infinite modes (21), the k_0 finite modes (22), the source pole contribution (18), and the two double steepest descent paths, (16) and (17).

c. *Modified Double Deformation*

We apply the modified double deformation technique to this problem, just as we did in the coated perfect conductor case. We do two integrations by parts:

$$\tilde{I}(k_0) = \int_0^\infty d\tau'\, I(\tau')e^{ik_0\tau'} = -\frac{1}{k_0^2}\int_0^\infty d\tau'\, I''(\tau')e^{ik_0\tau'} \tag{23}$$

and split the Fourier transform of the current into two parts:

$$\tilde{I}(k_0) = -\frac{1}{k_0^2}\left[\int_0^\gamma d\tau'\, I''(\tau')e^{ik_0\tau'} + \int_\gamma^\infty d\tau'\, I''(\tau')e^{ik_0\tau'}\right] \tag{24}$$

$$\equiv -\frac{1}{k_0^2}\left[\tilde{I}_{2b}(k_0,\gamma) + \tilde{I}_{2a}(k_0,\gamma)\right] \tag{25}$$

We substitute this into our expressions for SDP$_0$ (6) with $\gamma = \tau - \rho$ and into SDP$_2$ (7) with $\gamma = \tau - \sqrt{\epsilon_2}\,\rho$. We then deform the k_0 integral of the parts containing $\tilde{I}_{2a}$ upward, the parts containing $\tilde{I}_{2b}$ downward. However, since we have introduced a pole at the origin, we must deform our integration paths in the k_0 up to $i\delta$ before separating, as we did in the halfspace case. If we also split the integrals over the poles in the k_ρ plane into a part containing $\tilde{I}_{2a}$ and one containing $\tilde{I}_{2b}$, it can be shown that the total for $\tilde{I}_{2a}$ is identically zero, which satisfies causality. If we confine ourselves to times $\tau < \sqrt{\epsilon_2}\,\rho$, we can concentrate on the part of SDP$_0$ containing $\tilde{I}_{2b}$:

$$\mathrm{SDP}_0 = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_{i\delta}^\infty dk_0\, e^{-ik_0\tau}\frac{\tilde{I}_{2b}(k_0,\tau-\rho)}{k_0^2}\right.$$
$$\left.\int_0^\infty dq\,\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \tag{26}$$

We will deform the k_0 integral downward, to $k_0 = -ip$. In so doing, we obtain three contributions: the double steepest descent path contributions, the residue from the pole at the origin, and the residues of the enclosed poles in the k_0 plane.

As is shown in Appendix B, DSDP$_{1u}$, DSDP$_{1d}$ and DSDP$_{2u}$ can be combined into one integral:

$$\text{DSDP} = -\frac{\eta_0}{8\pi^2}\text{Re}\left\{ i\int_0^\infty dq \int_{-q/(\sqrt{\epsilon_2}-1)}^{q/(\sqrt{\epsilon_2}+1)} dp\, e^{-p\tau}\frac{\tilde{I}_{2b}(-ip,\tau-\rho)}{p} \right.$$
$$\left. \frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \tag{27}$$

and, if we indent the contour above $p = 0$, this will take care of the contribution from the pole at the origin as well.

The contributions from the poles of (8) in the k_0 plane are very similar to the standard double deformation results, except that there is no source pole. We obtain

$$\sum_{m=0}^{\infty} -\frac{\eta_0}{2\pi(\epsilon_1-1)}\text{Re}\left\{ \int_{q_{am}}^\infty dq\, e^{-ik_0\tau}\frac{\tilde{I}_{2b}(k_0,\tau-\rho)}{k_0^2}H_1^{(1)}(k_\rho\rho) \right.$$
$$\left. \left[\frac{k_\rho^2 k_{1z}^2(q-2ik_0)}{k_\rho\left[k_0^2(\epsilon_1-1)d - ik_z(1+k_z/k_{2z})\right]+k_0 d k_{1z}^2}\right]\right\} \tag{28}$$

for the k_0 infinite poles, and

$$\sum_{m=1}^{\infty} -\frac{\eta_0}{2\pi(\epsilon_1-1)}\text{Re}\left\{ \int_0^{q_{bm}} dq\, e^{-ik_0\tau}\frac{\tilde{I}_{2b}(k_0,\tau-\rho)}{k_0^2}H_1^{(1)}(k_\rho\rho) \right.$$
$$\left. \left[\frac{k_\rho^2 k_{1z}^2(q-2ik_0)}{k_\rho\left[k_0^2(\epsilon_1-1)d - ik_z(1+k_z/k_{2z})\right]+k_0 d k_{1z}^2}\right]\right\} \tag{29}$$

for the k_0 finite poles.

We also have the contributions for the k_ρ poles, which are now

$$\sum_{m=0}^{\infty} \frac{\eta_0}{2\pi(\epsilon_1-1)}\text{Re}\left\{ i\int_{k_{cm}}^\infty dk_0\,\frac{\tilde{I}_{2b}(k_0,\tau-\rho)}{k_0^3}e^{-ik_0\tau}\left[\frac{k_\rho k_{1z}^2 H_1^{(1)}(k_\rho\rho)}{1/k_z+1/k_{2z}-id}\right]\right\} \tag{30}$$

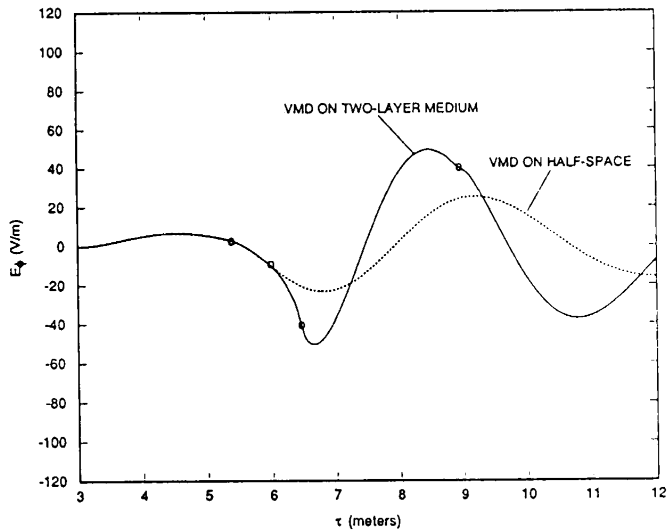

Figure 6.3 Total double deformation and halfspace solution; VMD on two-layer medium, source, Eq. (14); circles show various ray arrivals.

for the guided poles and

$$\sum_{m=1}^{\infty} \frac{\eta_0}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ i \int_{k_{am}}^{k_{bm}} dk_0 \frac{\tilde{I}_{2b}(k_0, \tau - \rho)}{k_0^3} e^{-ik_0\tau} \left[\frac{k_\rho k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{1/k_z + 1/k_{2z} - id} \right] \right\} \tag{31}$$

for the leaky poles. The total modified double deformation solution is therefore given by the sum of the k_ρ guided modes (30), the k_ρ leaky modes (31), the k_0 infinite modes (28) the k_0 finite modes (29), and the double steepest descent path contributions (27).

d. Results

Results are shown for the configuration with $\rho = 3$, $d = 1$, $\epsilon_1 = 3.2$, and $\epsilon_2 = 80$. Two sources are used; the first is the damped sinusoid (14) with $I_0 = 1$, $\omega_0 = 1$ and $\alpha_0 = 0.5$. The second is the smoothed pulse from Section 1.

First, we use the damped sinusoid source with standard double deformation. The results (eleven modes), compared to the halfspace case, are shown in Figure 6.3; the source pole contribution and the first two modes are shown in Figure 6.4. One can see, with the large

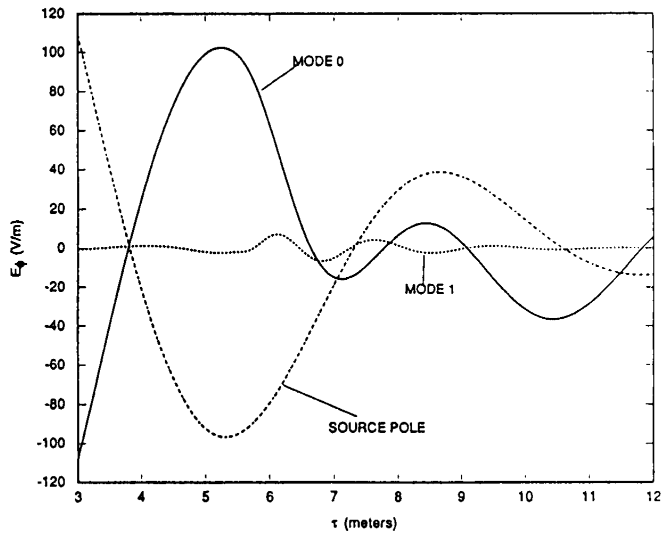

Figure 6.4 Modes 0 and 1, and source pole contribution; VMD on two-layer medium, source, Eq. (14).

value of ϵ_2 used, that the electric field does not differ very much from the coated perfect conductor result, given in Section 4. Indeed, since the travel time for electromagnetic waves in medium 2 is so long, the first few ray arrivals for this case are the same as the coated perfect conductor times from Section 4, as shown below:

Wave	Arrival Time
Direct (air)	3.000
Direct (dielectric)	5.367
First Lateral	5.966
First Reflected	6.450
Second Lateral	8.932
Second Reflected	8.944
Third Lateral	11.900

It turns out that the double steepest descent path contributions, (16) and (17), while not zero, are very small for the values considered, and thus may be safely ignored. It should also be pointed out that these results are the first instance of the successful application of the double deformation technique to the case of a vertical magnetic dipole on the surface of a two-layer medium.

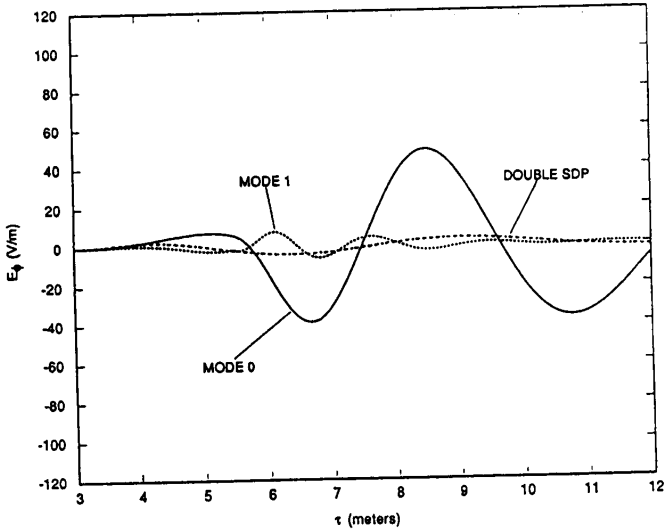

Figure 6.5 MDD: Modes 0 and 1, and double SDP contribution; VMD on two-layer medium, source, Eq. (14).

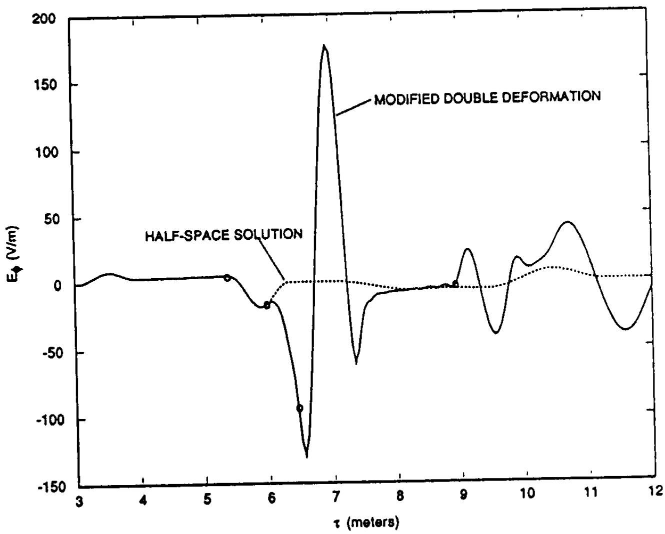

Figure 6.6 Modified double deformation result and halfspace solution; VMD on two-layer medium, source, Eq. (16); circles show various ray arrivals.

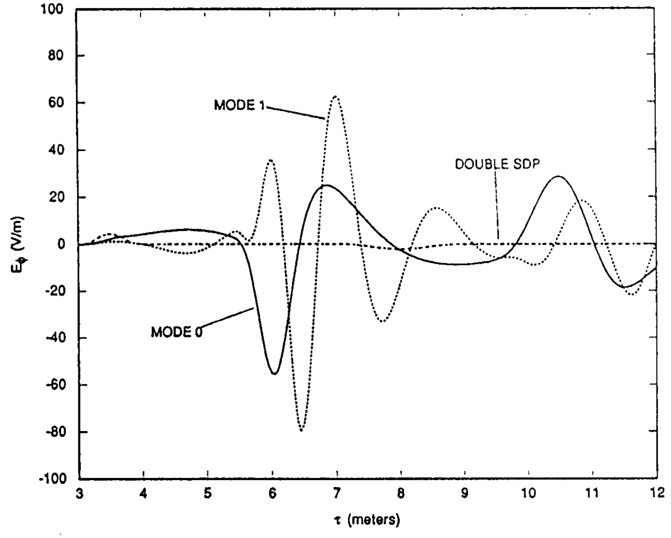

Figure 6.7 MDD: Modes 0 and 1, and double SDP contribution; VMD on two-layer medium, source, Eq. (16).

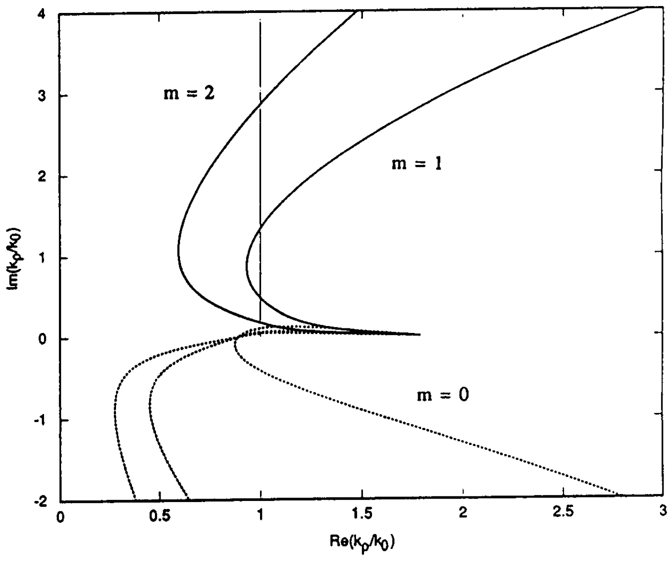

Figure 6.8 k_ρ pole loci, $m = 0, 1, 2$; VMD on lossy two-layer medium; vertical line at $\text{Re}(k_\rho/k_0) = 1$ is steepest descent path.

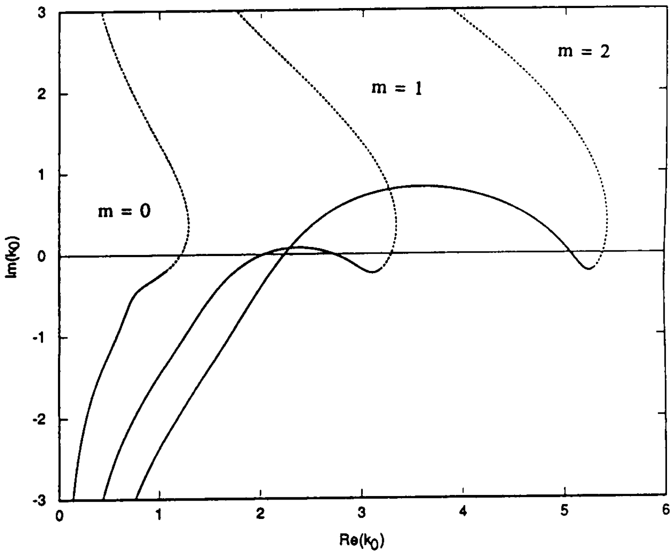

Figure 6.9 k_0 pole loci, $m = 0, 1, 2$; VMD on lossy two-layer medium.

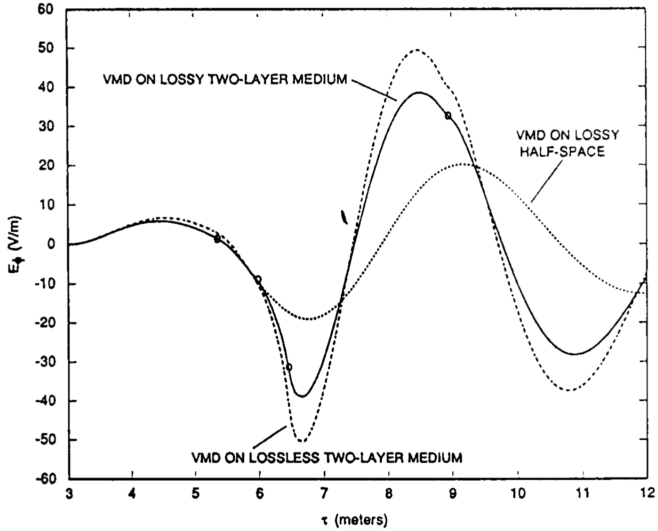

Figure 6.10 Total double deformation and halfspace solution; VMD on lossy two-layer medium, source, Eq. (14); circles show various ray arrivals.

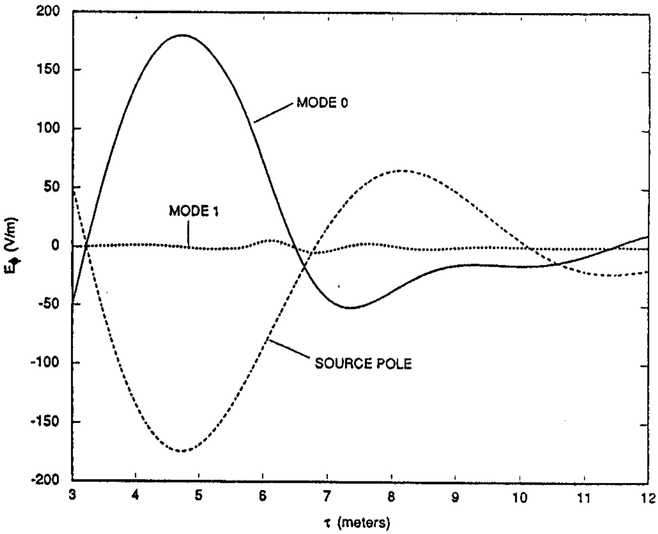

Figure 6.11 Modes 0 and 1, and source pole contribution; VMD on lossy two-layer medium, source, Eq. (14).

Next, we solve the same case using the modified double deformation technique. The results would overlay the double deformation solution; the double steepest descent path contribution (27) and the first two modes are shown in Figure 6.5. The agreement with the standard result is again virtually perfect. The DSDP contribution is significant in modified double deformation as opposed to the standard case because of the added pole singularity at the origin, near the integration path.

Finally, we use the pulse source with modified double deformation. The results (eleven modes), compared to the halfspace case, are shown in Figure 6.6; the double steepest descent path contribution and the first two modes are shown in Figure 6.7. Again, all the arrival times are clear, even with the increased importance of the higher order modes due to the high-frequency components of the pulse source.

e. VMD over Lossy Two-Layer Medium

One of the advantages of the double deformation technique is its ability to treat lossy or otherwise dispersive media, without any special consideration being necessary. If we consider our standard configuration, with the bounded layer now having a dielectric constant $\epsilon_1 = 3.2$ and a conductivity $\sigma = 10^{-3}$ $1/(\Omega\text{-m})$, and thus a complex dielectric constant $\underline{\epsilon}_1 = \epsilon_1 + i\sigma\eta_0/k_0$, there are only two changes. The first is in the pole loci, which shift slightly compared to the lossless case; the new loci are shown in Figures 6.8 and 6.9. Secondly, because of the frequency dependence of $\underline{\epsilon}_1$, the expressions for the k_0 poles change slightly. For standard double deformation, if we let $\sigma_1 = \sigma\eta_0$, we have

$$\sum_{m=0}^{\infty} \frac{\eta_0}{2\pi} \mathrm{Re}\left\{ \int_{q_{am}}^{\infty} dq\, n(q) e^{-ik_0\tau} \tilde{I}(k_0) H_1^{(1)}(k_\rho \rho) \frac{k_0 k_\rho^2 k_{1z}^2}{(\epsilon_1 - 1)k_0 + i\sigma_1} \left[ik_0 d[(\epsilon_1 - 1)k_0 - iq + i\sigma_1/2] - \frac{(\epsilon_1 - 1)k_0 iqk_\rho - i\sigma_1 q^2/2}{k_z[(\epsilon_1 - 1)k_0 + i\sigma_1]} - \frac{(\epsilon_1 - \epsilon_2)k_0 iqk_\rho - i\sigma_1/2[q^2 - (\epsilon_2 - 1)k_0^2]}{k_z[(\epsilon_1 - \epsilon_2)k_0 + i\sigma_1]} \right]^{-1} \right\} \tag{32}$$

for the k_0 infinite poles, and

$$\sum_{m=0}^{\infty} \frac{\eta_0}{2\pi} \mathrm{Re}\left\{ \int_{q_{am}}^{\infty} dq\, n(q) e^{-ik_0\tau} \tilde{I}(k_0) H_1^{(1)}(k_\rho \rho) \frac{k_0 k_\rho^2 k_{1z}^2}{(\epsilon_1 - 1)k_0 + i\sigma_1} \left[ik_0 d[(\epsilon_1 - 1)k_0 - iq + i\sigma_1/2] - \frac{(\epsilon_1 - 1)k_0 iqk_\rho - i\sigma_1 q^2/2}{k_z[(\epsilon_1 - 1)k_0 + i\sigma_1]} - \frac{(\epsilon_1 - \epsilon_2)k_0 iqk_\rho - i\sigma_1/2[q^2 - (\epsilon_2 - 1)k_0^2]}{k_z[(\epsilon_1 - \epsilon_2)k_0 + i\sigma_1]} \right]^{-1} \right\} \tag{33}$$

for the k_0 finite poles.

We examine the same cases as we did for the lossless configuration. First, we use the damped sinusoid source with standard double deformation. The results (eleven modes), compared to the halfspace case and to the lossless case, are shown in Figure 6.10; the source pole contribution and the first two modes are shown in Figure 6.11. In Appendix C, the case for lossy halfspace is considered, and a closed-form solution, similar to that developed for the lossless halfspace in Section 2, is given. Next, we solve the same case using the modified

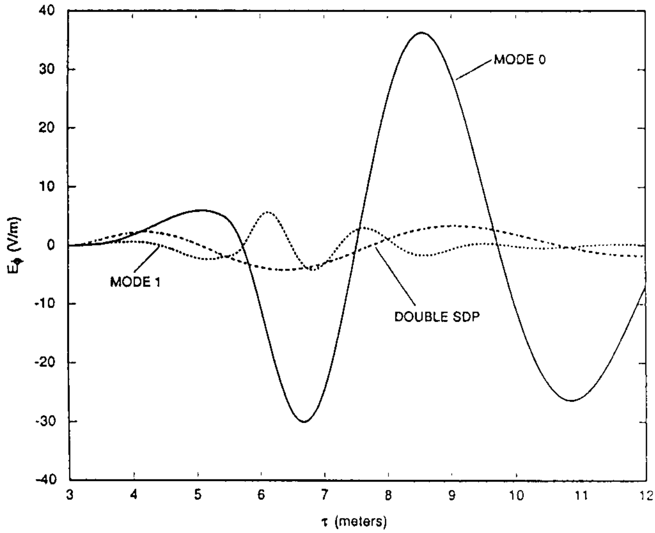

Figure 6.12 MDD: Modes 0 and 1, and double SDP contribution; VMD on lossy two-layer medium, source, Eq. (14).

double deformation technique. The results again overlay; the double steepest descent path contribution and the first two modes are shown in Figure 6.12. Next, we use the pulse source with modified double deformation. The results (eleven modes), compared to the halfspace case, are shown in Figure 6.13; the double steepest descent path contribution and the first two modes are shown in Figure 6.14. In all of these cases, the match with the halfspace solution is very good. Since the ray arrivals are due to the high frequency components of the waveforms, and since the lossy medium becomes lossless at high frequencies, we expect the arrival times to be the same for both the lossless and lossy configurations. Indeed, in all of the above results, the expected arrivals can be seen.

Finally, as a demonstration of the versatility of the modified double deformation technique, we consider two more cases, both with the lossy medium and the pulse source. In the first, we let $\rho = 9$ while d remains 1. The expected arrival times in this case are

Direct (free space)	9.000
First Lateral	11.966
Second Lateral	14.933
Direct (medium 1)	16.100
First Reflected	16.492
Second Reflected	17.618
Third Lateral	17.899

The results (eleven modes), compared to the halfspace case, are shown in Figure 6.15; the double steepest descent path contribution and the first two modes are shown in Figure 6.16. As can be seen, the results are quite good, although not as good as when $\rho = 3$. However, the arrivals can be seen even more clearly for this case, since there are fewer of them, and they tend to be spread more apart.

In the second case, we let $d = 0.5$ while ρ remains 3. The expected arrival times in this case are

Direct (free space)	3.000
First Lateral	4.483
Direct (medium 1)	5.367
First Reflected	5.657
Second Lateral	5.966
Second Reflected	6.450
Third Lateral	7.450
Third Reflected	7.589
Fourth Lateral	8.932
Fourth Reflected	8.944
Fifth Lateral	10.416
Fifth Reflected	10.431
Sixth Lateral	11.900

The results (eleven modes), compared to the halfspace case, are shown in Figure 6.17; the double steepest descent path contribution and the first two modes are shown in Figure 6.18. As can be seen, the match is very good, although there are too many ray arrivals to distinguish them all in the response.

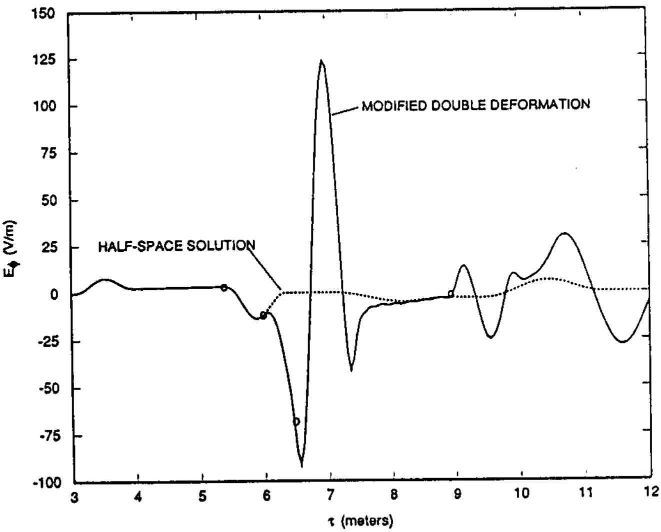

Figure 6.13 Modified double deformation result and halfspace solution; VMD on lossy two-layer medium, source, Eq. (16); circles show various ray arrivals.

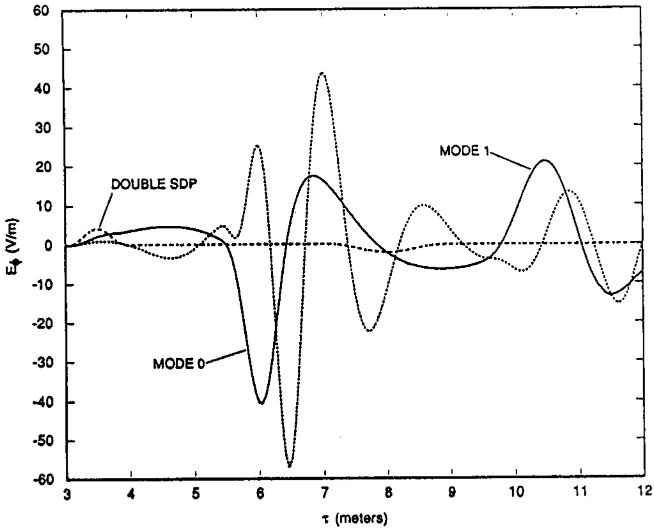

Figure 6.14 MDD: Modes 0 and 1, and double SDP contribution; VMD on lossy two-layer medium, source, Eq. (16).

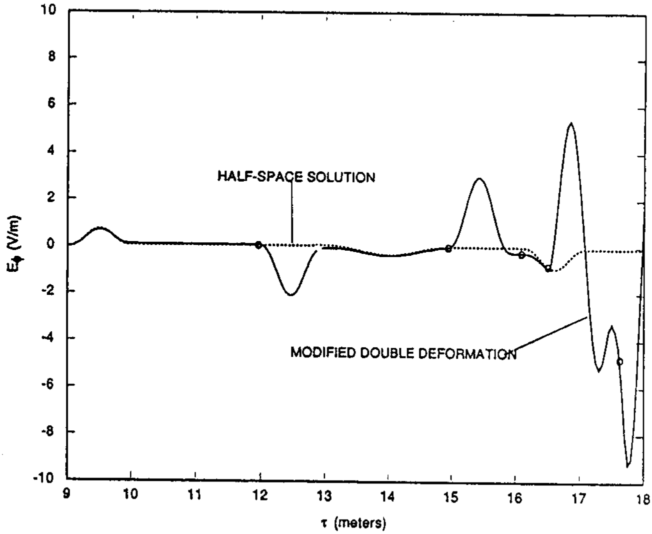

Figure 6.15 Modified double deformation result and halfspace solution, $\rho = 9$; VMD on lossy two-layer medium, source, Eq. (16); circles show various ray arrivals.

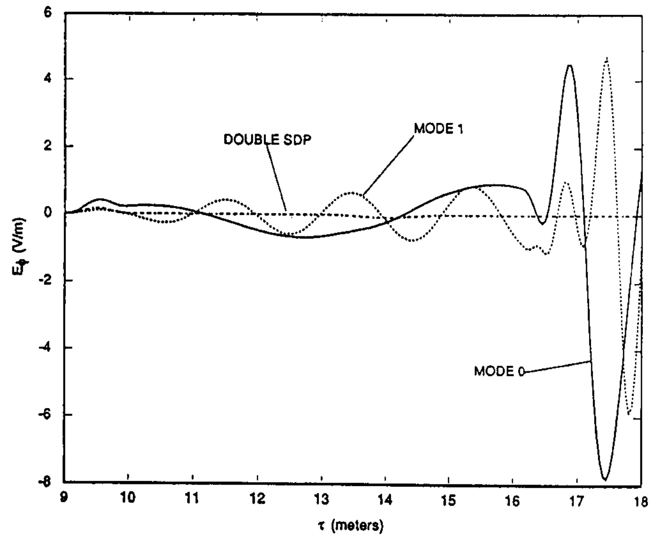

Figure 6.16 MDD: Modes 0 and 1, and double SDP contribution, $\rho = 9$; VMD on lossy two-layer medium, source, Eq. (16).

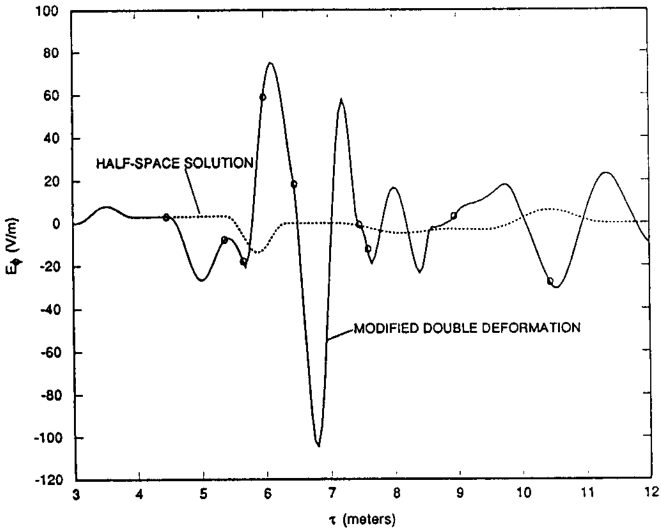

Figure 6.17 Modified double deformation result and halfspace solution, $d = 0.5$; VMD on lossy two-layer medium, source, Eq. (16); circles show various ray arrivals.

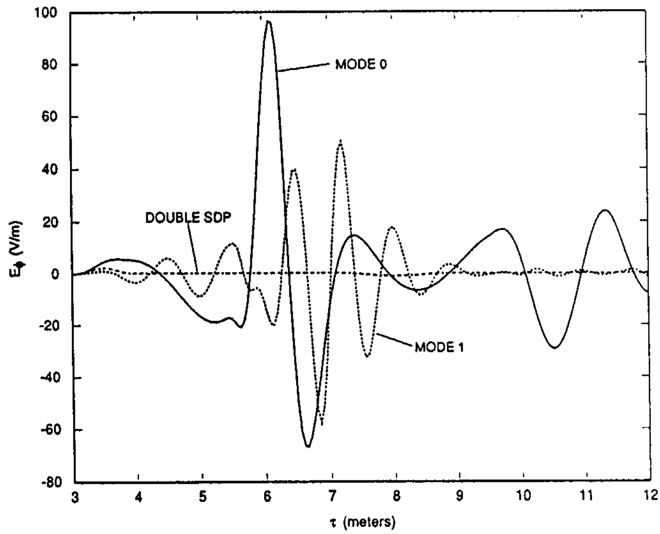

Figure 6.18 MDD: Modes 0 and 1, and double SDP contribution, $d = 0.5$; VMD on lossy two-layer medium, source, Eq. (16).

7. Modified Double Deformation for Vertical Electric Dipole over Two-Layer Medium

a. Single Deformation

The magnetic field on the surface of a two-layer medium from a vertical electric dipole also on the surface a distance ρ away is

$$H_\phi = \frac{1}{8\pi^2}\mathrm{Re}\left\{ i \int_0^\infty dk_0\, e^{-ik_0\tau}\tilde{I}(k_0) \int_{\mathrm{SIP}} dk_\rho \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)[1 + R^{TM}] \right\} \quad (1)$$

where

$$R^{TM} = \frac{R_{01} + R_{12}e^{2ik_{1z}d}}{1 + R_{01}R_{12}e^{2ik_{1z}d}}, \qquad R_{01} = \frac{\epsilon_1 k_z - k_{1z}}{\epsilon_1 k_z + k_{1z}}, \qquad R_{12} = \frac{\epsilon_2 k_{1z} - \epsilon_1 k_{2z}}{\epsilon_2 k_{1z} + \epsilon_1 k_{2z}} \quad (2)$$

and

$$k_z = \sqrt{k_0^2 - k_\rho^2}, \quad k_{1z} = \sqrt{\epsilon_1 k_0^2 - k_\rho^2}, \quad k_{2z} = \sqrt{\epsilon_2 k_0^2 - k_\rho^2} \quad (3)$$

The details of the double deformation technique for the VED are very similar to the VMD case, so we will concentrate on the differences. We first deform the k_ρ integral in (1) to the standard steepest descent path (SDP). Calling the height on SDP_0 q and the height on SDP_2 r, we obtain

$$\mathrm{SDP}_0 = \frac{1}{8\pi^2}\mathrm{Re}\left\{ \int_0^\infty dk_0\, e^{-ik_0\tau}\tilde{I}(k_0) \int_0^\infty dq \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho) \frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \quad (4)$$

where $k_\rho = k_0 + iq$, and

$$\mathrm{SDP}_2 = -\frac{1}{8\pi^2}\mathrm{Re}\left\{ \int_0^\infty dk_0\, e^{-ik_0\tau}\tilde{I}(k_0) \int_0^\infty dr \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho) \frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \quad (5)$$

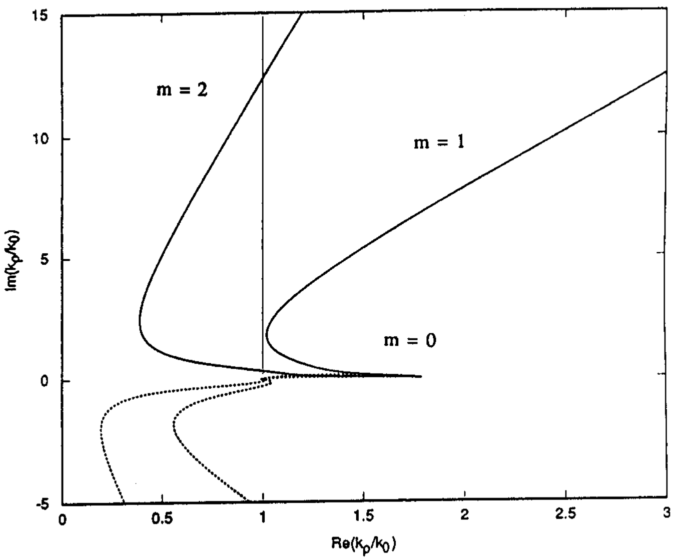

Figure 7.1 k_ρ **pole loci,** $m = 0, 1, 2$**; VED on two-layer medium; vertical line at** $\mathrm{Re}(k_\rho/k_0) = 1$ **is steepest descent path.**

where $k_\rho = \sqrt{\epsilon_2}\, k_0 + iq$. In both expressions, k_z and k_{2z} are evaluated on the UU Riemann sheet. The poles enclosed in the k_ρ plane are zeros of the denominator of (1),

$$1 + R_{01} R_{12} e^{2ik_{1z}d} = 0 \tag{6}$$

We again use a modal approach to study these poles. We let

$$R_{01} = e^{i\phi} \tag{7}$$

and

$$R_{12} = e^{i\psi} \tag{8}$$

where ϕ and ψ are complex numbers whose real parts lie between 0 and 2π . We thus obtain the modal equation

$$\phi + \psi + 2k_{1z}d = (2m+1)\pi, \qquad m = 0, 1, 2, \ldots \tag{9}$$

where the different m 's distinguish the different modes.

Some typical loci of poles ($m = 0, 1, 2$) are shown in Figure 7.1, where, as before, solid lines show poles on the UU Riemann sheet, and dashed lines those on the LU Riemann sheet. The behavior is similar to that for the VMD over two-layer medium case; we have the standard intersection points of the locus with SDP_0 at $k_0 = k_{cm}$, $k_0 = k_{bm}$ and $k_0 = k_{am}$. This gives us the same guided and leaky wave poles; however, the loci for modes 0 and 1 are different. These loci only intersect SDP_0 once, when $k_0 = k_{cm}$. The contribution to the total field from these k_ρ poles is

$$\sum_{m=0}^{\infty} \frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\Bigg\{ \int_{k_{cm}}^{\infty} dk_0\, e^{-ik_0\tau} \tilde{I}(k_0) \left[\frac{k_\rho k_z k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 + ik_z d[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2] + \frac{\epsilon_1\epsilon_2 k_0^2 k_z}{k_{2z}} \frac{[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]}{[\epsilon_1\epsilon_2 k_0^2 - (\epsilon_2 + \epsilon_1)k_\rho^2]}} \right] \Bigg\} \tag{10}$$

for the guided poles, where k_z is evaluated on the LU Riemann sheet, and

$$\sum_{m=2}^{\infty} \frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\Bigg\{ \int_{k_{am}}^{k_{bm}} dk_0\, e^{-ik_0\tau} \tilde{I}(k_0) \left[\frac{k_\rho k_z k_{1z}^2 H_1^{(1)}(k_\rho \rho)}{\epsilon_1 k_0^2 + ik_z d[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2] + \frac{\epsilon_1\epsilon_2 k_0^2 k_z}{k_{2z}} \frac{[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]}{[\epsilon_1\epsilon_2 k_0^2 - (\epsilon_2 + \epsilon_1)k_\rho^2]}} \right] \Bigg\} \tag{11}$$

for the leaky poles, where k_z is evaluated on the UU Riemann sheet. The total field is thus given by the sum of (4), (5), (10), and (11).

b. Double Deformation

Using our standard damped sinusoidal source,

$$I(\tau) = I_0 \tau \sin \omega_0 \tau e^{-\alpha_0 \tau} \tag{12}$$

with its Fourier transform

$$\tilde{I}(k_0) = \frac{iI_0}{2} \left[\frac{1}{(k_0 + i\alpha_0 + \omega_0)^2} - \frac{1}{(k_0 + i\alpha_0 - \omega_0)^2} \right] \tag{13}$$

we deform the k_0 integral in (4) upward to the positive imaginary k_0 axis for $\tau < \rho$ and downward to the negative imaginary k_0 axis for

$\tau > \rho$; likewise, we deform the k_0 integral in (5) upward to the positive imaginary k_0 axis for $\tau < \sqrt{\epsilon_2}\,\rho$ and downward to the negative imaginary k_0 axis for $\tau > \sqrt{\epsilon_2}\,\rho$. The total for $\tau < \rho$ is analytically zero; the details of this are almost identical to the VMD case and are therefore omitted. As before, we will concentrate on the case $\tau < \sqrt{\epsilon_2}\,\rho$. Deforming (4) downward and (5) upward, then, our answer is in two parts: the double steepest descent path contributions, and the residues of the enclosed poles in the k_0 plane.

The derivation of the double steepest descent path integrals is almost identical to the VMD case and are omitted. For DSDP$_0$, we obtain

$$\mathrm{DSDP}_0 = -\frac{1}{8\pi^2}\mathrm{Re}\left\{ i \int_0^\infty dp\, e^{-p\tau}\tilde{I}(-ip) \right.$$

$$\left. \int_{(\sqrt{\epsilon_2}+1)p}^\infty dq\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho) \frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \tag{14}$$

and for DSDP$_2$, we obtain

$$\mathrm{DSDP}_2 = -\frac{1}{8\pi^2}\mathrm{Re}\left\{ i \int_0^\infty ds\, e^{s\tau}\tilde{I}(is) \right.$$

$$\left. \int_{(\sqrt{\epsilon_2}-1)s}^\infty du\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho\rho) \frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \tag{15}$$

The poles enclosed are due to singularities of $\tilde{I}(k_0)$ and zeros of (6) of (4). For this source, the source pole contribution is:

$$-\frac{I_0}{8\pi}\mathrm{Re}\left\{ \int_0^\infty dq\, \frac{\partial}{\partial k_0} \left[e^{-ik_0\tau} \frac{k_\rho^2}{k_z} \right.\right.$$

$$\left.\left. H_1^{(1)}(k_\rho\rho) \frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right]_{k_0=\omega_0-i\alpha_0} \right\} \tag{16}$$

We now consider the contribution due to the zeros of (6). Using the same modal technique, we can find loci of these poles. These are shown, for $m = 0, 1,$ and 2, in Figure 7.2. For all modes $m \geq 2$, two sections of the locus are enclosed by the deformation of the integration path to the negative imaginary axis: the part from $k_0 = k_{cm}$ towards $k_0 = k_{bm}$,

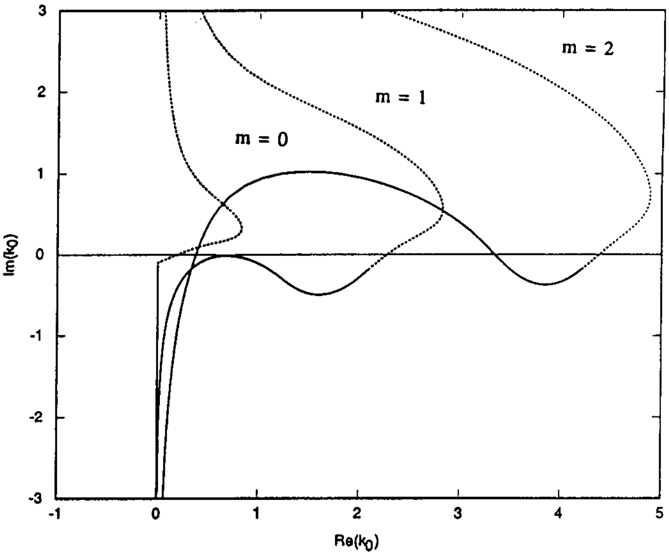

Figure 7.2 k_0 **pole loci,** $m = 0, 1, 2$**; VED on two-layer medium.**

which we call "k_0 finite", and from $k_0 = k_{am}$ to $k_0 \to -i\infty$, which we call "k_0 infinite". For $m \leq 1$, since the pole locus in the k_ρ plane never crosses SDP$_0$, there are no finite poles. It turns out that part of the $m = 0$ locus in the k_0 plane lies along the negative imaginary axis; these poles are only half-enclosed by the deformation. The integrals for the k_0 poles are therefore

$$\sum_{m=1}^{\infty} -\frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ \int_{q_{am}}^{\infty} dq\, n(q) e^{-ik_0\tau} \tilde{I}(k_0) \cdot k_\rho^2 k_{1z}^2 k_z H_1^{(1)}(k_\rho \rho) \cdot \right.$$
$$\left[\epsilon_1 k_0 k_\rho q + k_z d[(\epsilon_1 - 1)k_0 - iq][\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2] \right.$$
$$\left.\left. + \frac{\epsilon_1 \epsilon_2 k_0 k_\rho k_z q}{k_{2z}} \frac{[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]}{[\epsilon_1 \epsilon_2 k_0^2 - (\epsilon_2 + \epsilon_1)k_\rho^2]} \right]^{-1} \right\} \tag{17}$$

for the k_0 infinite poles, and

$$\sum_{m=2}^{\infty} -\frac{\epsilon_1}{2\pi(\epsilon_1 - 1)} \mathrm{Re}\left\{ \int_{q_{cm}}^{q_{bm}} dq\, n(q) e^{-ik_0\tau} \tilde{I}(k_0) \cdot k_\rho^2 k_{1z}^2 k_z H_1^{(1)}(k_\rho \rho) \cdot \right.$$

$$\left[\epsilon_1 k_0 k_\rho q + k_z d[(\epsilon_1 - 1)k_0 - iq][\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]\right.$$

$$\left. + \frac{\epsilon_1 \epsilon_2 k_0 k_\rho k_z q}{k_{2z}} \frac{[\epsilon_1 k_0^2 - (\epsilon_1 + 1)k_\rho^2]}{[\epsilon_1 \epsilon_2 k_0^2 - (\epsilon_2 + \epsilon_1)k_\rho^2]}\right]^{-1} \Bigg\} \tag{18}$$

for the k_0 finite poles, where q_{am}, q_{bm}, and q_{cm} are the values of q where the locus in the k_ρ plane crosses SDP$_0$. The function $n(q)$ has magnitude 1 when the pole is fully enclosed, and magnitude $1/2$ when the pole is only half enclosed. Furthermore, $n(q)$ is positive when the pole is on the UU Riemann sheet, and negative on the LU sheet.

The complete double deformation solution is thus made up of seven parts: the k_ρ guided modes (10), the k_ρ leaky modes (11), the k_0 infinite modes (17), the k_0 finite modes (18), the source pole contribution (16), and the two double steepest descent paths (14) and (15).

c. *Modified Double Deformation*

Applying the modification as we did in the halfspace case, with one integration by parts necessary to satisfy Jordan's Lemma,

$$\tilde{I}(k_0) = \int_0^\infty d\tau'\, I(\tau') e^{ik_0\tau'} = \frac{i}{k_0}\int_0^\infty d\tau'\, I'(\tau') e^{ik_0\tau'} \tag{19}$$

We now split the Fourier transform of the current into two parts:

$$\tilde{I}(k_0) = \frac{i}{k_0}\left[\int_0^\gamma d\tau'\, I'(\tau') e^{ik_0\tau'} + \int_\gamma^\infty d\tau'\, I'(\tau') e^{ik_0\tau'}\right] \tag{20}$$

$$\equiv \frac{i}{k_0}\left[\tilde{I}_{1b}(k_0,\gamma) + \tilde{I}_{1a}(k_0,\gamma)\right] \tag{21}$$

We substitute this into our expression for SDP$_0$ (4) with $\gamma = \tau - \rho$ and into SDP$_2$ (5) with $\gamma = \tau - \sqrt{\epsilon_2}\,\rho$. We then deform the k_0 integral of the parts containing $\tilde{I}_{1a}$ upward, the parts containing $\tilde{I}_{1b}$ downward. However, since we have introduced a pole at the origin, we must deform our integration paths in the k_0 up to $i\delta$ before separating, as we did in the coated perfect conductor case. If we also split the integrals over the poles in the k_ρ plane into a part containing $\tilde{I}_{1a}$ and one containing $\tilde{I}_{1b}$, the total for $\tilde{I}_{1a}$ is identically zero, which satisfies causality; the proof is similar to the VMD over a two-layer medium case and is omitted. If we confine ourselves to times $\tau < \sqrt{\epsilon_2}\,\rho$, we can concentrate on the part of SDP$_0$ containing $\tilde{I}_{1b}$. We will deform the k_0 integral downward, to $k_0 = -ip$. In so doing, we obtain three contributions: the residue from the double steepest descent path contributions, the pole at the origin, and the residues of the enclosed poles in the k_0 plane. As in the VMD case, we can combine the double steepest descent path contributions with the origin pole to obtain:

$$\text{DSDP} = \frac{1}{8\pi^2}\text{Re}\left\{ i\int_0^\infty dq \int_{-q/(\sqrt{\epsilon_2}-1)}^{q/(\sqrt{\epsilon_2}+1)} dp\, e^{-p\tau}\frac{\tilde{I}_{1b}(-ip,\tau-\rho)}{p} \right.$$
$$\left. \frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \tag{22}$$

where the integration path in the p plane detours around the origin pole.

The contributions from the poles of (6) in the k_0 plane are very similar to the standard double deformation results, except that there is no source pole. We obtain

$$\sum_{m=1}^{\infty} -\frac{\epsilon_1}{2\pi(\epsilon_1-1)}\text{Re}\left\{ i\int_{q_{am}}^\infty dq\, n(q)\frac{\tilde{I}_{1b}(k_0,\tau-\rho)}{k_0}e^{-ik_0\tau}\cdot \right.$$
$$k_\rho^2 k_{1z}^2 k_z H_1^{(1)}(k_\rho\rho)\cdot$$
$$\left[\epsilon_1 k_0 k_\rho q + k_z d[(\epsilon_1-1)k_0 - iq][\epsilon_1 k_0^2 - (\epsilon_1+1)k_\rho^2]\right.$$
$$\left.\left. + \frac{\epsilon_1\epsilon_2 k_0 k_\rho k_z q}{k_{2z}}\frac{[\epsilon_1 k_0^2-(\epsilon_1+1)k_\rho^2]}{[\epsilon_1\epsilon_2 k_0^2 - (\epsilon_2+\epsilon_1)k_\rho^2]}\right]^{-1}\right\} \tag{23}$$

for the k_0 infinite poles,

$$\sum_{m=1}^{\infty} -\frac{\epsilon_1}{2\pi(\epsilon_1-1)}\text{Re}\left\{ i\int_0^{q_{bm}} dq\, n(q)\frac{\tilde{I}_{1b}(k_0,\tau-\rho)}{k_0}e^{-ik_0\tau}\cdot \right.$$
$$k_\rho^2 k_{1z}^2 k_z H_1^{(1)}(k_\rho\rho)\cdot$$
$$\Big[\epsilon_1 k_0 k_\rho q + k_z d[(\epsilon_1-1)k_0 - iq][\epsilon_1 k_0^2 - (\epsilon_1+1)k_\rho^2]$$

$$+\frac{\epsilon_1\epsilon_2 k_0 k_\rho k_z q}{k_{2z}}\frac{[\epsilon_1 k_0^2-(\epsilon_1+1)k_\rho^2]}{[\epsilon_1\epsilon_2 k_0^2-(\epsilon_2+\epsilon_1)k_\rho^2]}\Bigg]^{-1}\Bigg\} \tag{24}$$

and for the k_0 finite poles. We also have the contributions for the k_ρ poles, which are now

$$\sum_{m=0}^{\infty}\frac{\epsilon_1}{2\pi(\epsilon_1-1)}\mathrm{Re}\Bigg\{i\int_{k_{cm}}^{\infty}dk_0\,\frac{\tilde{I}_{1b}(k_0,\tau-\rho)}{k_0}e^{-ik_0\tau}\left[\frac{k_\rho k_z k_{1z}^2 H_1^{(1)}(k_\rho\rho)}{\epsilon_1 k_0^2+ik_z d[\epsilon_1 k_0^2-(\epsilon_1+1)k_\rho^2]}\right]\Bigg\} \tag{25}$$

for the guided poles and and

$$\sum_{m=1}^{\infty}\frac{\epsilon_1}{2\pi(\epsilon_1-1)}\mathrm{Re}\Bigg\{\int_{k_{am}}^{k_{bm}}dk_0\,\frac{\tilde{I}_{1b}(k_0,\tau-\rho)}{k_0}e^{-ik_0\tau}\left[\frac{k_\rho k_z k_{1z}^2 H_1^{(1)}(k_\rho\rho)}{\epsilon_1 k_0^2+ik_z d[\epsilon_1 k_0^2-(\epsilon_1+1)k_\rho^2]}\right]\Bigg\} \tag{26}$$

for the leaky poles. The total modified double deformation solution is therefore given by the sum of the k_ρ guided modes (25), the k_ρ leaky modes (26), the k_0 infinite modes (23) the k_0 finite modes (24), and the double steepest descent path contributions (22).

d. Results

Results are shown for the configuration with $\rho=3$, $d=1$, and $\epsilon_1=3.2$. Two sources are used; the first is the damped sinusoid (12) with $I_0=1$, $\omega_0=1$ and $\alpha_0=0.5$. The second is the smoothed pulse, discussed in Section 1.

First, we solve the damped sinusoid with the standard double deformation technique. The total answer, with eleven modes, is shown in Figure 7.3, where it is compared to the closed-form halfspace result; the source pole contribution and the first two modes are in Figure 7.4. As before, all the arrivals can be clearly distinguished in the final result.

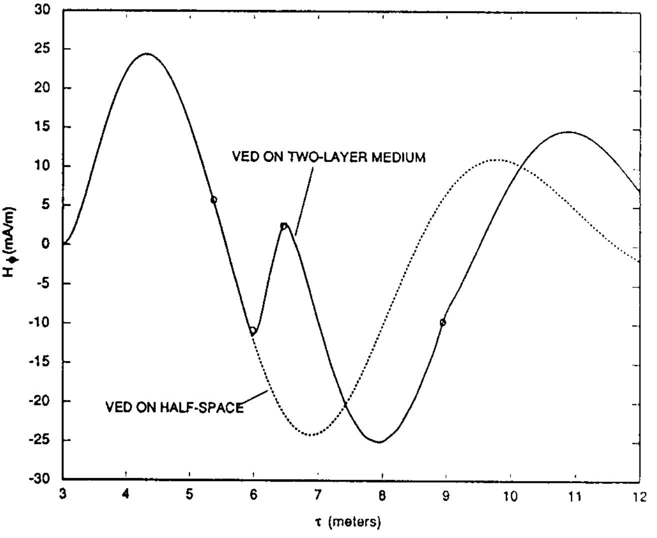

Figure 7.3 Total double deformation and halfspace solution; VED on two-layer medium, source, Eq. (12); circles show various ray arrivals.

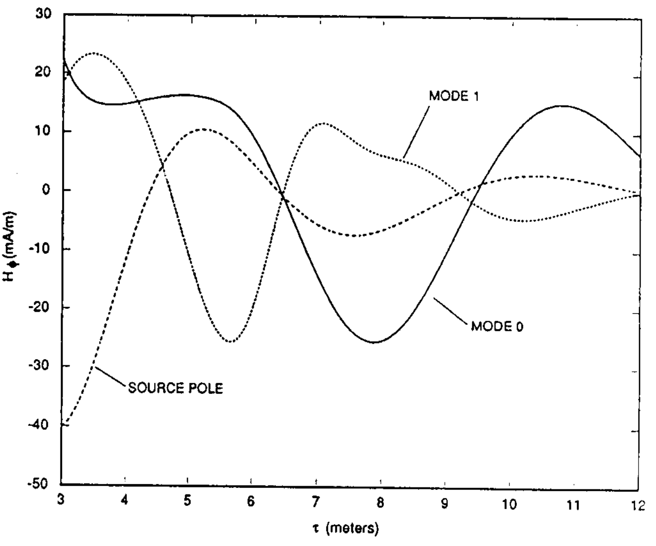

Figure 7.4 Modes 0 and 1, and source pole contribution; VED on two-layer medium, source, Eq. (12).

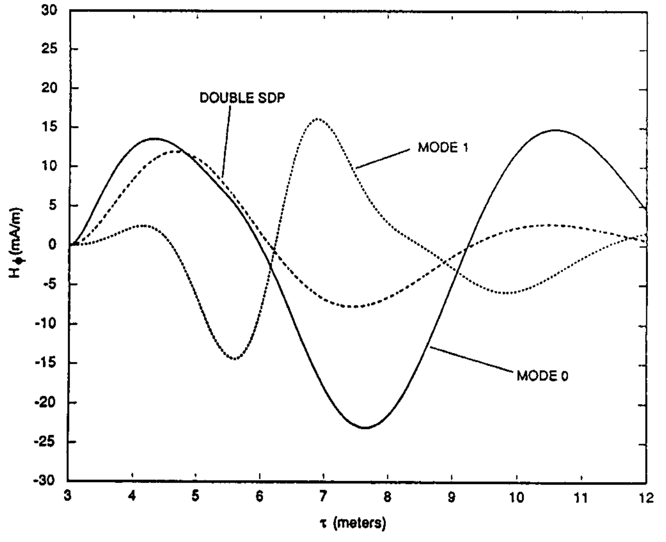

Figure 7.5 MDD: Modes 0 and 1, and origin pole contribution; VED on two-layer medium, source, Eq. (12).

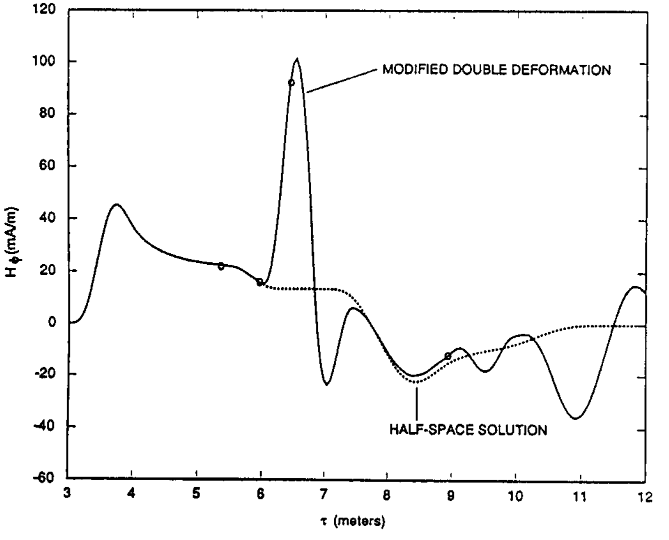

Figure 7.6 Modified double deformation result and halfspace solution; VED on two-layer medium, source, Eq. (16); circles show various ray arrivals.

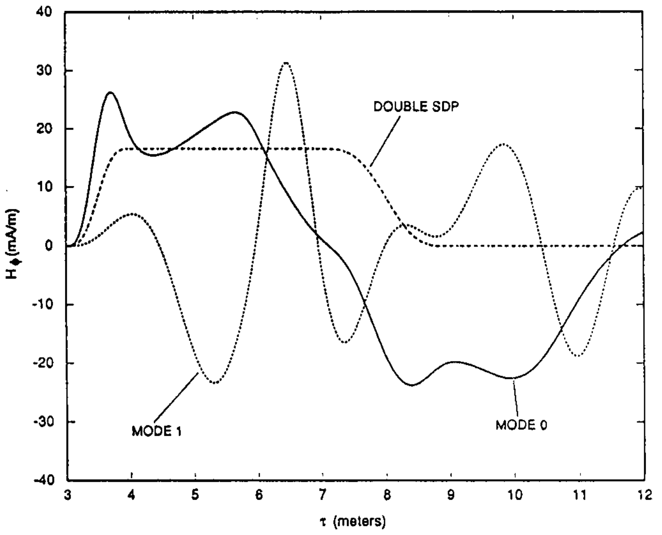

Figure 7.7 MDD: Modes 0 and 1, and origin pole contribution; VED on two-layer medium, source, Eq. (16).

Next, we solve the same case, but using the modified double deformation technique. The total result overlays the double deformation result; the double steepest descent path contribution and the first two modes are in Figure 7.5. As usual, the agreement between the modified and standard results is very good.

Finally, we use zT deformation to solve the case with a pulse source. Again, the total response (eleven modes, Figure 7.6) agrees with the closed-form halfspace solution very well up until the arrival of the first lateral wave. The double steepest descent path and the first two modes are shown in Figure 7.7.

Appendix A: Causality for VMD over Coated Perfect Conductor.

In the double deformation case, the expression for the total electric field for time $\tau < \rho$ can be written as follows:

$$\sum_{m=0}^{\infty} \frac{\eta_0}{4\pi} \mathrm{Re}\left\{ i \int_{k_{cm}}^{\infty} dk_0\, k_0 e^{-ik_0\tau} \tilde{I}(k_0) \mathrm{Res}\left[\frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)[1 + R^{TE}] \right] \right\} \tag{A1}$$

for the guided poles (k_z on bottom Riemann sheet),

$$\sum_{m=1}^{\infty} \frac{\eta_0}{4\pi} \mathrm{Re}\left\{ i \int_{k_{am}}^{k_{bm}} dk_0\, k_0 e^{-ik_0\tau} \tilde{I}(k_0) \mathrm{Res}\left[\frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)[1 + R^{TE}] \right] \right\} \tag{A2}$$

for the leaky poles, (k_z on top Riemann sheet),

$$\sum_{m=0}^{\infty} \frac{\eta_0}{4\pi} \mathrm{Re}\left\{ \int_{0}^{\infty} dq\, \mathrm{Res}\left[k_0 e^{-ik_0\tau} \tilde{I}(k_0) \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)[1 + R^{TE}] \right] \right\} \tag{A3}$$

for the k_0 infinite poles, (k_z on bottom Riemann sheet),

$$\sum_{m=1}^{\infty} \frac{\eta_0}{4\pi} \mathrm{Re}\left\{ \int_{q_{bm}}^{q_{am}} dq\, \mathrm{Res}\left[k_0 e^{-ik_0\tau} \tilde{I}(k_0) \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)[1 + R^{TE}] \right] \right\} \tag{A4}$$

for the k_0 finite poles (k_z on top Riemann sheet), and

$$\mathrm{DSDP} = \frac{\eta_0}{8\pi^2} \mathrm{Re}\left\{ i \int_0^{\infty} dp\, p e^{-p\tau} \tilde{I}(ip) \int_0^{\infty} dq\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1 + R_{01})^2 (1 - e^{2ik_{1z}d})^2}{(1 - R_{01} e^{2ik_{1z}d})(R_{01} - e^{2ik_{1z}d})} \right\} \tag{A5}$$

where $k_\rho = i(q + p)$, $k_z = \sqrt{q^2 + 2pq}$, and $k_{1z} = \sqrt{q^2 + 2pq - (\epsilon_1 - 1)p^2}$. The change in sign in the k_0 pole integrals from (19) is due to the deformation up instead of down. There is no source pole, because any physical source must remain finite as $\tau' \to \infty$, and so cannot have any poles in the upper half k_0 plane.

We will show that each integral in $(A1)$ cancels with the corresponding one in $(A3)$, each one in $(A2)$ with one in $(A4)$, and that the double SDP integral, $(A5)$, vanishes identically. Since all of the singularities are single poles, if we write

$$1 + R^{TE} = \frac{(1 + R_{01})(1 - e^{2ik_{1z}d})}{1 - R_{01}e^{2ik_{1z}d}} = \frac{f(k_0, k_\rho)}{g(k_0, k_\rho)} \tag{A6}$$

the residues are just given by taking the partial derivative of the denominator. This gives:

$$\sum_{m=0}^{\infty} \frac{\eta_0}{4\pi}\mathrm{Re}\left\{ i \int_{k_{cm}}^{\infty} dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{f(k_0,k_\rho)}{\frac{\partial g(k_0,k_\rho)}{\partial k_\rho}} \right\} \tag{A7}$$

$$\sum_{m=1}^{\infty} \frac{\eta_0}{4\pi}\mathrm{Re}\left\{ i \int_{k_{am}}^{k_{bm}} dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{f(k_0,k_\rho)}{\frac{\partial g(k_0,k_\rho)}{\partial k_\rho}} \right\} \tag{A8}$$

$$\sum_{m=0}^{\infty} \frac{\eta_0}{4\pi}\mathrm{Re}\left\{ \int_{0}^{\infty} dq\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{f(k_0,k_\rho)}{\frac{\partial g(k_0,k_\rho)}{\partial k_0}} \right\} \tag{A9}$$

$$\sum_{m=1}^{\infty} \frac{\eta_0}{4\pi}\mathrm{Re}\left\{ \int_{q_{bm}}^{q_{am}} dq\, k_0 e^{-ik_0\tau}\tilde{I}(k_0)\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{f(k_0,k_\rho)}{\frac{\partial g(k_0,k_\rho)}{\partial k_0}} \right\} \tag{A10}$$

Now we change the q integrals to k_0 integrals. For every value of q along the paths of integration, we have a value of k_0 that is the pole for that value of q. Let us call this value k_{0p}. For every value of q along this path of integration, we have

$$g(k_0, k_\rho) = 1 - R_{01}e^{2ik_{1z}d} = 0 \tag{A11}$$

This means that

$$\frac{d}{dq}(1 - R_{01}e^{2ik_{1z}d}) = 0 \tag{A12}$$

and thus

$$\begin{aligned}\frac{d}{dq}(1 - R_{01}e^{2ik_{1z}d}) &= \frac{\partial}{\partial k_\rho}(1 - R_{01}e^{2ik_{1z}d})\frac{dk_\rho}{dq} + \frac{\partial}{\partial k_0}(1 - R_{01}e^{2ik_{1z}d})\frac{dk_0}{dq} \\ &= 0\end{aligned} \tag{A13}$$

Since $dk_\rho/dq = i$ we can solve $(A13)$ for dq in terms of dk_0. We obtain

$$dq = i\frac{\frac{\partial}{\partial k_0}(1 - R_{01}e^{2ik_{1z}d})}{\frac{\partial}{\partial k_\rho}(1 - R_{01}e^{2ik_{1z}d})}dk_0 \tag{A14}$$

With this change of variable, it is clear that, since the integrands are the same in $(A7)$ and $(A9)$ and since the two paths form a closed contour enclosing no singularities, they cancel each other. We can also deform the integrals in $(A10)$ to those in $(A8)$ merely by changing sign and adding a contour at infinity whose contribution vanishes by Jordan's Lemma. So the only part of the field left is $(A5)$.

Now, we need to consider various parts of the integrand of $(A5)$. First we notice that $\tilde{I}(ip)$ and $k_z = \sqrt{q^2 + 2pq}$ are purely real. We have that $k_\rho = i(q + p)$ is positive imaginary, so

$$H_1^{(1)}(i(q+p)\rho) = -\frac{2}{\pi}K_1((q+p)\rho) \tag{A15}$$

is purely real. Only $k_{1z} = \sqrt{q^2 + 2pq - (\epsilon_1 - 1)p^2}$ will have a critical point, at $q = (\sqrt{\epsilon_1} - 1)p$, and thus we have only two regions to consider.

Region 1: $0 \le q \le (\sqrt{\epsilon_1} - 1)p$. Here, k_{1z} is complex. This means that R_{01} is a complex number with a magnitude of unity, or that we can write $R_{01} = e^{2i\psi}$, where ψ is real. if we let $e^{2ik_{1z}d} \equiv X$, where X is also real, and we obtain the same result as in (21), which gives that the contribution from this region to $(A5)$ is zero.

Region 2: Here, $q \ge (\sqrt{\epsilon_1} - 1)p$. Now, both k_z and k_{1z} are real. This means that R_{01} is real, but that $|e^{2ik_{1z}d}| = 1$. We obtain the same result as in (22), and here also the contribution to $(A5)$ is zero. Thus, the total field for $\tau < \rho$ is zero, which satisfies the causality requirement.

The above argument also holds for the part of the modified double deformation results that contains $\tilde{I}_{2a}$, by just substituting $-(1/k_0^2)\tilde{I}_{2a}(k_0, \tau - \rho)$ for $\tilde{I}(k_0)$. The change in the p integration range from 0 to ∞ to δ to ∞ due to the detour around the pole at the origin does not alter the result for the double steepest descent path. Thus, causality is satisfied for modified double deformation, as well, and in the stronger form we saw for the halfspace case.

Appendix B: Double Steepest Descent Path Contributions for VMD over Two-Layer Medium

We have four separate double steepest descent paths to consider: DSDP$_0$ when we deform the k_0 integral up, DSDP$_0$ when we deform the k_0 integral down, DSDP$_2$ when we deform the k_0 integral up, and DSDP$_2$ when we deform the k_0 integral down. Let us examine DSDP$_0$, deformed up, first. After manipulation of (4) from Section 6, we obtain

$$\mathrm{DSDP}_{0u} = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty ds\, se^{s\tau}\tilde{I}(is) \right.$$
$$\left. \int_0^\infty dq\,\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B1)$$

where $k_\rho = i(q+s)$, $k_z = \sqrt{q^2+2sq}$, $k_{1z} = \sqrt{q^2+2sq-(\epsilon_1-1)s^2}$, and
$k_{2z} = \sqrt{q^2+2sq-(\epsilon_2-1)s^2}$. First, we notice that $\tilde{I}(is)$ and k_z are purely real, and that

$$H_1^{(1)}(i(q+s)\rho) = -\frac{2}{\pi}K_1((q+s)\rho) \quad (B2)$$

is also real. There will be two critical points on the q integration path: $q = (\sqrt{\epsilon_1}-1)s$, where k_{1z} shifts from real to imaginary, and $q = (\sqrt{\epsilon_2}-1)s$, where k_{2z} shifts from real to imaginary. We then have three distinct regions in q to examine.

Region 1: $0 \le q \le (\sqrt{\epsilon_1}-1)s$. We have that k_{1z} and k_{2z} are purely imaginary. This means that R_{01} is a complex number with a magnitude of unity, or that we can write $R_{01} = e^{2i\psi}$, where ψ is real. Let $R_{12}e^{2ik_{1z}d} \equiv X$, where X is also real, and we obtain:

$$\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} = \frac{(1+e^{2i\psi})^2(1+X)^2}{(1+e^{2i\psi}X)(e^{2i\psi}+X)}$$
$$= \frac{(e^{i\psi}-e^{-i\psi})^2(1+X)^2}{(e^{-i\psi}+e^{i\psi}X)(e^{i\psi}+e^{-i\psi}X)} \quad (B3)$$

By inspection, we see that this expression is equal to its complex conjugate and is therefore real. Since k_z is real, we can see by examining $(B1)$ that the contribution from this range is zero.

Region 2: Here, $(\sqrt{\epsilon_1}-1)s \le q \le (\sqrt{\epsilon_2}-1)s$. Now, both k_z and k_{1z} are real. This means that R_{01} is real, but that both R_{12} and $e^{2ik_{1z}d}$ are complex numbers with magnitude unity. Writing $R_{12}e^{2ik_{1z}d} = e^{2i\xi}$, we write

$$\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} = -\frac{(1+R_{01})^2(e^{i\xi}-e^{i\xi})^2}{(R_{01}e^{i\xi}-e^{-i\xi})(R_{01}e^{-i\xi}-e^{i\xi})} \tag{B4}$$

Again, we can see that this expression equals its complex conjugate and is therefore real, and therefore the q integration for this range is zero.

Region 3: Unfortunately, the contribution from this region does not vanish. we are left with

$$\mathrm{DSDP}_{0u} = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty ds \int_0^{(\sqrt{\epsilon_2}-1)s} dq\, s e^{s\tau}\tilde{I}(is) \frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \tag{B5}$$

Next, let us consider DSDP $_0$, deformed down. We obtain

$$\mathrm{DSDP}_{0d} = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dp\, p e^{-p\tau}\tilde{I}(-ip) \int_0^\infty dq\, \frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \tag{B6}$$

where $k_\rho = i(q-p)$, $k_z = \sqrt{q^2-2pq}$, $k_{1z} = \sqrt{q^2-2pq-(\epsilon_1-1)p^2}$, and
$k_{2z} = \sqrt{q^2-2pq-(\epsilon_2-1)p^2}$. We have $\tilde{I}(-ip)$ purely real, as before. There will be three critical points on the q integration path, $q=p$, $q=2p$, $q=(1+\sqrt{\epsilon_1})p$, and $q=(1+\sqrt{\epsilon_2})p$, and thus five separate regions to consider.

Region 1: $0 \le q \le p$. Here, $k_\rho = -i(p-q)$, so

$$H_1^{(1)}(-i(p-q)\rho) = \frac{2}{\pi}K_1((p-q)\rho) - 2iI_1((p-q)\rho). \tag{B7}$$

Also, $k_z = -i\sqrt{2pq - q^2}$, $k_{1z} = -i\sqrt{(\epsilon_1 - 1)p^2 + 2pq - q^2}$, and $k_{2z} = -i\sqrt{(\epsilon_2 - 1)p^2 + 2pq - q^2}$ are negative imaginary. Thus, R_{01}, R_{12}, and $e^{2ik_{1z}d}$ are real. This leaves us with

$$\mathrm{DSDP}_{01} = \frac{\eta_0}{4\pi^3}\left\{\int_0^\infty dp\, p e^{-p\tau}\tilde{I}(-ip)\int_0^p dq\,\frac{(p-q)^2}{\sqrt{2pq-q^2}}K_1((p-q)\rho)\right.$$
$$\left.\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B8)$$

Region 2: $p \le q \le 2p$. Here, $k_\rho = i(q-p)$, so

$$H_1^{(1)}(i(q-p)\rho) = -\frac{2}{\pi}K_1((q-p)\rho). \quad (B9)$$

However, k_z, k_{1z}, and k_{2z} all remain negative imaginary. This gives

$$\mathrm{DSDP}_{02} = -\frac{\eta_0}{4\pi^3}\left\{\int_0^\infty dp\, p e^{-p\tau}\tilde{I}(-ip)\int_p^{2p} dq\,\frac{(q-p)^2}{\sqrt{2pq-q^2}}K_1((q-p)\rho)\right.$$
$$\left.\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B10)$$

If we make the substitution $q = 2p - u$, we can write $(B10)$ as

$$\mathrm{DSDP}_{02} = -\frac{\eta_0}{4\pi^3}\left\{\int_0^\infty dp\, p e^{-p\tau}\tilde{I}(-ip)\int_0^{p} du\,\frac{(p-u)^2}{\sqrt{2pu-u^2}}K_1((p-u)\rho)\right.$$
$$\left.\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B11)$$

where, in terms of u, $k_z = -i\sqrt{2pu - u^2}$, $k_{1z} = -i\sqrt{(\epsilon_1 - 1)p^2 + 2pu - u^2}$, and $k_{2z} = -i\sqrt{(\epsilon_2 - 1)p^2 + 2pu - u^2}$. It is clear, by comparison of $(B11)$ with $(B8)$, that the contributions from regions 1 and 2 exactly cancel each other.

Region 3: $2p \le q \le (1+\sqrt{\epsilon_1})p$. Here, k_ρ is positive imaginary, as before, so $H_1^{(1)}$ is purely real. The change is that k_z is real, while k_{1z} and k_{2z} remain complex. As for DSDP $_{1u}$, region 1, this means that the contribution from this range is zero.

Region 4: Here, $(1+\sqrt{\epsilon_1})p \leq q \leq (1+\sqrt{\epsilon_2})p$. Now, both k_z and k_{1z} are real. As for DSDP$_{1u}$, region 2, this means that the q integration from this range is also zero.

Region 5: The contribution from this region does not vanish. Interchanging the p and q integrations, we are left with

$$\text{DSDP}_{0d} = \frac{\eta_0}{8\pi^2}\text{Re}\left\{ \int_0^\infty dq\, i \int_0^{q/(\sqrt{\epsilon_2}+1)} dp\, p e^{-p\tau} \tilde{I}(-ip) \right.$$
$$\left. \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \quad (B12)$$

We turn now to DSDP$_2$. First, deforming upward, we obtain

$$\text{DSDP}_{2u} = -\frac{\eta_0}{8\pi^2}\text{Re}\left\{ i \int_0^\infty ds\, s e^{s\tau} \tilde{I}(is) \right.$$
$$\left. \int_0^\infty dr\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \quad (B13)$$

where $k_\rho = i(r+\sqrt{\epsilon_2}\, s)$, $k_z = \sqrt{(\epsilon_2-1)s^2+2\sqrt{\epsilon_2}\, rs+r^2}$, $k_{1z} = \sqrt{(\epsilon_2-\epsilon_1)s^2+2\sqrt{\epsilon_2}\, rs+r^2}$, and $k_{2z} = \sqrt{r^2+2\sqrt{\epsilon_2}\, rs}$. Since k_ρ is imaginary, but all the others are real over the entire r integration range, we have only one region. If we make the transformation $r = u-(\sqrt{\epsilon_2}-1)s$, we obtain

$$\text{DSDP}_{2u} = -\frac{\eta_0}{8\pi^2}\text{Re}\left\{ i \int_0^\infty ds\, s e^{s\tau} \tilde{I}(is) \int_{(\sqrt{\epsilon_2}-1)s}^\infty du\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \right.$$
$$\left. \frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \right\} \quad (B14)$$

where $k_\rho = i(u+s)$, $k_z = \sqrt{u^2+2su}$, $k_{1z} = \sqrt{u^2+2su-(\epsilon_1-1)s^2}$, and
$k_{2z} = \sqrt{u^2+2su-(\epsilon_2-1)s^2}$. Comparing this with $(B5)$, we see that we can combine these two terms:

$$\text{DSDP}_u = \frac{\eta_0}{8\pi^2}\text{Re}\left\{ i \int_0^\infty ds\, s e^{s\tau} \tilde{I}(is) \int_{(\sqrt{\epsilon_2}-1)s}^\infty du\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \right.$$

$$\left[\frac{(1+R_{01})^2(1+R_{12}e^{2ik_{1z}d})^2}{(1+R_{01}R_{12}e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} - \frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right]\Bigg\} \quad (B15)$$

Since k_z, k_{1z} and k_{2z} are all real, we can write this as follows:

$$\mathrm{DSDP}_u = \frac{\eta_0}{8\pi^2}\mathrm{Re}\Bigg\{ i\int_0^\infty ds\, se^{s\tau}\tilde{I}(is)\int_{(\sqrt{\epsilon_2}-1)s}^\infty du\,\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)$$

$$\frac{(1+R_{01})[(e^{2ik_{1z}d}+e^{-2ik_{1z}d})(1+R_{01})R_{12}+2(1+R_{01}R_{12}^2)]}{(e^{-ik_{1z}d}+R_{01}R_{12}e^{ik_{1z}d})(e^{ik_{1z}d}+R_{01}R_{12}e^{-ik_{1z}d})}\Bigg\} \quad (B16)$$

By inspection, we see that the quantity inside {} is equal to the negative of its complex conjugate, and this term is identically zero. In other words,

$$DSDP_{1u} = -DSDP_{2u} \quad (B17)$$

thus, for deformation of both steepest descent paths up, for $\tau < \rho$, we obtain no contribution, which is required by the causality condition.

Finally, we turn to DSDP$_2$ deformed downward. We obtain

$$\mathrm{DSDP}_{2d} = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\Bigg\{ i\int_0^\infty dp\, pe^{-p\tau}\tilde{I}(-ip)$$

$$\int_0^\infty dr\,\frac{k_\rho^2}{k_z}H_1^{(1)}(k_\rho\rho)\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\Bigg\} \quad (B18)$$

where $k_\rho = i(r-\sqrt{\epsilon_2}p)$, $k_z = \sqrt{(\epsilon_2-1)p^2-2\sqrt{\epsilon_2}rp+r^2}$, $k_{1z} = \sqrt{(\epsilon_2-\epsilon_1)p^2-2\sqrt{\epsilon_2}rp+r^2}$, and $k_{2z} = \sqrt{r^2-2\sqrt{\epsilon_2}rp}$. In this case, we have six critical points, $r = (\sqrt{\epsilon_2}-\sqrt{\epsilon_1})p$, $r = (\sqrt{\epsilon_2}-1)p$, $r = \sqrt{\epsilon_2}p$, $r = (\sqrt{\epsilon_2}+1)p$, $r = (\sqrt{\epsilon_2}+\sqrt{\epsilon_1})p$, and $r = 2\sqrt{\epsilon_2}p$, and thus seven regions to consider.

Region 1: $0 \le r \le (\sqrt{\epsilon_2}-\sqrt{\epsilon_1})p$. Here, k_ρ is negative imaginary, so $H_1^{(1)}$ is complex. Also, k_z is imaginary, while k_{1z} and k_{2z} are real. This means that R_{01} and $e^{2ik_{1z}d}$ are complex with magnitude unity, while R_{12} is real. Thus,

$$\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \quad (B19)$$

is purely imaginary. We can therefore reduce the r integral in this region to

$$\mathrm{DSDP}_{2d1} = -\frac{\eta_0}{4\pi^2}\left\{\int_0^\infty dp\, pe^{-p\tau}\tilde{I}(-ip)\int_0^{(\sqrt{\epsilon_2}-\sqrt{\epsilon_1})p} dr \right.$$
$$\left. I_1((\sqrt{\epsilon_2}p-r)\rho)\frac{k_\rho^2}{k_z}\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B20)$$

Region 2: $(\sqrt{\epsilon_2}-\sqrt{\epsilon_1})p \le r \le (\sqrt{\epsilon_2}-1)p$. The difference here is that k_{1z} is imaginary rather than real. This means that R_{01} and $e^{2ik_{1z}d}$ are real, while R_{12} is complex with magnitude unity. However,

$$\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})} \quad (B21)$$

is still purely imaginary. We can therefore reduce the r integral in this region to

$$\mathrm{DSDP}_{2d2} = -\frac{\eta_0}{4\pi^2}\left\{\int_0^\infty dp\, pe^{-p\tau}\tilde{I}(-ip)\int_{(\sqrt{\epsilon_2}-\sqrt{\epsilon_1})p}^{(\sqrt{\epsilon_2}-1)p} dr\, I_1((\sqrt{\epsilon_2}p-r)\rho)\right.$$
$$\left. \frac{k_\rho^2}{k_z}\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B22)$$

Region 3: $(\sqrt{\epsilon_2}-1)p \le r \le \sqrt{\epsilon_2}\,p$. The difference here is that k_{2z} is imaginary rather than real. This means that R_{01}, R_{12} and $e^{2ik_{1z}d}$ are all real, and so neither the real part nor the imaginary part of the integrand is zero. We obtain

$$\mathrm{DSDP}_{2d3} = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{i\int_0^\infty dp\, pe^{-p\tau}\tilde{I}(-ip)\int_{(\sqrt{\epsilon_2}-1)p}^{\sqrt{\epsilon_2}p} dr\right.$$
$$\frac{k_\rho^2}{k_z}\left[\frac{2}{\pi}K_1((\sqrt{\epsilon_2}p-r)\rho)) - 2iI_1((\sqrt{\epsilon_2}p-r)\rho)\right]$$
$$\left.\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B23)$$

Region 4: $\sqrt{\epsilon_2}\,p \le r \le (\sqrt{\epsilon_2}+1)p$. The only difference here is that $H_1^{(1)}$ is real rather than complex, so we obtain

$$\mathrm{DSDP}_{2d4} = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dp\, p e^{-p\tau}\tilde{I}(-ip)\int_{(\sqrt{\epsilon_2}-1)p}^{\sqrt{\epsilon_2}p} dr \left[-\frac{2}{\pi}K_1((\sqrt{\epsilon_2}p-r)\rho))\right]\frac{k_\rho^2}{k_z} \frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B24)$$

If we make the substitution $r = \sqrt{\epsilon_2}p + u$ here and $r = \sqrt{\epsilon_2}p - u$ in $(B23)$, we see that, since $k_\rho^2 = -u^2$, $k_z = \sqrt{u^2-p^2}$, $k_{1z} = \sqrt{u^2-\epsilon_1 p^2}$, and $k_{2z} = \sqrt{u^2-\epsilon_2 p^2}$ in each, the terms containing K_1 differ only by a sign, and therefore cancel each other.

Region 5: $(\sqrt{\epsilon_2}+1)p \le r \le (\sqrt{\epsilon_2}+\sqrt{\epsilon_1})p$. The only difference between this region and region 2 is that $H_1^{(1)}$ is real rather than complex. Since it is only the imaginary part of $H_1^{(1)}$ that contributes in region 2, the contribution from region 5 is zero.

Region 6: $(\sqrt{\epsilon_2}+\sqrt{\epsilon_1})p \le r \le 2\sqrt{\epsilon_2}\,p$. The only difference between this region and region 1 is that $H_1^{(1)}$ is real rather than complex; therefore, this region gives zero as well.

Thus, the total contribution from regions 1 through 6 is

$$\mathrm{DSDP}_{2d1-6} = -\frac{\eta_0}{4\pi^2}\left\{\int_0^\infty dp\, p e^{-p\tau}\tilde{I}(-ip) \int_0^{\sqrt{\epsilon_2}p} dr\, I_1((\sqrt{\epsilon_2}p-r)\rho)\frac{k_\rho^2}{k_z}\frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B25)$$

Region 7: $2\sqrt{\epsilon_2}\,p \le r$. Here, k_z is real. We have no cancellations, so we obtain

$$\mathrm{DSDP}_{2d7} = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dp\, p e^{-p\tau}\tilde{I}(-ip) \int_{2\sqrt{\epsilon_2}p}^{\infty} dr\,\frac{k_\rho^2}{k_z}\left[-\frac{2}{\pi}K_1((\sqrt{\epsilon_2}p-r)\rho))\right] \frac{(1-R_{01}^2)(1-R_{12}^2)e^{2ik_{1z}d}}{(R_{01}R_{12}+e^{2ik_{1z}d})(R_{01}+R_{12}e^{2ik_{1z}d})}\right\} \quad (B26)$$

The total result for DSDP $_2$ when we deform the k_0 integral downward is given by $(B25)$ and $(B26)$.

All of the above results hold for modified double deformation as well, with $-(1/k_0^2)\tilde{I}_{2a}$ replacing $\tilde{I}$ in the deformations up, and $-(1/k_0^2)\tilde{I}_{2a}$ replacing $\tilde{I}$ in the deformations down. However, there is one simplification we can make to the results for the time span $\rho < \tau < \sqrt{\epsilon_2}\,\rho$. In this case, $\tilde{I}_{2b}(k_0, \tau - \sqrt{\epsilon_2}\,\rho) = 0$, and therefore DSDP $_{2d}$ doesn't contribute. This fact also implies that $\tilde{I}_{2a}(k_0, \tau - \sqrt{\epsilon_2}\,\rho) = \tilde{I}_{2a}(k_0, \tau - \rho) + \tilde{I}_{2b}(k_0, \tau - \rho)$. If we make this substitution in $(B14)$, we can split it into parts. The part containing $\tilde{I}_{2a}(k_0, \tau - \rho)$ will cancel with DSDP $_{1u}$ by $(B17)$; the part containing $\tilde{I}_{2b}(k_0, \tau - \rho)$ can be converted into

$$\mathrm{DSDP}_u = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty ds \int_0^{(\sqrt{\epsilon_2}-1)s} dq\, e^{s\tau}\, \frac{\tilde{I}_{2b}(is, \tau - \rho)}{s} \right.$$
$$\left. \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1 + R_{01})^2(1 + R_{12}e^{2ik_{1z}d})^2}{(1 + R_{01}R_{12}e^{2ik_{1z}d})(R_{01} + R_{12}e^{2ik_{1z}d})} \right\} \qquad (B27)$$

By changing the variable of integration from s to $-p$, exchanging the order of the p and q integrations, and combining with the modified form of $(B12)$, we obtain

$$\mathrm{DSDP} = -\frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dq \int_{-q/(\sqrt{\epsilon_2}-1)}^{q/(\sqrt{\epsilon_2}+1)} dp\, e^{-p\tau}\, \frac{\tilde{I}_{2b}(-ip, \tau - \rho)}{p} \right.$$
$$\left. \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho) \frac{(1 + R_{01})^2(1 + R_{12}e^{2ik_{1z}d})^2}{(1 + R_{01}R_{12}e^{2ik_{1z}d})(R_{01} + R_{12}e^{2ik_{1z}d})} \right\} \qquad (B28)$$

Of course, the integration contour in the p plane will have to be indented, as usual, to avoid the pole at the origin.

Appendix C: Modified Double Deformation for VMD over Lossy Halfspace

The original expression for the electric field from a vertical magnetic dipole (VMD) on an infinite halfspace with dielectric constant ϵ_1 and conductivity σ is:

$$E_\phi(\tau) = \frac{\eta_0}{8\pi^2}\mathrm{Re}\left\{ \int_0^\infty dk_0\, k_0 e^{-ik_0\tau}\tilde{I}(k_0) \int_{\mathrm{SIP}} dk_\rho\, \frac{k_\rho^2}{k_z} H_1^{(1)}(k_\rho \rho)\left[1+R^{TE}\right] \right\} \tag{C1}$$

where

$$R^{TE} = \frac{k_z - k_{1z}}{k_z + k_{1z}} \tag{C2}$$

and

$$k_z = \sqrt{k_0^2 - k_\rho^2}, \quad k_{1z} = \sqrt{\underline{\epsilon}_1 k_0^2 - k_\rho^2} \tag{C3}$$

Here, we let $\underline{\epsilon}_1 = \epsilon_1 + i\sigma_1/k_0$, where $\sigma_1 = \sigma\eta_0$.

Deforming to the steepest descent path in the k_ρ plane, which loops around the branch points at $k_\rho = k_0$ and $k_\rho = \sqrt{k_0(\epsilon_1 k_0 + i\sigma_1)}$. Our expression can thus be divided into two parts,

$$E_\phi = \mathrm{SDP}_0 + \mathrm{SDP}_1 \tag{C4}$$

where

$$\mathrm{SDP}_0 = \frac{\eta_0}{2\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dk_0\, \frac{\tilde{I}(k_0)}{(\epsilon_1 - 1)k_0 + i\sigma_1} e^{-ik_0\tau} \int_0^\infty dq\, k_\rho^2 k_z H_1^{(1)}(k_\rho\rho) \right\} \tag{C5}$$

with $k_\rho = k_0 + iq$, and

$$\mathrm{SDP}_1 = -\frac{\eta_0}{2\pi^2}\mathrm{Re}\left\{ i\int_0^\infty dk_0\, \frac{\tilde{I}(k_0)}{(\epsilon_1 - 1)k_0 + i\sigma_1} e^{-ik_0\tau} \right.$$
$$\left. \cdot \int_0^\infty dq\, k_\rho^2 k_{1z} H_1^{(1)}(k_\rho\rho) \right\} \tag{C6}$$

with $k_\rho = \sqrt{k_0(\epsilon_1 k_0 + i\sigma_1)} + iq$.

We then modify the current source, as usual:

$$\tilde{I}(k_0) = \int_0^\infty d\tau'\, I(\tau')e^{ik_0\tau'} = -\frac{1}{k_0^2}\int_0^\infty d\tau'\, I''(\tau')e^{ik_0\tau'} \tag{C7}$$

to obtain

$$\tilde{I}(k_0) = -\frac{1}{k_0^2}\left[\int_0^{\tau-\rho} d\tau'\, I''(\tau')e^{ik_0\tau'} + \int_{\tau-\rho}^{\infty} d\tau'\, I''(\tau')e^{ik_0\tau'}\right] \qquad (C8)$$

$$\equiv -\frac{1}{k_0^2}\left[\tilde{I}_{2b}(k_0, \tau-\rho) + \tilde{I}_{2a}(k_0, \tau-\rho)\right] \qquad (C9)$$

We substitute this into our expression for SDP_0, making the standard quarter-circle deformation of the contours near the origin of the k_0 plane, and deform the k_0 integral of the half containing $\tilde{I}_{2a}$ upward, the half containing $\tilde{I}_{2b}$ downward. It is easy to show that the part containing $\tilde{I}_{2a}$ is identically zero; the demonstration is very similar to that for a VMD over lossless halfspace. What remains is to evaluate the part containing $\tilde{I}_{2b}$. We have

$$\text{SDP}_0 = -\frac{\eta_0}{2\pi^2}\text{Re}\left\{i\int_0^\infty dq \int_{i\delta}^\infty dk_0\, \frac{\tilde{I}_{2b}(k_0, \tau-\rho)}{k_0^2[(\epsilon_1 - 1)k_0 + i\sigma_1]} \cdot e^{-ik_0\tau}k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)\right\} \qquad (C10)$$

with $k_\rho = k_0 + iq$. We wish to deform the k_0 integral to the negative imaginary axis. This will leave three parts: the residue due to the double pole at the origin, the residue due to the pole at $k_0 = -ip_l$, where $p_l = \sigma_1/(\epsilon_1 - 1)$, and the integral along the negative imaginary k_0 axis. The integral along the axis vanishes in the same way as it did for the lossless case. For the double pole at the origin, we obtain $-\pi i$ times the residue, since we are detouring halfway around the pole clockwise:

$$Z_0 = -\frac{\eta_0}{2\pi}\text{Re}\left\{\int_0^\infty dq\, \frac{\partial}{\partial k_0}\left[e^{-ik_0\tau}\frac{\tilde{I}_{2b}(k_0, \tau-\rho)}{(\epsilon_1 - 1)k_0 + i\sigma_1}k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)\right]_{k_0=0}\right\} \qquad (C11)$$

Let

$$g(k_0) \equiv e^{-ik_0\tau}\tilde{I}_{2b}(k_0, \tau-\rho) \qquad (C12)$$

and

$$f(k_0, q) \equiv \frac{k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)}{(\epsilon_1 - 1)k_0 + i\sigma_1} \qquad (C13)$$

where $k_\rho = k_0 + iq$. Then we can express the Z_0 integral very simply as

$$\mathrm{SDP}_0 = -\frac{\eta_0}{2\pi}\mathrm{Re}\left\{ g(0)\int_0^\infty dq\, f'(0,q) + g'(0)\int_0^\infty dq\, f(0,q)\right\}. \qquad (C14)$$

We have, from Section 2, that

$$g(0) = I'(\tau-\rho), \qquad (C15)$$

and

$$g'(0) = -i[\rho I'(\tau-\rho) + I(\tau-\rho)], \qquad (C16)$$

We must also keep in mind that $k_\rho = iq$ and $k_z = q$ when $k_0 = 0$. Now,

$$f(0,q) = -\frac{2iq^3}{\pi\sigma_1}K_1(q\rho), \qquad (C17)$$

so

$$\int_0^\infty dq\, f(0,q) = -\frac{3i}{\sigma_1\rho^4}. \qquad (C18)$$

Now,

$$f'(0,q) = \frac{2}{\pi\sigma_1^2}\left\{q^3\rho\sigma_1 K_0(q\rho) + [(\epsilon_1-1)q^3 - 2\sigma_1]K_1(q\rho)\right\}$$

and

$$\int_0^\infty dq\, f'(0,q) = \frac{3(\epsilon_1-1)}{\rho^4\sigma_1^2}$$

Thus,

$$Z_0 = \frac{\eta_0}{2\pi\rho^4}\left\{\frac{3}{\sigma_1}I(\tau-\rho) - \frac{3(\epsilon_1-1-\rho\sigma_1)}{\sigma_1^2}I'(\tau-\rho)\right\}. \qquad (C19)$$

Now, we turn our attention to the loss pole. We have

$$L_0 = \frac{\eta_0}{2\pi(\epsilon_1-1)}\mathrm{Re}\left\{\int_0^\infty dq\, \frac{\tilde{I}_{2b}(-ip_l,\tau-\rho)}{p_l^2}e^{-p_l\tau}k_\rho^2 k_z H_1^{(1)}(k_\rho\rho)\right\} \qquad (C20)$$

This cannot be expressed in closed form, but we can simplify it somewhat:

$$L_0 = \frac{\eta_0}{\pi(\epsilon_1 - 1)} \frac{\tilde{I}_{2b}(-ip_l, \tau - \rho)e^{-p_l\tau}}{p_l^2} \left\{ \int_0^{p_l} dq\,(p_l - q)^2 \sqrt{q(2p_l - q)} I_1((p_l - q)\rho) + \frac{1}{\pi} \int_{2p_l}^{\infty} dq\,(q - p_l)^2 \sqrt{q(q - 2p_l)} K_1((q - p_l)\rho) \right\} \quad (C21)$$

Similarly, for SDP $_1$,

$$Z_1 = -\frac{\eta_0}{2\pi\rho^4} \left\{ \frac{3}{\sigma_1} I(\tau - \sqrt{\epsilon_1}\rho) - \frac{3(\epsilon_1 - 1 - \sqrt{\epsilon_1}\rho\sigma_1)}{\sigma_1^2} I'(\tau - \sqrt{\epsilon_1}\rho) \right\}, \quad (C22)$$

and

$$L_1 = -\frac{\eta_0}{\pi(\epsilon_1 - 1)} \frac{\tilde{I}_{2b}(-ip_l, \tau - \sqrt{\epsilon_1}\rho)e^{-r_l\tau}}{r_l^2} \left\{ \int_0^{r_l} dq\,(r_l - q)^2 \sqrt{q(2r_l - q)} I_1((r_l - q)\rho) + \frac{1}{\pi} \int_{2r_l}^{\infty} dq\,(q - r_l)^2 \sqrt{q(q - 2r_l)} K_1((q - r_l)\rho) \right\} \quad (C23)$$

where $r_l = \sqrt{\epsilon_1}\, p_l$.

References

[1] Sommerfeld, A., *Partial Differential Equations in Physics,* Academic Press Inc., New York, 1949.

[2] Baños, A., *Dipole Radiation in the Presence of a Conducting Halfspace,* Pergamon, New York, 1966.

[3] Kong, J. A., L. C. Shen, and L. Tsang, "Field of an antenna submerged in a dissipative dielectric medium," *IEEE Trans. Antennas Propagat.,*, Vol. AP-25, No. 6, 887–889, Nov. 1977.

[4] Bannister, P. R., "The image theory for electromagnetic fields of a horizontal electric dipole in the presence of a conducting half space," *Radio Sci.,* Vol. 17, No. 5, 1095–1102, Sept.–Oct. 1982.

[5] Lindell, I. V., and E. Alanen, "Exact image theory for the Sommerfeld halfspace problem, part I: vertical magnetic dipole," *IEEE Trans. Antennas Propagat.,* Vol. AP-32, No. 2, 126–133, Feb. 1984.

[6] Lindell, I. V., and E. Alanen, "Exact image theory for the Sommerfeld halfspace problem, part II: vertical electric dipole," *IEEE Trans. Antennas Propagat.,* Vol. AP-32, No. 8, 841–847, Aug. 1984.

[7] Ward, S. H., "Electromagnetic theory for geophysical applications," in *Mining Geophysics,* Vol. 2, part A: Tulsa, SEG, 10–196, 1967.

[8] Wait, J. R., *Electromagnetic Waves in Stratified Media,* Second Ed., Pergamon Press, New York, 1970.

[9] Wait, J. R., "The magnetic dipole over the horizontally stratified earth," *Can. J. Phys.,* Vol. 29, No. 6, 577–592, Nov. 1951.

[10] Wait, J. R., "Radiation from a vertical electric dipole over a stratified ground," *IRE Trans. Antennas Propagat.,* Vol. AP-1, No. 1, 9–11, July 1953.

[11] Wolf, A., "Electric field of an oscillating dipole on the surface of a two-layer earth," *Geophysics,* Vol. 11, No. 4, 518–534, Oct. 1946.

[12] El-Said, M. A. H., "Geophysical prospection of underground water in the desert by means of electromagnetic interference fringes," *Proc. IRE,* Vol. 44, No. 1, 24–30, Jan. 1956.

[13] Bhattacharyya, B. K., "Electromagnetic fields of a vertical magnetic dipole placed above the the Earth's surface," *Geophysics,* Vol. 28, No. 3, 408–425, June 1963.

[14] Kong, J. A., L. Tsang, and G. Simmons, "Geophysical subsurface probing with radio-frequency interferometry," *IEEE Trans. Antennas Propagat.,* Vol. AP-22, No. 4, 616–620, July 1974.

[15] Kong, J. A., *Theory of Electromagnetic Waves,* Wiley-Interscience, New York, 1975.

[16] Chan, W. C. M., J. A. Kong, and L. Tsang, "Geophysical subsurface probing with a horizontal magnetic dipole," *IEEE Trans. Antennas Propagat.,* Vol. AP-24, No. 2, 877–879, March 1976.

[17] Chew, W. C., "Mixed boundary value problems in microstrip and geophysical probing problems," Ph.D. thesis, Massachusetts Institute of Technology; Cambridge, MA, June 1980.

[18] Kong, J. A. (Ed.), *Research Topics in Electromagnetic Wave Theory,* Wiley-Interscience, New York, 1981.

[19] Chew, W. C., and J. A. Kong, "Electromagnetic field of a dipole on a two-layer earth," *Geophysics,* Vol. 46, No. 3, 309–315, March 1981.

[20] Chew, W. C., and J. A. Kong, "Asymptotic approximation of waves due to a dipole on a two-layer medium," *Radio Sci.,* Vol. 17, No. 3, 509–513, May–June 1982.

[21] Wait, J. R., "Electromagnetic fields of a dipole over an anisotropic halfspace," *Can. J. Phys.,* Vol. 44, No. 10, 2387–2401, Oct. 1966.

[22] Wait, J. R., "Electromagnetic fields of a dipole over a stratified anisotropic half space," *IEEE Trans. Antennas Propagat.,* Vol. AP-14, No. 6, 790–792, Nov. 1966.

[23] Sinha, A. K., and P. K. Bhattacharya, "Electric dipole over an anisotropic and inhomogeneous earth," *Geophysics,* Vol. 32, No. 4, 652–667, Aug. 1967.

[24] Kong, J. A., "Electromagnetic fields due to dipole antennas over stratified anisotropic media," *Geophysics,* Vol. 37, No. 6, 985–996, Dec. 1972.

[25] Cheng, D., J. A. Kong, and L. Tsang, "Geophysical subsurface probing of a two-layered uniaxial medium with a horizontal mag-

netic dipole," *IEEE Trans. Antennas Propagat.*, Vol. AP-25, No. 6, 766–769, Nov. 1977.

[26] Mao, S., S. Jones, and G. D. Vendelin, "Millimeter-wave integrated circuits," *IEEE Trans. Microwave Theory Tech.*, Vol. MTT-16, No. 2, 455–461, July 1968.

[27] Chew, W. C., J. A. Kong, and L. C. Shen, "Radiation characteristics of a circular microstrip antenna," *J. Appl. Phys.*, Vol. 51, No. 7, 3907–3915, July 1980.

[28] Poh, S. Y., W. C. Chew, and J. A. Kong, "Approximate formulas for line capacitance and characteristic impedance of microstrip line," *IEEE Trans. Microwave Theory Tech.*, Vol. MTT-29, No. 2, 135–142, Feb. 1981. *Correction* in Vol. MTT-29, No. 10, p. 1119, Oct. 1981.

[29] Hawley, P. F., "Transients in electrical prospecting," *Geophysics,* Vol. 3, No. 3, 247–257, July 1938.

[30] Bennett, C. L., and G. F. Ross, "Time-domain electromagnetics and its applications," *Proc. IEEE,* Vol. 66, No. 3, 299–318, March 1978.

[31] Patra, H. P., and K. Mallick, "Transient soundings," Ch. 5 in *Geosounding Principles*, Vol. 2, Elsevier, Amsterdam, 1980.

[32] Kaufman, A. A., and G. V. Keller, *Frequency and Transient Soundings,* Elsevier, Amsterdam, 1983.

[33] Wait, J. R., "Transient electromagnetic propagation in a conducting medium," *Geophysics,* Vol. 16, 213–221, 1951.

[34] Bhattacharyya, B. K., "Propagation of transient electromagnetic waves in a medium of finite conductivity," *Geophysics,* Vol. 22, No. 1, 75–88, Jan. 1957.

[35] Bhattacharyya, B. K., "Propagation of an electric pulse through a homogeneous and isotropic medium," *Geophysics,* Vol. 22, No. 4, 905–921, Oct. 1957.

[36] Poritsky, H., "Propagation of transient fields from dipoles near the ground," *British J. Appl. Phys.,* Vol. 6, 421–426, Dec. 1955.

[37] Van der Pol, B., "On discontinuous electromagnetic waves and the occurrence of a surface wave," *IRE Trans. Antennas Propagat.,* Vol. AP-4, No. 3, 288–293, July 1956.

[38] Pekeris, C. L., and Z. Alterman, "Radiation resulting from an impulsive current in a vertical antenna placed on a dielectric ground," *J. Appl. Phys.*, Vol. 28, No. 11, 1317–1323, Nov. 1957.

[39] Cagniard, L., *Reflection and Refraction of Progressive Seismic Waves,* E. A. Flinn and C. H. Dix, Eds., McGraw-Hill, New York, 1962.

[40] Bhattacharyya, B. K., "Electromagnetic fields of a transient magnetic dipole on the Earth's surface," *Geophysics,* Vol. 23, No. 1, 89–108, Feb. 1959.

[41] de Hoop, A. T., "A modification of Cagniard's method for solving seismic pulse problems," *Appl. Sci. Res.*, Vol. B8, 349–356, 1960.

[42] de Hoop, A. T., and H. J. Frankena, "Radiation of pulses generated by a vertical electric dipole above a plane non-conducting earth," *Appl. Sci. Res.*, Vol. B8, 369–377, 1960.

[43] Frankena, H. J., "Transient phenomena associated with Sommerfeld's horizontal dipole problem," *Appl. Sci. Res.*, Vol. B8, 357–368, 1960.

[44] Wait, J. R., "Propagation of electromagnetic pulses in homogeneous conducting earth," *Appl. Sci. Res.*, Vol. B8, 213–253, 1960.

[45] Bremmer, H., "The pulse solution connected with the Sommerfeld problem for a dipole in the interface between two dielectrics," in *Electromagnetic Waves,* edited by R. E. Langer, 39–64, University of Wisconsin Press, Madison, 1962.

[46] Hill, D., "The transient fields of a Hertzian dipole in the presence of a dielectric halfspace," *Radio Sci.*, Vol. 6, Nos. 8,9, 787–794, Aug.–Sept. 1971.

[47] Wait, J. R., and D. A. Hill, "Transient signals from a buried magnetic dipole," *J. Appl. Phys.*, Vol. 42, No. 10, 3866–3869, Sept. 1971.

[48] Wait, J. R., and D. A. Hill, "Transient magnetic fields produced by a step-function-excited loop buried in the earth," *Electron. Lett.*, Vol. 8, No. 11, 294–295, June 1972.

[49] Wait, J. R., and D. A. Hill, "Transient electromagnetic fields of a finite circular loop in the presence of a conducting half space," *J. Appl. Phys.*, Vol. 43, No. 11, 4532–4534, Nov. 1972.

[50] Wait, J. R., and R. H. Ott, "On calculating transient electromagnetic fields of a small current-carrying loop over a homogeneous earth," *P. Appl. Geophys.*, Vol. 95, 157–162, 1972.

[51] Fuller, J. A., and J. R. Wait, "A pulsed dipole in the earth," in *Transient Electromagnetic Fields,* L. B. Felsen, Ed., 237–269, Springer-Verlag, Berlin, 1976.

[52] de Hoop, A. T., "Pulse electromagnetic radiation from a line source in a two-media configuration," *Radio Sci.,* Vol. 14, No. 2, 253–268, March–April 1979.

[53] Haddad, H., and D. C. Chang, "Transient electromagnetic field generated by a vertical electric dipole on the surface of a dissipative earth," *Radio Sci.,* Vol. 16, No. 2, 169–178, March–April 1981.

[54] Baum, C. E., "The singularity expansion method," in *Transient Electromagnetic Fields,* L. B. Felsen, Ed., 130–176, Springer-Verlag, Berlin, 1976.

[55] Kuester, E. F., "The transient electromagnetic field of a pulsed line source located above a dispersively reflecting surface," *IEEE Trans. Antennas Propagat.,* Vol. AP-32, No. 11, 1154–1162, Nov. 1979.

[56] Wait, J. R., "Shielding of a transient electromagnetic dipole field by a conductive sheet," *Can. J. Phys.,* Vol. 34, No. 8, 890–893, Aug. 1956.

[57] Vanyan, L. L., *Electromagnetic depth soundings:* selected and translated by George V. Keller, Consultants Bureau, New York, 1967.

[58] Kaufman, A. A., and S. A. Terent'yev, "The nonstationary field of a vertical magnetic dipole in beds of limited thickness," *Izv., Earth Physics,* No. 9, 85–87, 1971.

[59] Wait, J. R., "On the theory of transient electromagnetic sounding over a stratified earth," *Can. J. Phys.,* Vol. 50, No. 11, 1055–1061, June 1972.

[60] Dey, A., and H. F. Morrison, "Electromagnetic coupling in frequency and time-domain induced polarization surveys over a multi-layered earth," *Geophysics*, Vol. 38, No. 2, 380–405, April 1973.

[61] Tsang, L., R. Brown, J. A. Kong, and G. Simmons, "Numerical evaluation of electromagnetic fields due to dipole antennas in the presence of stratified media," *J. Geophys. Res.*, Vol. 79, No. 14, 2077–2080, May 1974.

[62] Kaufman, A. A., "Exploration methods based on nonstationary electromagnetic fields in the near zone," *Izv. Acad. Sci. USSR Phys. Solid Earth,* No. 11, 43–53, 1973.

[63] Lee, T., and R. Lewis, "Transient E.M. response of a large loop on a layered ground," *Geophys. Prospect.*, Vol. 22, No. 3, 430–444, Sept. 1974.

[64] Lytle, R. J., and D. L. Lager, "Using the natural-frequency concept in remote probing of the earth," *Radio Sci.,* Vol. 11, No. 3, 199–209, March 1976.

[65] Botros, A. Z., and S. F. Mahmoud, "The transient fields of simple radiators from the point of view of remote sensing of the ground subsurface," *Radio Sci.,* Vol. 13, No. 2, 379–389, March–April 1978.

[66] Kaufman, A. A., "Harmonic and transient fields on the surface of a two-layered medium," *Geophysics,* Vol. 44, No. 7, 1208–1217, July 1979.

[67] Mahmoud, S. F., A. Z. Botros, and J. R. Wait, "Transient electromagnetic fields of a vertical magnetic dipole on a two-layer earth," *Proc. IEEE,* Vol. 67, No. 7, 1022–1029, July 1979.

[68] Ezzeddine, A., J. A. Kong, and L. Tsang, "Transient fields of a vertical electric dipole over a two-layer nondispersive dielectric," *J. Appl. Phys.,* Vol. 52, No. 3, 1202–1208, March 1981.

[69] Ezzeddine, A., J. A. Kong, and L. Tsang, "Time response of a vertical electric dipole over a two-layer medium by the double deformation technique," *J. Appl. Phys.*, Vol. 53, No. 2, 813–822, Feb. 1982.

[70] Tsang, L., and J. A. Kong, "Modified modal theory of transient response in layered media," *J. Math. Phys.,* Vol. 20, No. 6, 1170–1182, June 1979.

[71] Bowhill, S. A., *The Physics of The Ionosphere*, p. 308, The Physical Society, London, 1955.

[72] Rönnäng, B. O., "Transient wave propagation in symmetric ionized Epstein layers," *Proc. IEEE,* Vol. 54, No. 9, 1207–1209, Sept. 1966.

[73] Gray, K. G., "An exact solution for the impulse response of a uniform plasma half space," *IEEE Trans. Antennas Propagat.,* Vol. AP-22, No. 6, 819–821, Nov. 1974.

[74] Gray, K. G., and S. A. Bowhill, "Transient response of stratified media: Multiple scattering integral and differential equation for an impulsive incident plane wave," *Radio Sci.,* Vol. 9, No. 1, 57–62, Jan. 1974.

[75] Gray, K. G., and S. A. Bowhill, "Transient response of stratified media: Response to an arbitrary incident plane wave," *Radio Sci.,* Vol. 9, No. 1, 63–69, Jan. 1974.

[76] Gray, K. G., and S. A. Bowhill, "The impulse response of a cold stratified plasma in the presence of collisions and a vertical magnetic field by a multiple-scattering technique," *Radio Sci.,* Vol. 9, No. 5, 559–566, May 1974.

[77] Ott, E., and J. Shmoys, "Transient radiation in a plane stratified dispersive medium. I. Halfspace configuration," *Can. J. Phys.,* Vol. 46, No. 9, 1059–1071, May 1968.

[78] Ott, E., and J. Shmoys, "Transient radiation in a plane stratified dispersive medium. II. Exponential profile, plasma slab, and plasma gap," *Can. J. Phys.,* Vol. 47, No. 14, 1435–1446, July 1969.

[79] Chang, F. Y., "Transient analysis of lossless coupled transmission lines in a nonhomogeneous dielectric medium," *IEEE Trans. Microwave Theory Tech.,* Vol. MTT-18, No. 9, 616–626, Sept. 1970.

[80] Agrawal, A. K., H. M. Fowles, L. D. Scott, and S. H. Gurbaxani, "Application of modal analysis to the transient response of multiconductor transmission lines with branches," *IEEE Trans. Electromagn. Compat.,* Vol. EMC-21, No. 3, 256–262, Aug. 1979.

[81] Rosenbaum, J. H., "The long-time response of a layered elastic medium to explosive sound," *J. Geophys. Res.,* Vol. 65, No. 5, 1577–1613, May 1960.

[82] Poh, S. Y., and J. A. Kong, "Transient response of a vertical electric dipole on a two-layer medium" *Journal of Electromagnetic Waves and Applications,* Vol. 1, No. 2, 135–158, 1987.

[83] Kong, J. A., *Electromagnetic Wave Theory,* Wiley-Interscience, New York, 1986.