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$(3+1)$-Dimensional Nonparaxial Spatiotemporally Localized Waves in Transparent Dispersive Media

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ABSTRACT: Most of the analytical work on general transparent dispersive media to date has been confined to second-order dispersion within the framework of the paraxial approximation. It is the aim in this article to lift this restriction. Specifically, a detailed discussion is provided of modulated $(3+1)$-dimensional nonparaxial spatiotemporally localized waves in second-order transparent dispersive media. Novel infinite-energy invariant wavepackets and finite-energy almost undistorted solutions are discussed in detail. Illustrative numerical examples of the latter are given for normal dispersion in fused silica and for anomalous dispersion in a Lorentz plasma.

1. INTRODUCTION

In recent years, there has been increasing interest in novel classes of localized solutions to various hyperbolic equations governing acoustic, electromagnetic, and quantum mechanical wave phenomena. The bulk of the research along these lines has been performed in connection to the basic formulation, generation, propagation, guidance, scattering, and diffraction properties of localized waves (LWs) in free space (See [1–17] for pertinent literature). This interest has been sustained by advancements in ultrafast acoustical, optical, and electrical devices capable of generating and shaping very short, pulsed waves. These ultrashort pulses exhibit distinct advantages in their performance by comparison to conventional quasi-monochromatic signals. It has been shown that such pulses have extended ranges of localization in the near-to-far field regions. These properties, together with their uniform focused depth in the near field, render LW fields very useful in applications involving remote sensing, ground-penetrating radar, directed energy transfer and secure communications, nondestructive testing, secure signaling, and interference-free communications.

Examples of analytical LWs include infinite-energy nondiffracting wavepackets, such as the Focus Wave Mode (FWM) derived by Brittingham [1] (see, also, [2]), the X-wave introduced by Lu and Greenleaf [5], the Focus X Wave (FXW) derived by Besieris *et al.* [8], and the Bessel X wave formulated by Saari and Reivelt [7], as well as almost nondiffracting finite-energy ones, e.g., the Modified Power Spectrum (MPS) pulse deduced by Ziolkowski [3] and the Modified Focus X Wave (MFXW) derived by Besieris *et al.* [8]. Each LW pulse is an ultra-wideband wave field consisting of a highly focused central portion embedded in a sparse, low intensity background. Two scales, thus, characterize these pulsed wave fields: (a) an extremely small scale depicting the spatial extension and the temporal duration of the high intensity focused pulse; (b) a larger scale specifying the size of the low intensity background field. This double trait causes LW pulses to behave in an extraordinary fashion when they propagate in free space, or scatter and diffract from objects. Another distinct feature of all LW pulses is an unusual coupling between their spatial and temporal spectral components. This coupling manifests itself as a time-dependent (dynamic) initial excitation on the source plane of the generated pulse; specifically, distinct segments of the source plane should be excited at different times using various time sequences. One factor determining the sequential order of the excitation of the various source elements is the spatio-temporal spectral coupling. The unusual structure of the frequency content of LW pulses causes the spectral depletion of the peaks of such pulsed wave fields to be entirely different from that of conventional quasi-monochromatic signals, or other broadband signals. Thorough investigations of the spectral depletion of LW pulses generated from finite-time dynamic apertures have been undertaken. A finite-time dynamic aperture is an artifice developed for studying the decay of propagating finite-energy LW pulses by time-limiting known closed-form infinite energy LW solutions. This provides a well-established scheme for shaping the spectral components of the initial field in a manner that it can control the decay rate of a LW pulse traveling away from its source plane. Such an approach is dependent on the *a priori* knowledge of exact closed-form LW solutions. It does not matter whether the known exact LW pulses have infinite energy, if the power content of the initial excitation on the source plane is always finite. In most cases, finite energy pulses can be generated by appropriately time-windowing the infinite energy excitation field.

Experimental demonstrations of localized waves have been performed in the acoustical [18] and optical regimes [17, 19–23]. Work, however, was carried out at microwave frequencies recently [24–26].

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In parallel to the work on spatially localized and temporally ultra-short pulses in free space, research has been carried out on localized waves that can compensate for temporal dispersive media effects. The main goal has been to ascertain the degree of reduction of dispersive effects through proper shaping of the initial excitations. Such, almost nondispersive LWs, would find applications in diverse physical areas, such as massive particle physics, high resolution imaging, medical radiology, tissue characterization and photodynamic therapy.

One of the simplest analytical models incorporating dispersion is the Klein-Gordon equation, important in modeling cold plasma and in relativistic particle physics. Original work on localized waves involving this equation by MacKinnon [27, 28] dealt with the modeling of relativistic massive particles. A more systematic approach to the derivation of exact, nondispersive packet solutions to equations modeling relativistic massive particles was carried out by Shaarawi *et al.* [29], first in connection with the Klein-Gordon equation and then the Dirac equation. The bidirectional wave transformation [4] developed for the scalar wave equation was shown by Tippett and Ziolkowski [30] to have interesting extensions for first-order hyperbolic systems. The method they developed was applied to the linearized cold plasma equations and nondispersive LWs solutions were established. FWM-type solutions to the Klein-Gordon were developed by Hillion [31] and Borisov and Utkin [32]. Luminal, subluminal, and superluminal ideal (nondispersive) solutions to the Klein-Gordon equation were derived by Rodrigues and Maiorino [33]. A comprehensive study of FWM-type LWs in a collisionless plasma was undertaken by Abdel-Rahman *et al.* [34], with emphasis on their generation from finite-time dynamic apertures. A systematic derivation of finite energy FWM and X Wave-type LWs based on dimension-reduction techniques for the Klein-Gordon was provided by Besieris *et al.* [35, 36]. More recently, Kiselev-type (exponentially localized) nondispersive solutions to the Klein-Gordon equation were derived by Perel and Fialkowsky [37].

The Klein-Gordon equation, characterized by anomalous dispersion, is a very special case of the broad general class of normal and anomalous temporally dispersive media. A first attempt to address space-time localization in a more general dispersive medium, albeit confined to second-order dispersion, was made by Sõnjalg and Saari [38] and by Sõnjalg, Rätsep and Saari [39], who studied the suppression of temporal spread of ultrashort pulses and demonstrated experimentally the propagation of a Bessel-X pule with strong lateral and longitudinal localization in dispersive media. Subsequent work along these lines was undertaken by Porras [40], Orlov *et al.* [41], Zamboni-Rached *et al.* [42], Porras *et al.* [43], Porras and Gonzalo [44], Porras and Di Trapani [45], Longhi [46], Ciattoni and Di Porto [47], Orlov and Stabinis [48], Malaguti *et al.* [49], Porras *et al.* [50], Melaguti and Trillo [51], Salem and Bağci [52]. More recent work in this area has been undertaken by Hall and Abouraddy [53, 54], He *et al.* [55], Yessenov *et al.* [56], and Palastro *et al.* [57]. A tutorial review of spatiotemporal sculpturing of light has been provided by Zhan [58] recently.

Most of the analytical work on general dispersive media to date has been confined to second-order dispersion within the framework of the paraxial approximation. It is the aim in this article to lift this restriction. Specifically, a detailed discussion will be provided of modulated $(3+1)$-dimensional nonparaxial spatiotemporally localized waves in second-order transparent dispersive media, and novel infinite-energy invariant wavepackets and finite-energy almost undistorted nonparaxial solutions will be derived.

An outline of this work is as follows. In Section 2, the general theory of temporally dispersive media is given, and an expansion of a modulated scalar-valued solution to all orders of dispersion is provided. Terminating this expansion to second order, different approximations (e.g., paraxial, inability to account for single-cycle pulses, etc.) are discussed. Also, a framework for infinite-energy invariant nonparaxial localized waves is introduced. To view the different families of localized waves, both infinite-energy and finite energy, exact localized waves to the Klein-Gordon equation are given in Section 3. A wide class of infinite-energy invariant nonparaxial localized solutions for second-order dispersive media is given in Section 4. Finite-energy nonparaxial localized solutions for second-order dispersive media are given in Section 5. Numerical illustrations of subluminal, luminal, and superluminal finite-energy $(3+1)$-dimensional nonparaxial spatiotemporally localized waves in second-order transparent dispersive media are given in Section 6, for normal dispersion in fused silica and anomalous dispersion in a Lorenz plasma. Concluding remarks are made in Section 7.

2. BASIC THEORY

Electromagnetic wave propagation in a linear, homogeneous, lossy, temporally dispersive medium is governed by the scalar equation [59]

$$\nabla^2 u(\vec{r},t) + \beta_{op}^2(-i\partial/\partial t)u(\vec{r},t) = 0 \tag{1}$$

if polarization is neglected. In this expression, $u(\vec{r},t)$ is a real field and $\beta_{op}^2(-i\partial/\partial t)$ a real pseudo-differential operator. A physical interpretation of the latter is provided in the frequency domain; specifically,

$$F\left\{\beta_{op}^2(-i\partial/\partial t)u(\vec{r},t)\right\} = \beta_c^2(\omega)\tilde{u}(\vec{r},\omega), \tag{2}$$

where $F\{\cdot\}$ denotes Fourier transformation, and $\tilde{u}(\vec{r},\omega)$ is the Fourier transform of $u(\vec{r},t)$ with respect to time. The function $\beta_c(\omega)$ appearing at the right-hand side of Eq. (2) is a complex wavenumber, viz., $\beta_c(\omega) = \beta(\omega) - i\alpha(\omega)$, expressed in terms of the real wavenumber $\beta(\omega)$ and attenuation factor $\alpha(\omega)$. Strictly, the Kramers-Kronig relations require the presence of loss. In the sequel, however, it will be assumed that the loss in the medium is very small and can be neglected; therefore, $\alpha(\omega) = 0$.

For a physically convenient central radian frequency ω_0, the real field $u(\vec{r},t)$ is expressed as follows:

$$\begin{aligned} u_\mp(\vec{r},t) &= \psi_\mp(\vec{r},t)e^{i\omega_0 t \mp i\beta(\omega_0)z} + cc \\ &= \psi_\mp(\vec{r},t)e^{i\omega_0\left(t\mp\frac{z}{v_{ph}}\right)} + cc, \quad z \geq 0. \end{aligned} \tag{3}$$

Here, $\psi_{\mp}(\vec{r},t)$ are complex-valued envelope functions, and $v_{ph}=\omega_0/\beta(\omega_0)$ denotes the phase speed in the medium computed at the central carrier frequency ω_0. A formal introduction of Eq. (3) into Eq. (1) yields the following exact equations governing the envelope functions $\psi_{\mp}(\vec{r},t)$

$$\left[\nabla_\perp^2+\frac{\partial^2}{\partial z^2}\mp 2i\beta(\omega_0)\frac{\partial}{\partial z}-\beta^2(\omega_0)\right.$$
$$\left.+\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial\omega^m}\beta^2(\omega)\bigg|_{\omega=\omega_0}\left(-i\frac{\partial}{\partial t}\right)^m\right]\psi_{\mp}(\vec{r},t)=0, \quad (4)$$

with $\nabla_\perp^2$ denoting the transverse (with respect to z) Laplacian operator. The presence of time derivatives arises from an inverse temporal Fourier transformation following the expansion of $\beta^2(\omega)$ in a Taylor series around the carrier frequency ω_0.

Usually, at this stage in the study of wave propagation through dispersive media, one introduces the moving reference frame $\xi=z$, $\tau=t-(z/v_{gr})$, in terms of the group speed $v_{gr}=1/\beta_1$; $\beta_1\equiv d\beta(\omega)/d\omega|_{\omega=\omega_0}$. Then, Eq. (4) is transformed into [59, 60]

$$\left[\nabla_\perp^2+\frac{\partial^2}{\partial z^2}+\frac{1}{v_{gr}^2}\frac{\partial^2}{\partial\tau^2}-2\frac{1}{v_{gr}}\frac{\partial^2}{\partial z\partial\tau}\mp 2i\beta(\omega_0)\right.$$
$$\left.\left(\frac{\partial}{\partial z}-\frac{1}{v_{gr}}\frac{\partial}{\partial\tau}\right)-\beta^2(\omega_0)\right]\psi_{\mp}(\vec{r},\tau)$$
$$+\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial\omega^m}\beta^2(\omega)\bigg|_{\omega=\omega_0}\left(-i\frac{\partial}{\partial\tau}\right)^m\psi_{\mp}(\vec{r},\tau)=0. \quad (5)$$

Several techniques have been developed based on the type of approximations made to the exact equations (5). Among them, the most primitive is the *slowly varying envelope approximation* (SVEA), whereby one neglects the second derivative with respect to z (paraxial approximation), as well as the mixed derivative term involving z and τ, and retains dispersive effects to second order. Recent improvements, such as the *slowly evolving wave approximation* (SEWA) and the slightly altered *slowly evolving envelope approximation* (SEEA) can accommodate the propagation of ultra-short (few-cycle) pulses by retaining the mixed derivative term.

The approach in this exposition is fundamentally different. Solutions are sought for the envelope functions of the form

$$\psi_{\mp}(\vec{r},t)=\psi_{\mp}(x,y,\tau);\quad \tau=t-\frac{z}{v}, \quad (6)$$

where v is a fixed, yet unspecified, speed. For the sake of simplicity, azimuthal symmetry with respect to the z-axis is assumed. Under these conditions, Eq. (4) changes to

$$\left[\nabla_\rho^2+\frac{1}{v^2}\frac{\partial^2}{\partial\tau^2}\pm 2i\beta(\omega_0)\frac{1}{v}\frac{\partial}{\partial\tau}-\beta^2(\omega_0)\right]\psi_{\mp}(\vec{r},\tau)$$
$$+\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial\omega^m}\beta^2(\omega)\bigg|_{\omega=\omega_0}\left(-i\frac{\partial}{\partial\tau}\right)^m\psi_{\mp}(\vec{r},\tau)=0. \quad (7)$$

where ρ is the radial polar coordinate. Next, elementary solutions to Eq. (7) are assumed of the form $\psi_{\mp}^e(\rho,\tau)=J_0(\kappa\rho)e^{\mp i\alpha v\tau}$, with $J_0(\cdot)$ the zero-order ordinary Bessel function, and α,κ real parameters with units of m^{-1}. Such elementary solutions for the envelope functions are possible provided that the following constraint (dispersion) relationships are obeyed:

$$-\kappa^2-\alpha^2+2\alpha\beta(\omega_0)-\beta^2(\omega_0)$$
$$+\sum_{m=0}^{\infty}\frac{1}{m!}\frac{\partial^m}{\partial\omega^m}\beta^2(\omega)\bigg|_{\omega=\omega_0}(\mp\alpha v)^m=0. \quad (8)$$

The latter can be rewritten formally more compactly as

$$-\kappa^2-[\alpha-\beta(\omega_0)]^2+\beta^2(\omega_0\mp\alpha v)=0. \quad (9)$$

More general expressions for the envelope functions can be obtained by means of spectral superposition, viz.,

$$\psi_{\mp}(\rho,\tau)=\int_0^\infty d\alpha\int_0^\infty d\kappa\kappa J_0(\kappa\rho)e^{\mp i\alpha v\tau}$$
$$\delta\left\{-\kappa^2-[\alpha-\beta(\omega_0)]^2+\beta^2(\omega_0\mp\alpha v)\right\}\tilde{\psi}(\kappa,\alpha). \quad (10)$$

Consequently, solutions to the original problem [cf. Eq. (3)] are given as

$$u_{\mp}(\rho,z,t)=\psi_{\mp}(\rho,\tau)e^{i\omega_0\left(t\mp\frac{z}{v_{ph}}\right)}+cc,\quad z\geq 0. \quad (11)$$

The formally exact solutions in Eq. (11) involve two speeds: the phase speed v_{ph} of the forward and backward moving plane waves multiplying the envelope functions and the fixed speed v associated with the forward moving envelope functions. The wave fields $u_{\mp}(\rho,z,t)$ are essentially invariant; at worst, they can propagate along the positive z-direction with only local deformations and regenerate periodically. The solutions given in Eq. (11) cannot be physically realizable by virtue of their invariance; they must contain infinite energy initially. Nevertheless, based on experience with analogous idealized wave solutions in the absence of absorption and dispersion, physically realizable versions can be achieved by means of time-limited or size-limited aperture sources on the plane $z=0$. It is in this sense that exact invariant wave solutions, such as those in Eq. (11), are important.

For appropriate spectra $\tilde{\psi}(\kappa,\alpha)$ in Eq. (10), the envelope functions and, hence, the real fields $u_{\mp}(\rho,z,t)$ can be made compact both temporally and spatially around the pulse center at $z=vt$. The invariance and spatio-temporal localization of the exact solutions $u_{\mp}(\rho,z,t)$ in Eq. (11) is achieved by balancing two physical mechanisms; namely, diffraction and dispersion. A crucial role in attaining such a balance is played by the fixed speed v. How does one choose this speed? Can it be given a physical interpretation? A reasonable answer is that the speed v must be chosen so that with proper selections of spectra $\tilde{\psi}(\kappa,\alpha)$ the integrations in Eq. (10) converge to nonsingular, localized solutions for the envelope functions. Essentially, this means that although the solutions $u_{\mp}(\rho,z,t)$ in

Eq. (11) are general, their actual implementation and correct choice(s) for the speed v will depend on specific realizations of the wavenumber $\beta(\omega_0)$; in other words, on the type of medium one deals with. In the sequel, we shall provide specific illustrative examples along these lines.

3. EXACT MODULATED LOCALIZED WAVES IN A COLD PLASMA

There are several physical situations where $\beta^2(\omega)$ is a second-order polynomial in ω. In such cases, one can obtain exact analytical localized wave solutions. A particular example of this category will be provided in this section. Specifically, a study will be undertaken of localized waves in cold plasma. The main goal of this canonical problem is to illustrate in a relatively simple setting the procedure for deriving invariant localized waves by a judicious choice of the pivotal fixed speed v. Most of the exact localized wave solutions derived in this section are new. In addition to their own physical relevance, they will be used for comparison purposes with approximate localized waves in more complicated media in the next section.

Electromagnetic wave propagation in collisionless (cold) plasma is governed by the Klein-Gordon equation. In this case, $\beta^2(\omega) = (\omega/c)^2 - (\omega_p/c)^2$, where $c = 1/\sqrt{\varepsilon_0\mu_0}$ is the speed of light *in vacuo* and ω_p denotes the plasma frequency. For this medium, the dispersion relations given in Eq. (9) simplify to

$$-\kappa^2+\alpha^2\left(\frac{v^2}{c^2}-1\right)+2\alpha\beta(\omega_0)\left(1\mp\frac{\omega_0}{\beta(\omega_0)}\frac{v}{c^2}\right)=0. \quad (12)$$

The phase speed in cold plasma is given by $v_{ph} = \omega_0/\beta(\omega_0) = c/\sqrt{1-(\omega_p/\omega_0)^2}$. It is related to the group speed as follows: $v_{ph}v_{gr} = c^2$. As a result, the dispersion relations in Eq. (12) may be rewritten as

$$-\kappa^2+\alpha^2\left(\frac{v^2}{c^2}-1\right)+2\alpha\beta(\omega_0)\left(1\mp\frac{v}{v_{gr}}\right)=0, \quad (13)$$

and the envelope wave functions $\psi_\mp(\rho,\tau)$ in Eq. (10) specialize to

$$\psi_\mp(\rho,\tau)=\int_0^\infty d\alpha\int_0^\infty d\kappa\kappa J_0(\kappa\rho)e^{\mp i\alpha v\tau}$$
$$\delta\left\{-\kappa^2+\alpha^2\left(\frac{v^2}{c^2}-1\right)+2\alpha\beta(\omega_0)\left(1\mp\frac{v}{v_{gr}}\right)\right\}\tilde{\psi}(\kappa,\alpha). \quad (14)$$

It is clear, in this case, that the fixed speed v can be compared to three other speeds; the speed of light *in vacuo* c and the phase and group speeds in the medium. When such comparisons are made, one should take into consideration the relations $c < v_{ph} < \infty$ and $0 \le v_{gr} < c$ for $\omega_0 \ge \omega_p$. Several distinct cases will be considered in detail below.

Case A ($v > c$):

With $\tilde{\gamma} = 1/\sqrt{(v/c)^2-1}$, the integration over κ in Eq. (14) allows the envelope function $\psi_+(\rho,\tau)$ to be expressed as

$$\psi_+(\rho,\tau)=\int_0^\infty d\alpha\tilde{F}_+(\alpha)e^{i\alpha v\tau}$$
$$J_0\left[\frac{\rho}{\tilde{\gamma}}\sqrt{\alpha^2+2\alpha\tilde{\gamma}^2\beta(\omega_0)\left(1+\frac{v}{v_{gr}}\right)}\right];\ \tau=t-\frac{z}{v} \quad (15)$$

For the specific spectrum $\tilde{F}_+(\alpha) = \exp(-a_+\alpha)$, where a_+ is a real positive parameter, the integration can be carried out explicitly ([61], 4.15.19), resulting in the following exact solution to the 3D Klein-Gordon equation:

$$u_+(\rho,z,t)=e^{i\omega_0\left(t+\frac{z}{v_{ph}}\right)}\psi_+(\rho,\tau)$$
$$=e^{i\omega_0\left(t+\frac{z}{v_{ph}}\right)}\frac{1}{\sqrt{(\rho/\tilde{\gamma})^2+\left[a_+-iv\left(t-\frac{z}{v}\right)\right]^2}}$$
$$\times\exp\left\{\tilde{\gamma}^2\beta(\omega_0)\left(1+\frac{v}{v_{gr}}\right)\left[a_+-iv\left(t-\frac{z}{v}\right)\right]\right.$$
$$\left.-\sqrt{(\rho/\tilde{\gamma})^2+\left[a_+-iv\left(t-\frac{z}{v}\right)\right]^2}\right\}+cc. \quad (16)$$

This solution is clearly bidirectional. The envelope function $\psi_+(\rho,\tau)$ moves in the positive z-direction with the fixed superluminal speed v, while it is modulated by a plane wave propagating backwards with the phase speed v_{ph}. It should be noted that since the relation $v > c$ must hold for the validity of the solution, and furthermore, $c < v_{ph} < \infty$, a particular choice for v would be the phase speed v_{ph}. In that case, both the envelope function and the modulating plane wave in Eq. (16) move with the same speed, but in opposite directions. For $v = v_{ph}$, a closer study reveals that the wavepacket $u_+(\rho,z,t)$ can be expressed in terms of a different envelope function traveling forward at the phase speed v_{ph}, modulated by a plane wave also moving in the forward direction at the group speed v_{gr}; specifically,

$$u_+(\rho,z,t)=\exp\left[i\,2\beta(\omega_0)\frac{v_{ph}}{v_{gr}}\left(\frac{v_{ph}}{v_{gr}}-1\right)^{-1}(z-v_{gr}t)\right]$$
$$\frac{1}{\sqrt{(\rho/\tilde{\gamma})^2+\left[a_+-iv_{ph}\left(t-\frac{z}{v_{ph}}\right)\right]^2}}\times\exp\left\{-\tilde{\gamma}^2\beta(\omega_0)\right.$$
$$\left.\left(1+\frac{v_{ph}}{v_{gr}}\right)\sqrt{(\rho/\tilde{\gamma})^2+\left[a_+-iv_{ph}\left(t-\frac{z}{v_{ph}}\right)\right]^2}\right\}+cc. \quad (17)$$

In order to be able to interpret the general solution $u_+(\rho,z,t)$ given in Eq. (16) more easily, the latter is specialized to a solution of the 3D scalar wave equation by means of the restrictions $\beta(\omega_0) = \omega_0/c$, $v_{ph} = v_{gr} = c$. One, then, obtains

$$u^+_{\mathrm{SWE}}(\rho,z,t)=e^{i\omega_0\left(t+\frac{z}{c}\right)}\psi_+(\rho,\tau)$$

$$= e^{i\omega_0\left(t+\frac{z}{c}\right)} \frac{1}{\sqrt{(\rho/\tilde{\gamma})^2 + \left[a_+ - iv\left(t-\frac{z}{v}\right)\right]^2}} \times \exp\left\{\tilde{\gamma}^2 \frac{\omega_0}{c}\left(1+\frac{v}{c}\right)\left[a_+ - iv\left(t-\frac{z}{v}\right)\right] - \sqrt{(\rho/\tilde{\gamma})^2 + \left[a_+ - iv\left(t-\frac{z}{v}\right)\right]^2}\right\} + cc. \quad (18)$$

This solution is bidirectional. The envelope function $\psi_+(\rho, \tau)$ moves in the positive z-direction with the fixed superluminal speed v, while it is modulated by a plane wave propagating backwards with the speed of light. A closer study, however, reveals that the wavepacket can be expressed in terms of a different envelope function traveling forward at the superluminal speed v, modulated by a plane wave also moving in the forward direction at the subluminal speed c^2/v; specifically,

$$u^+_{\text{SWE}}(\rho, z, t) = \frac{1}{\sqrt{(\rho/\tilde{\gamma})^2 + \left[a_+ - iv\left(t-\frac{z}{v}\right)\right]^2}} \exp\left[i\frac{\omega_0}{c}\frac{(v/c)}{\left(\frac{v}{c}-1\right)}\left(z - \frac{c^2}{v}t\right)\right] \times \exp\left\{-\tilde{\gamma}^2\frac{\omega_0}{c}\left(1+\frac{v}{c}\right) \sqrt{(\rho/\tilde{\gamma})^2 + \left[a_+ - iv\left(t-\frac{z}{v}\right)\right]^2}\right\} + cc. \quad (19)$$

In the limit $v \to c$, it can be shown from either Eq. (18) or Eq. (19) that $u^+_{\text{SWE}}(\rho, z, t)$ is simplified as follows:

$$u^+_{\text{FWM}}(\rho, z, t) = e^{i\omega_0\left(t+\frac{z}{c}\right)}\psi_+(\rho, \tau) + cc = e^{i\omega_0\left(t+\frac{z}{c}\right)} \frac{1}{a_+ - ic\left(t-\frac{z}{c}\right)} e^{-\frac{\omega_0}{c}\frac{\rho^2}{a_+ - ic\left(t-\frac{z}{c}\right)}} + cc. \quad (20)$$

This is the original axisymmetric focus wave mode (FWM) solution to the 3D scalar wave equation in unbounded free space. In the form shown in Eq. (20), it was first formulated by Ziolkowski [3] who was motivated by Brittingham's work in 1983 [1]. The pure FWM consists of an envelope traveling along the positive z-direction with speed c modulated by a plane wave moving in the negative z-direction with speed c. The entire wave packet sustains only local deformations; more precisely, it regenerates periodically. The FWM is physically unrealizable because it contains infinite energy. Finite energy FWM-type localized waves in an unbounded space have been derived by various means, e.g., by a superposition of pure FWMs, by Ziolkowski [3], Besieris *et al.* [4], and others.

In the limit $\omega_0 \to 0$, one obtains from the solution $u^+_{\text{SWE}}(\rho, z, t)$ given in Eq. (19) the "pure" zero-order X wave solution

$$u_{XW}\left(\rho, t - \frac{z}{v}\right) = \frac{1}{\sqrt{(\rho/\tilde{\gamma})^2 + \left[a_+ - iv\left(t-\frac{z}{v}\right)\right]^2}}, \quad (21)$$

which was introduced by Lu and Greenleaf [5] and Ziolkowski *et al.* [6]. (In the latter reference, this solution was referred to as a *slingshot superluminal pulse*). It is an infinite energy localized wave (LW) pulse propagating without distortion along the z-direction with the superluminal speed.

The more general X-shaped wave packet $u^+_{\text{SWE}}(\rho, z, t)$ given in Eq. (19) combines features present in both the zero-order X wave [cf. Eq. (21)] and the FWM [cf. Eq. (20)]. For this reason, it has been called focused X wave (FXW) [8]. It resembles the zero-order X wave, except that its highly focused central region has a tight exponential localization, in contrast to the loose algebraic transverse dependence of the zero-order X wave.

With this background in mind, one can interpret the solution $u_+(\rho, z, t)$ of the 3D Klein-Gordon equation given in Eq. (16) as an extension of that of the 3D scalar wave equation given in Eq. (19). Thus, $u_+(\rho, z, t)$ is an exact modulated FXW in cold plasma.

Proceeding analogously, one can establish the following X-shaped unidirectional localized wave solution

$$u_-(\rho, z, t) = e^{i\omega_0\left(t-\frac{z}{v_{ph}}\right)}\psi_-(\rho, \tau) = e^{i\omega_0\left(t-\frac{z}{v_{ph}}\right)} \frac{\exp\left[-i\tilde{\gamma}^2\beta(\omega_0)\left(\frac{v}{v_{gr}}-1\right)v\left(t-\frac{z}{v}\right)\right]}{\sqrt{(\rho/\tilde{\gamma})^2 + \left[a_- - iv\left(t-\frac{z}{v}\right)\right]^2}} \times \exp\left\{-\tilde{\gamma}^2\beta(\omega_0)\left(\frac{v}{v_{gr}}-1\right) \sqrt{(\rho/\tilde{\gamma})^2 + \left[a_- - iv\left(t-\frac{z}{v}\right)\right]^2}\right\} + cc. \quad (22)$$

to the 3D Klein-Gordon equation, with a_- being a real positive parameter. As in the case of the solution $u_+(\rho, z, t)$, the speed v of the envelope can be chosen to equal the phase speed v_{ph}. Then, $u_-(\rho, z, t)$ can be rewritten in the new form

$$u_-\left(\rho, t - \frac{z}{v_{ph}}\right) = \frac{\exp\left\{-\beta(\omega_0)\sqrt{(\rho/\tilde{\gamma})^2 + \left[a_- - iv_{ph}\left(t-\frac{z}{v_{ph}}\right)\right]^2}\right\}}{\sqrt{(\rho/\tilde{\gamma})^2 + \left[a_- - iv_{ph}\left(t-\frac{z}{v_{ph}}\right)\right]^2}} + cc. \quad (23)$$

This special X-shaped localized wave has an interesting structure. The "modulating plane wave" has disappeared completely; only an envelope function remains traveling along the positive z-direction with the phase speed v_{ph}. Essentially, it is the analog of the pure XW solution [cf. Eq. (21)] for cold plasma.

Case B ($v = c$):

For $v = c$, the dispersion relations in Eq. (13) become

$$-\kappa^2 + 2\alpha\beta(\omega_0)\left(1 \mp \frac{c}{v_{gr}}\right) = 0. \quad (24)$$

For α and κ nonnegative, the only interesting envelope wave function is

$$\psi_+(\rho,\tau)=\int_0^\infty d\alpha\int_0^\infty d\kappa\kappa J_0(\kappa\rho)e^{i\alpha c\tau}$$
$$\delta\left[-\kappa^2+2\alpha\beta(\omega_0)\left(1+\frac{c}{v_{gr}}\right)\right]\tilde{\psi}(\kappa,\alpha). \quad (25)$$

Integration over α yields

$$\psi_+(\rho,\tau)=\int_0^\infty d\kappa\kappa J_0(\kappa\rho)$$
$$\exp\left[ic\tau\frac{\kappa^2}{2\beta(\omega_0)\left(1+c/v_{gr}\right)}\right]\tilde{F}_+(\alpha). \quad (26)$$

For the spectrum

$$\tilde{F}_+(\alpha)=\beta(\omega_0)\left(1+c/v_{gr}\right)$$
$$\exp\left[-a_+\frac{\kappa^2}{2\beta(\omega_0)\left(1+c/v_{gr}\right)}\right], \quad (27)$$

where a_+ is a real positive parameter, and the integration over κ in Eq. (26) can be carried out explicitly ([62], 6.631.4), resulting in the following exact solution to the 3D Klein-Gordon equation:

$$u_+(\rho,z,t)=e^{i\omega_0\left(t+\frac{z}{v_{ph}}\right)}\psi_+(\rho,\tau)=\frac{e^{i\omega_0\left(t+\frac{z}{v_{ph}}\right)}}{a_+-ic\left(t-\frac{z}{c}\right)}$$
$$\exp\left[-\frac{1}{2}\beta(\omega_0)(1+c/v_{gr})\frac{\rho^2}{a_+-ic\left(t-\frac{z}{c}\right)}\right]. \quad (28)$$

This is an FWM solution for cold plasma. It should be noted that when $u_+(\rho,z,t)$ is specialized to the case of free space, one obtains the FWM solution of the 3D scalar wave equation given in Eq. (20).

Case C ($v=-c$):

Proceeding analogously with the previous case, we obtain the FWM solution

$$u_-(\rho,z,t)=e^{i\omega_0\left(t-\frac{z}{v_{ph}}\right)}\psi_-(\rho,\tau)=\frac{e^{i\omega_0\left(t-\frac{z}{v_{ph}}\right)}}{a_--ic\left(t+\frac{z}{c}\right)}$$
$$\exp\left[-\frac{1}{2}\beta(\omega_0)(1+c/v_{gr})\frac{\rho^2}{a_--ic\left(t+\frac{z}{c}\right)}\right]. \quad (29)$$

for the Klein-Gordon equation. When $u_-(\rho,z,t)$ is specialized to the case of free space, one obtains the dual to the FWM solution of the 3D scalar wave equation given in Eq. (20); specifically,

$$u^-_{\text{FWM}}(\rho,z,t)=e^{i\omega_0\left(t-\frac{z}{c}\right)}\psi_-(\rho,\tau)+cc$$
$$=e^{i\omega_0\left(t-\frac{z}{c}\right)}\frac{1}{a_--ic\left(t+\frac{z}{c}\right)}e^{-\frac{\omega_0}{c}\frac{\rho^2}{a_--ic\left(t+\frac{z}{c}\right)}}+cc. \quad (30)$$

The importance of this solution is due to its connection to paraxial pulsed beam solutions to the 3D scalar wave equation. Specifically, if $t+(z/c)$ is formally replaced by $2z/c$ in $u^-_{\text{FWM}}(\rho,z,t)$, one obtains an exact solution to the paraxial pulsed beam equation

$$\left(\nabla^2_\rho+2\frac{\partial^2}{\partial\varsigma\partial z}\right)u_{PB}(\rho,z,t)=0,\quad \varsigma=z-ct. \quad (31)$$

One of the simplest monochromatic paraxial beams is derived from Eq. (30), viz.,

$$u_{PB}(\rho,z,t)=\frac{1}{a_+-i2z}e^{-i\omega_0\left(z-ct-i\frac{\rho^2}{a_+-i2z}\right)}. \quad (32)$$

It is important to point out that the procedure outlined above for deriving paraxial pulsed beams from exact FWMs does not apply to the Klein-Gordon equation.

Case D ($v<c$):

Recall the dispersion relations given in Eq. (13), viz.,

$$-\kappa^2+\alpha^2\left(\frac{v^2}{c^2}-1\right)+2\alpha\,\beta(\omega_0)\left(1\mp\frac{v}{v_{gr}}\right)=0 \quad (33)$$

and the envelope wave functions $\psi_\mp(\rho,\tau)$ in Eq. (14); specifically,

$$\psi_\mp(\rho,\tau)=\int_0^\infty d\alpha\int_0^\infty d\kappa\kappa J_0(\kappa\rho)e^{\mp i\alpha v\tau}$$
$$\delta\left\{-\kappa^2+\alpha^2\left(\frac{v^2}{c^2}-1\right)+2\alpha\beta(\omega_0)\left(1\mp\frac{v}{v_{gr}}\right)\right\}\tilde{\psi}(\kappa,\alpha). \quad (34)$$

For $v<c$, one can identify three special ranges: (i) $v_{gr}<v<c$; (ii) $v=v_{gr}$; (iii) $v<v_{gr}$. The structure of the solutions corresponding to these three special ranges is summarized below:

(i) The solutions $u_\mp(\rho,z,t)$ are localized and subluminal.

(ii) The solution $u_-(\rho,z,t)$ is unbounded, and $u_+(\rho,z,t)$ is localized and subluminal.

(iii) The solution $u_-(\rho,z,t)$ is unbounded, and $u_+(\rho,z,t)$ is localized and subluminal.

We shall provide next an illustrative example of a subluminal localized solution for $u_+(\rho,z,t)$. With $\gamma=1/\sqrt{(1-(v/c)^2}$, the integration over κ and a change of variables in Eq. (34) allows the envelope function $\psi_+(\rho,\tau)$ to be brought to the particular form

$$\psi_+(\rho,\tau)=\exp\left[iv\tau\beta(\omega_0)\left(1+v/v_{gr}\right)\right]$$
$$\times\int_0^\infty d\bar{\alpha}\tilde{F}_+(\bar{\alpha})\cos(\bar{\alpha}v\tau)J_0\left(\frac{\rho}{\gamma}\sqrt{b^2-\bar{\alpha}^2}\right);$$

$$b = \gamma^2 \beta(\omega_0)\left(1 + \frac{v}{v_{gr}}\right); \ \tau = t - \frac{z}{v}. \quad (35)$$

For the spectrum $\tilde{F}_+(\bar{\alpha}) = H(\bar{\alpha} - b)$, where $H(\cdot)$ is the Heaviside unit step function, the integration can be carried out explicitly ([62], 6.677.6), resulting in the following exact solution to the 3D Klein-Gordon equation:

$$u_+(\rho, z, t) = e^{i\omega_0\left(t + \frac{z}{v_{ph}}\right)} \psi_+(\rho, \tau)$$
$$= e^{i\omega_0\left(t + \frac{z}{v_{ph}}\right)} \exp\left[i\beta(\omega_0)\left(1 + \frac{v}{v_{ph}}\right) v\left(t - \frac{z}{v}\right)\right]$$
$$\times \frac{\sin\left\{b\sqrt{\rho^2 + \left[v\left(t - \frac{z}{v}\right)\right]^2}\right\}}{\sqrt{\rho^2 + \left[v\left(t - \frac{z}{v}\right)\right]^2}} + cc. \quad (36)$$

This expression is a nonsingular localized wave consisting of an envelope function moving in the positive z-direction with the subluminal ($v < c$) speed and modulated by a plane wave propagating in the opposite direction with the phase speed v_{ph}.

4. INVARIANT MODULATED NONPARAXIAL LOCALIZED WAVES IN ARBITRARY LOSSLESS DISPERSIVE MEDIA: SECOND-ORDER DISPERSIVE EFFECTS

Our aim in this section is to examine the feasibility of novel nonparaxial invariant localized wave solutions in arbitrary lossless dispersive media, with dispersive effects considered only up to second order. Using the notation

$$\beta_n = [(d^n/d\omega^n)\beta(\omega)]_{\omega=\omega_0}, \ v_{ph} = \omega_0/\beta_0, \ v_{gr} = 1/\beta_1, \quad (37)$$

the dispersion relations in Eq. (8) become

$$-\kappa^2 + \alpha^2\left[\left(\frac{v}{v_{gr}}\right)^2 \left(1 + \beta_0\beta_2 v_{gr}^2\right) - 1\right]$$
$$+2\alpha\beta_0\left(1 \mp \frac{v}{v_{gr}}\right) = 0. \quad (38)$$

For convenience, the following definition is made:

$$\frac{1}{g^2} = (v/v_{gr})^2\left(1 + \beta_0\beta_2 v_{gr}^2\right) - 1. \quad (39)$$

In the sequel, it will be tacitly stipulated that g is a real number. This condition could be satisfied for both normal ($\beta_2 > 0$) and anomalous ($\beta_2 < 0$) dispersion if v were larger than a critical speed v_c determined by setting the right-hand side of Eq. (38) equal to zero. If, however, $v = v_{gr}$, one has the relation $1/g^2 = \beta_0\beta_2 v_{gr}^2$. This means that the reality of g can be satisfied only for normal dispersion ($\beta_2 > 0$). Of course, these statements assume that one chooses the central frequency ω_0 so that β_0 is positive.

The envelope functions corresponding to the dispersion relations in Eq. (38), with the definition in Eq. (39) considered, can be written as

$$\psi_\mp(\rho, \tau) = \int_0^\infty d\alpha \int_0^\infty d\kappa\kappa J_0(\kappa\rho) e^{\mp i\alpha v\tau}$$
$$\delta\left\{-\kappa^2 + \frac{\alpha^2}{g^2} + 2\alpha\beta_0\left(1 \mp \frac{v}{v_{gr}}\right)\right\}\tilde{\psi}(\kappa, \alpha). \quad (40)$$

For $v \neq v_{gr}$ and $v > v_c$, these envelope functions are identical to those studied in Section 3, Case A, except for the change $\gamma \to g$. Consequently, the FXW solutions found there can be transferred to the case under consideration; specifically,

$$u_+(\rho, z, t) = e^{i\omega_0\left(t + \frac{z}{v_{ph}}\right)} \frac{1}{\sqrt{(\rho/g)^2 + \left[a_+ - iv\left(t - \frac{z}{v}\right)\right]^2}}$$
$$\times \exp\left\{g^2\beta_0\left(1 + \frac{v}{v_{gr}}\right)\left[a_+ - iv\left(t - \frac{z}{v}\right)\right]\right.$$
$$\left. -\sqrt{(\rho/g)^2 + \left[a_+ - iv\left(t - \frac{z}{v}\right)\right]^2}\right\} + cc. \quad (41)$$

and

$$u_-(\rho, z, t) = e^{i\omega_0\left(t - \frac{z}{v_{ph}}\right)} \frac{\exp\left[-ig^2\beta_0\left(\frac{v}{v_{gr}} - 1\right) v\left(t - \frac{z}{v}\right)\right]}{\sqrt{(\rho/g)^2 + \left[a_- - iv\left(t - \frac{z}{v}\right)\right]^2}}$$
$$\times \exp\left\{-g^2\beta_0\left(\frac{v}{v_{gr}} - 1\right)\right.$$
$$\left.\sqrt{(\rho/\gamma)^2 + \left[a_- - iv\left(t - \frac{z}{v}\right)\right]^2}\right\} + cc. \quad (42)$$

Caution must be exercised when implementing the solution $u_-(\rho, z, t)$. Two conditions for its validity have already been mentioned; namely, $v \neq v_{gr}$ and $v > v_c$. In this case, however, an additional restriction must be imposed; namely, $v > v_{gr}$.

Recall that the relation $1/g^2 = 0$ holds at the critical speed v_c. In this case, the dispersion relations in Eq. (38) are identical to those examined in Section 3, Cases B and C, except for the changes $\pm c \to \pm v_c$. This means that the medium under consideration can support FWM solutions; specifically,

$$u_+(\rho, z, t) = \frac{e^{i\omega_0\left(t + \frac{z}{v_{ph}}\right)}}{a_+ - iv_c\left(t - \frac{z}{v_c}\right)}$$
$$\exp\left[-\frac{1}{2}\beta_0(1 + v_c/v_{gr})\frac{\rho^2}{a_+ - iv_c\left(t - \frac{z}{v_c}\right)}\right], \quad (43)$$

and

$$u_-(\rho, z, t) = \frac{e^{i\omega_0\left(t - \frac{z}{v_{ph}}\right)}}{a_- - iv_c\left(t + \frac{z}{v_c}\right)}$$

$$\exp\left[-\frac{1}{2}\beta_0(1+v_c/v_{gr})\frac{\rho^2}{a_- - iv_c\left(t+\frac{z}{v_c}\right)}\right]. \quad (44)$$

An intriguing situation arises when $g(\text{real}) \neq 0$ and $v = v_{gr}$. As mentioned above, this condition can be met only for normal dispersion ($\beta_2 > 0$). In that case, the $(-)$ dispersion relation in Eq. (38) is reduced to

$$-\kappa^2 + \frac{\alpha^2}{g^2} = 0; \quad 1/g^2 = \beta_0\beta_2 v_{gr}^2, \quad (45)$$

and the corresponding envelope function assumes the simple form

$$\psi_-(\rho,\tau) = \int_0^\infty d\alpha \int_0^\infty d\kappa\kappa J_0(\kappa\rho)e^{-i\alpha v_{gr}\tau} \delta\left(-\kappa^2+\frac{\alpha^2}{g^2}\right)\tilde{\psi}(\kappa,\alpha). \quad (46)$$

The integration over α gives rise to the envelope function

$$\psi_-(\rho,\tau) = \int_0^\infty d\kappa\kappa J_0(\kappa\rho)\tilde{G}(\kappa)e^{-i\kappa g v_{gr}\tau}. \quad (47)$$

Choosing the specific spectrum $\tilde{G}(\kappa) = (1/\kappa)\exp(-a_-\kappa g)$, one obtains the following solution:

$$u_-(\rho,z,t) = e^{i\omega_0\left(t-\frac{z}{v_{ph}}\right)}\psi_-(\rho,\tau) = e^{i\omega_0\left(t-\frac{z}{v_{ph}}\right)} \frac{1}{\sqrt{(\rho/g)^2+\left[a_- - iv_{gr}\left(t-\frac{z}{v_{gr}}\right)\right]^2}}. \quad (48)$$

This interesting, localized wave structure is a *modulated X wave*. It consists of an envelope traveling forwards with the group speed v_{gr}, and it is modulated by a plane wave also moving forwards, however with the phase speed v_{ph}. A solution analogous to the one given in Eq. (48), but derived on the basis of a paraxial approximation, was reported by Porras and Gonzalo [44] recently. It is important to recall that no bounded unidirectional localized wave solution [analogous to that in Eq. (48)] could be found in Section 3 for the Klein-Gordon equation with an envelope traveling at the group speed, simply because cold plasma is an anomalously dispersive medium ($\beta_2 < 0$), and the restriction $0 < v_{gr} < c$ is applied in that case.

Thus far, the discussion has assumed that the quantity g defined in Eq. (39) is real. For $v < v_c$, g is purely imaginary, i.e., $g = i\bar{g}$ with $\bar{g}$ real. Then, the dispersion relations in Eq. (38) change to

$$-\kappa^2 - \frac{\alpha^2}{\bar{g}^2} + 2\alpha\beta_0\left(1 \mp \frac{v}{v_{gr}}\right) = 0, \quad (49)$$

which are analogous to those studied in Section 3, Case D. Therefore, possibilities exist for "subluminal" localized waves; one such structure is given below:

$$u_+(\rho,z,t) = e^{i\omega_0\left(t+\frac{z}{v_{ph}}\right)}\exp\left[i\beta_0\left(1+\frac{v}{v_{gr}}\right)v\left(t-\frac{z}{v}\right)\right] \times\frac{\sin\left\{b\sqrt{\rho^2+\left[v\left(t-\frac{z}{v}\right)\right]^2}\right\}}{\sqrt{\rho^2+\left[v\left(t-\frac{z}{v}\right)\right]^2}}+cc; \; b\equiv\bar{g}^2\beta_0\left(1+\frac{v}{v_{gr}}\right). \quad (50)$$

The condition for subluminal localized waves is satisfied for $v = v_{gr}$ in media characterized by anomalous dispersion. In that case, one has $\bar{g}^2 = \beta_0|\beta_2|v_{gr}^2$, and the solution above becomes

$$u_+(\rho,z,t) = e^{i\omega_0\left(t+\frac{z}{v_{ph}}\right)}\exp\left[i2\beta_0 v_{gr}\left(t-\frac{z}{v_{gr}}\right)\right] \frac{\sin\left\{b\sqrt{\rho^2+\left[v_{gr}\left(t-\frac{z}{v_{gr}}\right)\right]^2}\right\}}{\sqrt{\rho^2+\left[v_{gr}\left(t-\frac{z}{v_{gr}}\right)\right]^2}}+cc, \quad (51)$$

with $b \equiv 2\beta_0^2|\beta_2|v_{gr}^2$. Of course, it is known that in general anomalously dispersive media the group speed can take any value from $-\infty$ to ∞. Therefore, one cannot ascertain without additional information whether the envelope of the wave packet in Eq. (51) moves in the positive or negative z-direction.

5. FINITE-ENERGY MODULATED NONPARAXIAL LOCALIZED WAVES IN ARBITRARY LOSSLESS DISPERSIVE MEDIA: SECOND-ORDER DISPERSIVE EFFECTS

It has already been mentioned that physically realizable localized waves in the presence of dispersion can be achieved by means of time-limited or size-limited aperture sources on the plane $z = 0$. It is in this respect that the exact invariant wave solutions derived in Section 3 for the Klein-Gordon equation and the approximate ones obtained in Section 4 for arbitrary lossless dispersive media could be important. In this section, an attempt will be made to determine novel nonparaxial analytical expressions of finite-energy localized waves in arbitrary lossless dispersive media.

Toward this goal, it is convenient to return to Eq. (4) for the envelope functions and simplify it under the assumption that absorption can be neglected, and dispersive effects are retained up to second order:

$$\left[\nabla_\perp^2+\frac{\partial^2}{\partial z^2}\mp i2\beta_0\frac{\partial}{\partial z}-i\,2\beta_0\beta_1\frac{\partial}{\partial t} -\left(\beta_1^2+\beta_0\beta_2\right)\frac{\partial^2}{\partial t^2}\right]\psi_\mp(\vec{r},t)=0. \quad (52)$$

General solutions for the envelope functions can be represented in terms of an ordinary Fourier synthesis as follows:

$$\psi_\mp(\vec{r},t) = \int_{R_3} d\vec{k}\int_{R_1} d\omega\delta\Big[-k_x^2-k_y^2-k_z^2+\left(\beta_1^2+\beta_0\beta_2\right)$$

$$\omega^2 \mp 2\beta_0 k_z + 2\beta_0\beta_1\omega\Big] \times e^{-i\vec{k}\cdot\vec{r}} e^{i\omega t}\tilde{\psi}(\vec{k},\omega). \quad (53)$$

Next, the squares of the functions involving k_z and ω within the argument of the Dirac delta function are completed. A subsequent change of variables recasts Eq. (53) into the new form

$$\psi_{\mp}(\vec{r},t) = \exp\left[-i\beta_0\beta_1 v_{eff}^2 \left(t \mp \frac{z}{\beta_1 v_{\mathit{eff}}^2}\right)\right]\phi(\vec{r},t), \quad (54)$$

with the function $\varphi(\vec{r},t)$ obeying the 3D Klein-Gordon equation

$$\left(\nabla^2 - \frac{1}{v_{\mathit{eff}}^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_{\mathit{eff}}^2}{v_{eff}^2}\right)\phi(\vec{r},t) = 0, \quad (55)$$

which involves the *effective speed*

$$v_{\mathit{eff}} = \frac{1}{\sqrt{\beta_1^2 + \beta_0\beta_2}} \quad (56)$$

and the *effective "plasma frequency"*

$$\omega_{\mathit{eff}} = \beta_0 v_{\mathit{eff}}\sqrt{(v_{\mathit{eff}}/v_{gr})^2 - 1} \quad (57)$$

In the case of the Klein-Gordon equation with $\beta(\omega) = (1/c)\sqrt{\omega^2 - \omega_p^2}$, the quantities above are simplified to $v_{\mathit{eff}} = c$ and $\omega_{\mathit{eff}} = \omega_p$.

To derive finite-energy localized wave solutions, it is convenient to introduce the following new coordinates: $\varsigma = z - vt$, $\eta = z + v_{\mathit{eff}}t$. One, then, has in the place of Eq. (55)

$$\left[\nabla_\rho^2 - \left(\frac{v^2}{v_{\mathit{eff}}^2} - 1\right)\frac{\partial^2}{\partial\varsigma^2} + 2\left(1 + \frac{v}{v_{\mathit{eff}}}\right)\frac{\partial^2}{\partial\varsigma\partial\eta} - \frac{\omega_{\mathit{eff}}^2}{v_{\mathit{eff}}^2}\right]$$
$$\phi(\rho,\varsigma,\eta) = 0, \quad (58)$$

where azimuthal symmetry has been assumed for simplicity. A general solution to this equation can be represented spectrally as follows:

$$\phi(\rho,\varsigma,\eta) = \int_0^\infty d\alpha \int_0^\infty d\beta \int_0^\infty d\kappa\kappa J_0(\kappa\rho) e^{-i\alpha\varsigma} e^{i\beta\eta}$$
$$\times\delta\left[-\kappa^2 + \left(\frac{v^2}{v_{\mathit{eff}}^2} - 1\right)\alpha^2 + 2\left(1 + \frac{v}{v_{\mathit{eff}}}\right)\alpha\beta - \frac{\omega_{\mathit{eff}}^2}{v_{\mathit{eff}}^2}\right]\tilde{\phi}(\kappa,\alpha,\beta). \quad (59)$$

For a given central radian frequency ω_0, the effective speed v_{eff} and effective frequency ω_{eff} are fixed. In the sequel, it will be assumed that v_{eff} is a real positive quantity. By virtue of its definition, however, ω_{eff} may be real or purely imaginary. To proceed further with the analysis, specific assumptions will have to be made.

5.1. Finite-Energy FWM-Type Localized Waves

Let ω_{eff} be real and $v = v_{\mathit{eff}}$. Then, Eq. (59) is simplified to

$$\phi(\rho,\varsigma,\eta) = \int_0^\infty d\beta \int_0^\infty d\kappa\kappa J_0(\kappa\rho) e^{i\beta\eta}$$
$$e^{-i\left(\kappa^2 + \frac{\omega_{\mathit{eff}}^2}{v_{\mathit{eff}}^2}\right)\frac{\varsigma}{4\beta}}\phi_1(\kappa,\beta) \quad (60)$$

after the integration over α has been performed. Choosing the specific spectrum

$$\tilde{\phi}_1(\kappa,\beta) = \frac{1}{2\sqrt{\pi}\beta^{3/2}} e^{-a_2\beta} e^{-a_1\frac{\kappa^2}{4\beta}} e^{-a_3\frac{\omega_{\mathit{eff}}^2}{v_{\mathit{eff}}^2}\frac{1}{4\beta}}, \quad (61)$$

where $a_{1,2,3}$ are positive real parameters, leads to the final solutions [recall Eqs. (54) and (3)]

$$u_{\mp}(\rho,z,t) = e^{i\omega_0\left(t \mp \frac{z}{v_{ph}}\right)} e^{-i\beta_0\beta_1 v_{\mathit{eff}}^2\left(t \mp \frac{z}{\beta_1 v_{\mathit{eff}}^2}\right)}$$
$$\frac{1}{a_1 + i\varsigma}\frac{1}{\sqrt{a_2 - i\eta + \frac{\rho^2}{a_1 + i\varsigma}}}$$
$$\times\exp\left[-\frac{\omega_{\mathit{eff}}}{v_{\mathit{eff}}}\sqrt{(a_3 + i\varsigma)\left(a_2 - i\eta + \frac{\rho^2}{a_1 + i\varsigma}\right)}\right]. \quad (62)$$

To be interpreted more easily, they are specialized to free space. In that case, one has the finite-energy *splash mode* solution [3]

$$u(\rho,z,t) = \frac{1}{a_1 + i\varsigma}\frac{1}{\sqrt{a_2 - i\eta + \frac{\rho^2}{a_1 + i\varsigma}}};$$
$$\varsigma = z - ct, \ \eta = z + ct \quad (63)$$

to the 3D scalar wave equation in vacuum, which consists of a superposition of FWMs. In this sense, one can say that the finite-energy solutions $u_{\mp}(\rho,z,t)$ given in Eq. (62) are the splash mode analogs for an arbitrary lossless dispersive medium, with dispersive effects considered up to second order. Three distinct speeds appear in Eq. (62): the phase speed v_{ph}, speed $\beta_1 v_{\mathit{eff}}^2 = v_{\mathit{eff}}^2/v_{gr}$, and effective speed v_{eff}.

5.2. Finite-Energy X-Shaped Localized Waves

In Eq. (59), it is assumed that $v > v_{\mathit{eff}}$ and ω_{eff} is real. Then, with $g \equiv 1/\sqrt{(v/v_{\mathit{eff}})^2 - 1}$ and upon integration with respect to κ, one has

$$\phi(\rho,\varsigma,\eta) = \int_0^\infty d\alpha \int_0^\infty d\beta e^{-i\alpha\varsigma} e^{i\beta\eta} J_0$$
$$\left[\frac{\rho}{g}\sqrt{\alpha^2 + 2g^2\left(\frac{v}{v_{\mathit{eff}}} + 1\right)\alpha\beta - g^2\frac{\omega_{\mathit{eff}}^2}{v_{\mathit{eff}}^2}}\right]\tilde{\phi}_2(\alpha,\beta). \quad (64)$$

Completion of the square with respect to α and a change of variable brings Eq. (64) to the new form

$$\phi(\rho,\varsigma,\eta)=\int_0^\infty d\beta e^{i\beta\eta}e^{ig^2\beta\left(\frac{v}{v_{eff}}+1\right)\varsigma}\int_0^\infty d\bar{\alpha}e^{-i\bar{\alpha}\varsigma}$$

$$J_0\left[\frac{\rho}{g}\sqrt{\bar{\alpha}^2-B^2}\right]\tilde{\phi}_2(\bar{\alpha},\beta);$$

$$B\equiv g\left(\frac{v}{v_{eff}}+1\right)\sqrt{\beta^2+\left(\frac{v}{v_{eff}}+1\right)^{-2}\frac{\omega_{eff}^2}{v_{eff}^2}}. \quad (65)$$

The spectrum $\tilde{\varphi}(\bar{a},\beta)$ is chosen as follows: $\tilde{\varphi}_2(\bar{\alpha},\beta)=\tilde{F}(\beta)\exp(-a_1\bar{\alpha})H(\bar{\alpha}-B)$. Then, the integration over $\bar{\alpha}$ can be carried out exactly ([61], 4.15.9), resulting in the expression

$$u_{\mp}(\rho,z,t)=e^{i\omega_0\left(t\mp\frac{z}{v_{ph}}\right)}e^{-i\beta_0\beta_1v_{eff}^2\left(t\mp\frac{z}{\beta_1v_{eff}^2}\right)}$$
$$\frac{1}{\sqrt{(\rho/g)^2+(a_1+i\varsigma)^2}}$$
$$\times\int_0^\infty d\beta e^{i\beta A}\exp\left[-Q\sqrt{\beta^2+W^2}\right]\tilde{F}(\beta), \quad (66)$$

where

$$A=\frac{v}{veff}\left(1+\frac{v}{veff}\right)^{-1}\left(z-\frac{v_{eff}^2}{v}t\right),$$

$$W=\left(\frac{v}{v_{eff}}+1\right)^{-2}\frac{\omega_{eff}^2}{v_{eff}^2}, \quad (67)$$

$$Q=g\left(\frac{v}{v_{eff}}+1\right)\sqrt{(\rho/g)^2+(a_1+i\varsigma)^2}.$$

For the singular spectrum $\tilde{F}(\beta)=\delta(\beta)$, these expressions are reduced to

$$u_{\mp}(\rho,z,t)=e^{i\omega_0\left(t\mp\frac{z}{v_{ph}}\right)}e^{-i\beta_0\beta_1v_{eff}^2\left(t\mp\frac{z}{\beta_1v_{eff}^2}\right)}$$
$$\times\frac{1}{\sqrt{(\rho/g)^2+(a_1+i\varsigma)^2}}$$
$$\exp\left[-g\frac{\omega_{eff}}{v_{eff}}\sqrt{(\rho/g)^2+(a_1+i\varsigma)^2}\right], \quad (68)$$

which are variations of the FXW solutions found in Section 4. Using appropriate nonsingular spectra $\tilde{F}(\beta)$, one can determine from Eq. (66) finite-energy X-shaped localized waves analogous to the modified focus X wave (MFXW) solutions to the 3D scalar wave equation in free space [8]. One such solution arises from the Fourier cosine transform

$$u_{\mp}(\rho,z,t)=e^{i\omega_0\left(t\mp\frac{z}{v_{ph}}\right)}e^{-i\beta_0\beta_1v_{eff}^2\left(t\mp\frac{z}{\beta_1v_{eff}^2}\right)}$$

$$\frac{1}{\sqrt{(\rho/g)^2+(a_1+i\varsigma)^2}}\times\int_0^\infty d\beta\cos(\beta A)\exp\left[-a\sqrt{\beta^2+W^2}\right]$$

$$\pi^{-1/2}\sqrt{2}\beta\left[\left(W^2+\beta^2\right)^{1/2}-W\right]\left(W^2+\beta^2\right)^{-1/2}; \quad (69)$$

specifically,

$$u_{\mp}(\rho,z,t)=e^{i\omega_0\left(t\mp\frac{z}{v_{ph}}\right)}e^{-i\beta_0\beta_1v_{eff}^2\left(t\mp\frac{z}{\beta_1v_{eff}^2}\right)}$$

$$\frac{1}{\sqrt{(\rho/g)^2+(a_1+i\varsigma)^2}}\times\left[Q+\left(Q^2+A^2\right)^{1/2}\right]^{1/2}$$

$$\left(Q^2+A^2\right)^{-1/2}\exp\left[-W\left(Q^2+A^2\right)^{1/2}\right]. \quad (70)$$

5.3. Finite-Energy Subluminal Localized Waves

In Eq. (59), it is assumed that $v<v_{eff}$, and ω_{eff} is real. Then, with $g\equiv1/\sqrt{1-(v/v_{eff})^2}$ and upon integration with respect to κ, one has

$$\phi(\rho,\varsigma,\eta)=\int_0^\infty d\alpha\int_0^\infty d\beta e^{-i\alpha\varsigma}e^{i\beta\eta}$$

$$J_0\left[\frac{\rho}{g}\sqrt{-\alpha^2+2g^2\left(\frac{v}{v_{eff}}+1\right)\alpha\beta+g^2\frac{\omega_{eff}^2}{v_{eff}^2}}\right]\times\tilde{\phi}_2(\alpha,\beta)(71)$$

Completion of the square with respect to α and a change of variable brings Eq. (71) to the new form

$$\phi(\rho,\varsigma,\eta)=\int_0^\infty d\beta e^{i\beta\eta}e^{-ig^2\beta\left(\frac{v}{v_{eff}}+1\right)\varsigma}\int_0^\infty d\bar{\alpha}e^{-i\bar{\alpha}\varsigma}$$

$$J_0\left[\frac{\rho}{g}\sqrt{\bar{B}^2-\bar{\alpha}^2}\right]\tilde{\phi}_2(\bar{\alpha},\beta);\ \bar{B}\equiv g^2\left(\frac{v}{v_{eff}}+1\right)$$

$$\sqrt{\beta^2-g^{-2}\left(\frac{v}{v_{eff}}+1\right)^{-2}\frac{\omega_{eff}^2}{v_{eff}^2}}. \quad (72)$$

The spectrum $\tilde{\phi}(\bar{a},\beta)$ is chosen as follows: $\tilde{\phi}_2(\bar{\alpha},\beta)=H(\bar{B}-\bar{\alpha})$. Then, the integration over $\bar{\alpha}$ can be carried out exactly ([62], 6.677.6]), resulting in the expression

$$u_{\mp}(\rho,z,t)=e^{i\omega_0\left(t\mp\frac{z}{v_{ph}}\right)}e^{-i\beta_0\beta_1v_{eff}^2\left(t\mp\frac{z}{\beta_1v_{eff}^2}\right)}$$

$$\frac{1}{\sqrt{(\rho/g)^2+\varsigma^2}}\times\int_0^\infty d\beta e^{-i\beta\frac{v}{v_{eff}}\left(1-\frac{v}{v_{eff}}\right)^{-1}\left(z-\frac{v_{eff}^2}{v}t\right)}$$

$$\sin\left(a\sqrt{\beta^2-b^2}\right)\tilde{F}(\beta), \quad (73)$$

where

$$a = g^2 \left(\frac{v}{v_{eff}} + 1 \right) \sqrt{(\rho/g)^2 + \varsigma^2},$$
$$b = g^{-1} \left(\frac{v}{v_{eff}} + 1 \right)^{-1} \frac{\omega_{eff}}{v_{eff}}. \quad (74)$$

Choosing the spectrum $\tilde{F}(\beta) = \sqrt{2}\pi^{-1/2} e^{-a_1 \beta} (\beta + b)^{-1/2}$, $a_1 > 0$, $\beta > b$, results ([61], 5.6.29) in the finite-energy subluminal localized solution

$$u_{\mp}(\rho, z, t) = e^{i\omega_0 \left(t \mp \frac{z}{v_{ph}}\right)} e^{-i\beta_0 \beta_1 v_{eff}^2 \left(t \mp \frac{z}{\beta_1 v_{eff}^2}\right)}$$
$$\frac{1}{\sqrt{(\rho/g)^2 + \varsigma^2}} \times a r^{-1} R^{-1/2} e^{-br};$$
$$r = \left(p^2 + a^2\right)^{1/2}, \; p = a_1 + i \frac{v}{v_{eff}} \left(1 - \frac{v}{v_{eff}}\right)^{-1}$$
$$\times \left(z - \frac{v_{eff}^2}{v} t\right), \; R = p + r. \quad (75)$$

There are several remaining issues associated with nonparaxial finite-energy localized wave solutions in arbitrary lossless dispersive media. For example, the case where ω_{eff} is purely imaginary in Eq. (59). This case will not be explored in this exposition.

6. ILLUSTRATIVE NUMERICAL EXAMPLES

6.1. Finite-Energy Localized Waves in a Lorentz Plasma under Anomalous Dispersion Conditions

In this subsection, numerical examples will be presented of three types of finite-energy localized waves propagating in a lossless Lorentz medium characterized by the index of refraction

$$n(\omega) = \sqrt{1 + \frac{\omega_p^2}{\omega_r^2 - \omega^2}}, \quad (76)$$

with specific values $\omega_p = 4.36 \times 10^{10}$ rad/s and $\omega_r = 2 \times 10^{10}$ rad/s for the plasma and resonant frequencies, respectively. For these parameters, the index of refraction is purely imaginary in the frequency range $\omega_r < \omega < \omega_c$, where $\omega_c = 4.79683 \times 10^{16}$ rad/s. In the frequency regime where $n(\omega)$ is real, one has the properties $n(\omega) > 1$ for $\omega < \omega_r$ and $n(\omega) < 1$ for $\omega > \omega_c$. Correspondingly, the phase and group speeds are respectively superluminal and subluminal for $\omega < \omega_r$, and both subluminal for $\omega > \omega_c$.

The central frequency is chosen as $\omega_0 = 5 \times 10^{10}$ rad/s. At this frequency, the relevant parameters are given as follows:

$$v_{ph} = 9.74453 \times 10^8 \text{ m/s}, \; v_{gr} = 7.87767 \times 10^7 \text{ m/s};$$
$$\beta_0 = 51.3109 \text{ rad/m}, \; \beta_1 = 1.26941 \times 10^{-8} \text{ s/m},$$
$$\beta_2 = -1.5322 \times 10^{-24} \text{ s}^2\text{/m};$$
$$v_{eff} = 1.10082 \times 10^8 \text{ m/s}, \; \omega_{eff} = 5.5135 \times 10^9 \text{ rad/s}.$$

Since $\beta_2 < 0$, the medium is characterized by anomalous dispersion for frequencies near the chosen central frequency ω_0.

6.1.1. Splash Mode

The finite-energy, FWM-type localized wave $u_-(\rho, z, t)$, shown analytically in Eq. (62), will be used for the first numerical illustration. The envelope of this wavepacket propagates bidirectionally along the z-direction with the effective speeds $\pm v_{eff}$, defined in Eq. (56), and it is modulated by the product of two plane waves traveling in the positive z-direction with speeds v_{ph} and $v_m \equiv v_{eff}^2/v_{gr}$, respectively. In addition to the central frequency ω_0, two other frequencies appear in the expression for the finite energy FWM localized wave. The frequency $\omega_m \equiv \beta_0 v_{gr}$, associated with the second modulating plane wave, and the effective frequency ω_{eff}, defined in Eq. (57), associated with the envelope function. The expression for $u_-(\rho, z, t)$ in Eq. (62) has been derived under the condition that ω_{eff} is a real quantity. For Lorentz plasma, this condition is met if the central frequency is larger than the critical frequency defined at the beginning of this section. But this is precisely the anomalous dispersion regime. [It should be noted that for a central frequency smaller than the resonant frequency (normal dispersion region), the effective frequency is purely imaginary, and the Klein-Gordon equation in Eq. (55) changes into a Proca (or De Broglie) equation. Finite-energy localized waves can be determined under these conditions; however, they will not be pursued in this exposition].

Surface and temporal plots for the finite-energy splash mode solution given in Eq. (62) are shown in Figure 1.

6.1.2. Modified Focus X Wave (Superluminal)

The finite-energy modified focus X wave solution is given in Eq. (70). Surface and temporal plots are shown in Figure 2 for a speed v slightly larger than v_{eff}.

6.1.3. MacKinnon Wavepacket (Subluminal)

The finite-energy subluminal MacKinnon solution is given in Eq. (75). Surface and temporal plots are shown in Figure 3 for a speed v slightly smaller than v_{eff}.

6.2. Finite-Energy Localized Waves in Fused Silica under Normal Dispersion Conditions

In this subsection, numerical examples will be presented of two types of finite-energy localized waves propagating in lossless fused silica characterized by an index of refraction defined by the Sellmeier formula [40]

$$n(\omega) = \sqrt{1 + \sum_{m=1}^{3} \frac{(2\pi c)^2 B_m}{1 - \lambda_m^2/\lambda^2}};$$

$$\lambda_m = 2\pi c/\omega_m, \; \lambda = 2\pi c/\omega, \quad (77)$$

with specific values

$$B_1 = 0.6961663, \; B_2 = 0.4079426, \; B_3 = 0.8974794$$
$$\lambda_1 = 0.0684043, \; \lambda_2 = 0.1162414, \; \lambda_3 = 9.896161.$$

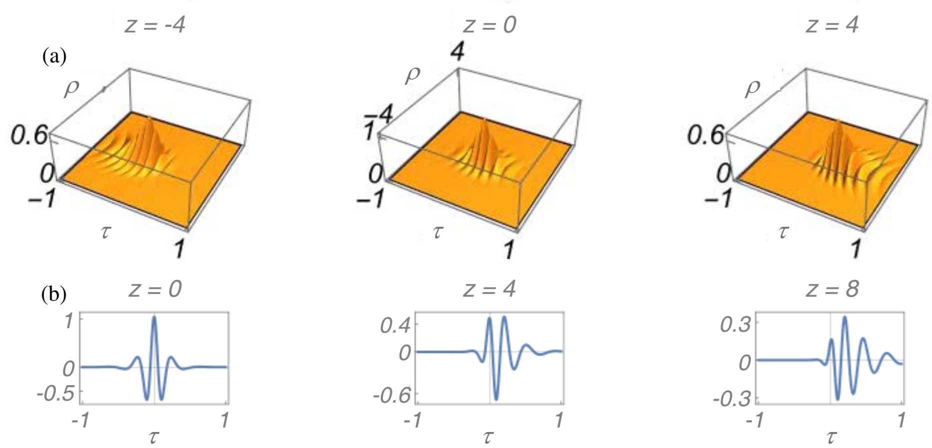

FIGURE 1. (a) Surface plots of $|\mathrm{Re}\{u_{+}(\rho, z, t)\}|$ vs. $\tau = t - z/v_{eff} \in (-1, 1)10^{-10}$ s and $\rho \in (-4, 4)10^{-1}$ m for three values of the range $z = -4$, 0 and 4 m; (b) Temporal plots of $\mathrm{Re}\{u_{+}(0, z, t)\}$ vs. $\tau = t - z/v_{eff} \in (-1, 1)10^{-10}$ s for $z = 0$, 4 and 8 m. Parameter values: $a_1 = 2 \times 10^{-3}\,\mathrm{m}^{-1}$ and $a_2 = 10\,\mathrm{m}^{-1}$.

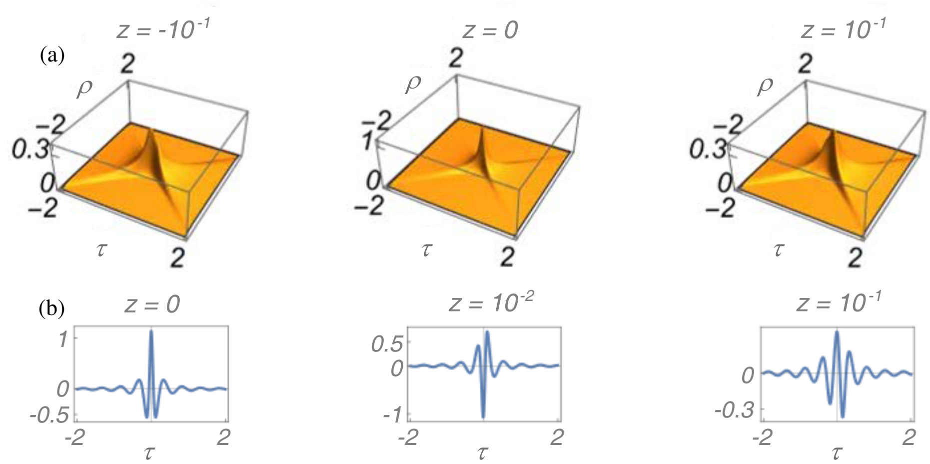

FIGURE 2. (a) Surface plots of $|\mathrm{Re}\{u_{-}(\rho, z, t)\}|$ vs. $\tau = t - z/v \in (-2, 2)10^{-11}$ s and $\rho \in (-2, 2)10^{-2}$ m for three values of the range $z = -10^{-1}, 0$ and 10^{-1} m. (b) Temporal plots of $\mathrm{Re}\{u_{+}(0, z, t)\}$ vs. $\tau = t - z/v \in (-2, 2)10^{-11}$ s for $z = 0, 10^{-2}$ and 10^{-1} m. Parameter values: $a_1 = 10^{-3}\,\mathrm{m}^{-1}$ and $v = 1.10815 \times 10^{8} > v_{eff}$.

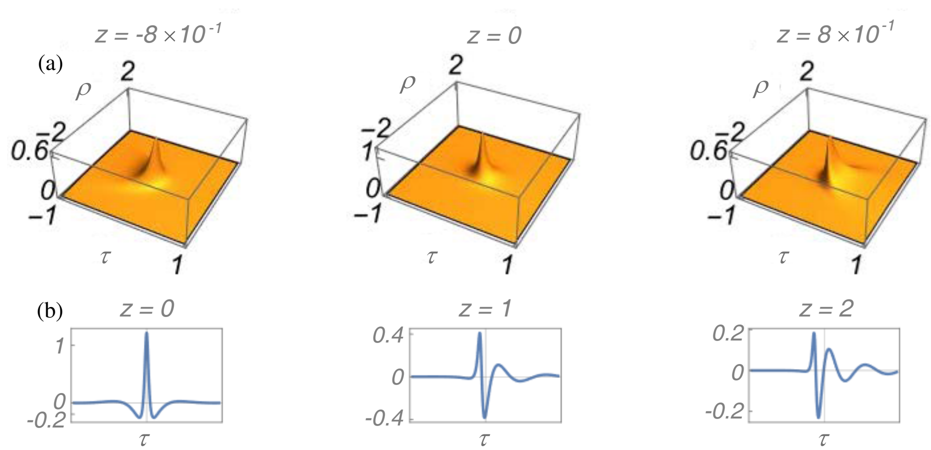

FIGURE 3. (a) Surface plots of $|\mathrm{Re}\{u_{+}(\rho, z, t)\}|$ vs. $\tau = t - z/v \in (-1, 1)10^{-10}$ s and $\rho \in (-2, 2)10^{-1}$ m for three values of the range $z = -8 \times 10^{-1}, 0$ and 8×10^{-1} m. (b) Temporal plots of $\mathrm{Re}\{u_{+}(0, z, t)\}$ vs. $\tau = t - z/v \in (-1, 1)10^{-10}$ s for $z = 0, 1$ and 2 m. Parameter values: $a_1 = 1\,\mathrm{m}^{-1}$ and $v = 1.1 \times 10^{8} < v_{eff}$.

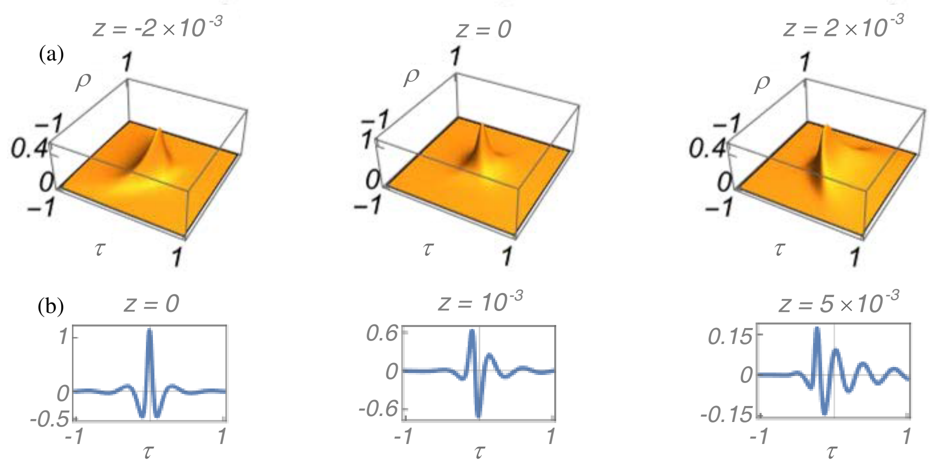

FIGURE 4. (a) Surface plots of $|\psi_-(\rho,z,\tau)|$ vs. $\tau = t - z/v_{gr} \in (-1,1)5\times10^{-13}$ s and $\rho \in (-1,1)2\times10^{-3}$ m for three values of the range $z = -2\times10^{-3}, 0$ and 2×10^{-3} m. (b) Temporal plots of Re$\{\psi_-(0,z,\tau)\}$ vs. $\tau = t - z/v_{gr} \in (-1,1)5\times10^{-13}$ s for $z = 0, 10^{-3}$ and 5×10^{-3} m. Parameter value: $a_1 = 1\,\text{m}^{-1}$.

The central frequency is chosen as $\omega_0 = 2\pi c\,10^4$ rad/s. At this frequency, the relevant parameters are given as follows:

$$\begin{aligned} v_{ph} &= 1.72904\times10^8\,\text{m/s},\ v_{gr} = 1.7239\times10^8\,\text{m/s};\\ \beta_0 &= 109{,}018\,\text{rad/m},\ \beta_1 = 5.8079\times10^{-9}\,\text{s/m},\\ \beta_2 &= 2.7756\times10^{-24}\,\text{s}^2/\text{m}. \end{aligned}$$

Since $\beta_2 > 0$, the medium is characterized by normal dispersion for frequencies near the chosen central frequency ω_0.

6.2.1. Finite-Energy X-Shaped Localized Wave

The nonparaxial Eq. (5) with dispersive effects retained to second order is simplified to

$$\left(-2i\beta_0\frac{\partial}{\partial z}+\frac{\partial^2}{\partial z^2}-\frac{1}{v_{gr}}\frac{\partial^2}{\partial z\partial\tau}-\beta_0\beta_2\frac{\partial^2}{\partial\tau^2}+\nabla_t^2\right)$$

$$\psi(\rho,z,\tau)=0;\ \tau=t-z/v_{gr}. \tag{78}$$

It has already been mentioned that in the *slowly varying envelope approximation* (SVEA) one neglects the second derivative with respect to z (paraxial approximation), as well as the mixed derivative term involving z and τ, and retains dispersive effects to second order. Recent improvements, such as the *slowly evolving wave approximation* (SEWA) and the slightly altered *slowly evolving envelope approximation* (SEEA) can accommodate the propagation of ultra-short (few-cycle) pulses by retaining the mixed derivative term. In the following, both second derivative with respect to z and second derivative with respect to z and τ will be retained.

A finite-energy X-shaped solution to Eq. (78) is given as follows:

$$\psi(\rho,z,\tau)=\frac{1}{\sqrt{M}\sqrt{b^2M+\tau^2}}\sqrt{b\sqrt{M}+\sqrt{b^2M+\tau^2}}$$

$$e^{-iZ}e^{-\frac{1}{b}\sqrt{b^2M+\tau^2}};$$

$$\begin{aligned} A&=1-v_{gr}^2\beta_0\beta_2,\ b=\frac{\sqrt{A}}{v\beta_{0gr}};\\ Z&=z\left(v_{gr}\beta_0\right)^2/A+\tau v_{gr}\beta_0/A,\\ M&=\rho^2v_{gr}^2\beta_0^3\beta_2+(a_1+iZ). \end{aligned} \tag{79}$$

Surface and temporal plots are shown in Figure 4.

6.2.2. Finite-Energy MacKinnon-Type Localized Wave

A finite-energy MacKinnon-type solution to Eq. (78) is given as follows:

$$\psi(\rho,z,\tau)=\frac{1}{R}e^{iZ}\left(\frac{1}{r_1}\sqrt{R_1}\,e^{-r_1}-\frac{1}{r_2}\sqrt{R_2}\,e^{-r_2}\right);$$

$$\begin{aligned} A&=1+v_{gr}^2\beta_0\beta_2,\\ Z&=z\left(v_{gr}\beta_0\right)^2\beta_2/A-\tau\,v_{gr}\beta_0/A,\\ R&=\sqrt{\rho^2\beta_0^3v_{gr}^2\beta_2/A+Z^2}.\\ r_1&=\sqrt{(a_1-iR)^2+(\tau\beta_0v_{gr})^2/A},\\ r_2&=\sqrt{(a_1+iR)^2+(\tau\beta_0v_{gr})^2/A};\\ R_1&=a_1-iR+r_1,\ R_2=a_1+iR+r_2. \end{aligned} \tag{80}$$

Surface and temporal plots are shown in Figure 5.

7. DISCUSSION

Instead of the expansion approach used in Section 2, the following alternative method has been used in the literature. Based on the dispersion relationship [see Eq. (2)]

$$-\kappa^2-k_z^2+\frac{\omega^2}{c^2}n^2(\omega)=-\kappa^2-k_z^2+\beta^2(\omega)=0, \tag{81}$$

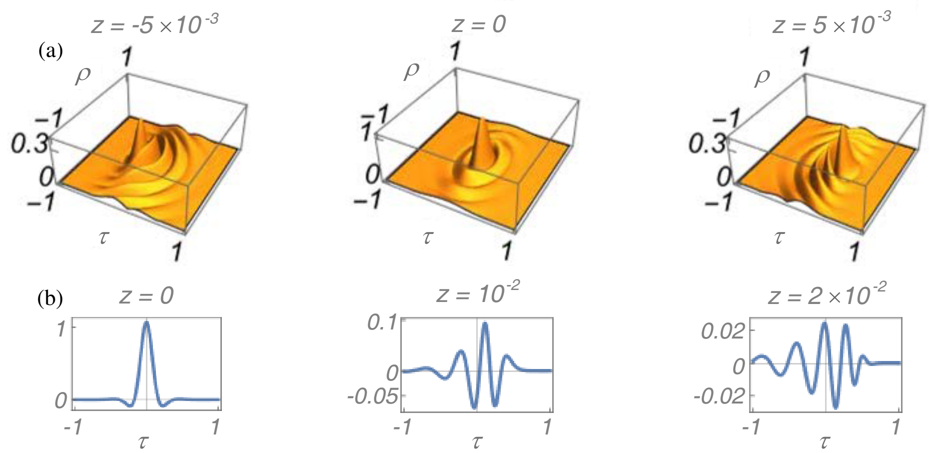

FIGURE 5. (a) Surface plots of $|\psi_{-}(\rho, z, \tau)|$ vs. $\tau = t - z/v_{gr} \in (-1, 1)10^{-12}$ s and $\rho \in (-1, 1)10^{-3}$ m for three values of the range $z = -5 \times 10^{-3}, 0$ and 5×10^{-3} m. (b) Temporal plots of $\mathrm{Im}\{\psi_{-}(0, z, \tau)\}$ vs. $\tau = t - z/v_{gr} \in (-1, 1)10^{-12}$ s for $z = 0, 10^{-2}$ and 2×10^{-2} m. Parameter value: $a_1 = 10\,\mathrm{m}^{-1}$.

where $n(\omega)$ denotes the index of refraction, and the Fourier synthesis below is used to obtain an azimuthally symmetric real wavefunction

$$u(r, z, t) = \int_{-\infty}^{\infty} dk_z e^{-ik_z z} \int_{-\infty}^{\infty} d\omega e^{i\omega t} \int_{0}^{\infty} d\kappa \kappa J_0(\kappa\rho)$$
$$\delta\left[-\kappa^2 - k_z^2 + \beta^2(\omega)\right] \times \tilde{u}(\kappa, k_z, \omega - \omega_0), \quad (82)$$

with ω_0 being a fixed carrier frequency. Next, a new variable $\Omega = \omega - \omega_0$ is introduced, together with the constraint $k_z = (\Omega/v) - \beta_0$. Expanding $\beta^2(\Omega + \omega_0)$ in a Taylor series and retaining up to second-order dispersion terms we obtain

$$\psi_{-}(r, z, t) = \int_{-\infty}^{\infty} d\Omega e^{-i(\Omega/v)(z - vt)} \int_{0}^{\infty} d\kappa\kappa J_0(\kappa\rho)$$
$$\times\delta\left[-\kappa^2 \frac{\Omega^2}{v^2}\left(\frac{v^2}{v_{gr}^2}\left(1 + \beta_0\beta_2 v_{gr}^2\right) - 1\right) + 2\frac{\Omega}{v}\beta_0\left(1 - \frac{v}{v_{gr}}\right)\right]. (83)$$

The argument of the Dirac delta function is identical to the expression in Eq. (38) with the definition $\Omega = \alpha v$.

This approach has been used both in free space (e.g., [17, 22, 23]) and in dispersive media (e.g., [46, 49, 52, 53, 55, 57]). However, with a few exceptions (e.g., [46, 49, 52]), all the work is limited to deriving invariant wavepackets in dispersive media based on the paraxial (narrow angular spectrum) approximation. In contrast, nonparaxial infinite-energy and finite-energy spatiotemporally localized pulses have been derived in this article.

The scope in this article has been limited to azimuthally symmetric solutions. This restriction can be lifted easily by replacing $J_0(\kappa\rho)$ with $\exp(im\phi)J_m(\kappa\rho)$ in the Fourier synthesis and choosing appropriate spectra. Another method allowing azimuthal dependence is based on the following specific *ansatz*. Given the following azimuthally symmetric solution to the Klein-Gordon equation for a cold plasma

$$w(\rho, z, t) = e^{ik_z z} \frac{\exp\left[\sqrt{k_z^2 + \omega_p^2/c^2}\sqrt{\rho^2 + (a - ict)^2}\right]}{\sqrt{\rho^2 + (a - ict)^2}},$$

$$a > 0, \quad (84)$$

the expression

$$W(\rho, \phi, z, t) = w(\rho, z, t)\left(\frac{\rho e^{i\phi}}{(a - ict) + \sqrt{\rho^2 + (a - ict)^2}}\right)^m \quad (85)$$

is an azimuthally asymmetric solution to the Klein-Gordon equation. Of course, this also applies to the "effective" Klein-Gordon equation derived in Section 5, specifically Eq. (55), where v_{eff} replaces c, and ω_{eff} replaces the plasma frequency ω_p. Two extensions to the new solution above are possible: (a) Integration over the wavenumber k_z with appropriate spectra will yield new solutions; (b) the Klein-Gordon equation is Lorentz invariant. Therefore, Lorentz transformations, say involving the variables z and t, will result in new solutions.

In this article, the solutions are limited to scalar-valued ones. One method of lifting this restriction is as follows. Let the scalar potential $\Phi(\vec{r}, t)$ and the vector potential $\vec{A}(\vec{r}, t)$ obey the Klein-Gordon equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\omega_p^2}{c^2}\right)\left\{\begin{array}{c}\Phi(\vec{r}, t) \\ \vec{A}(\vec{r}, t)\end{array}\right\} = 0, \quad (86)$$

as well as the Lorentz gauge

$$\nabla \cdot \vec{A}(\vec{r}, t) + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi(\vec{r}, t) = 0. \quad (87)$$

If the electric and magnetic fields are defined as $\vec{E} = -\nabla\Phi - \partial\vec{A}/\partial t$ and $\vec{B} = \nabla \times \vec{A}$, respectively, they obey the Maxwell-

Proca equations

$$\nabla \times \vec{E} = -\frac{\partial}{\partial t}\vec{B},$$
$$\nabla \times \vec{B} = \frac{1}{c^2}\frac{\partial}{\partial t}\vec{E} - \frac{\omega_p^2}{c^2}\vec{A}, \tag{88}$$
$$\nabla \cdot \vec{E} = -\frac{\omega_p^2}{c^2}\Phi,$$

In this case, the fields $\vec{E}$ and $\vec{B}$ individually obey the Klein-Gordon equation.

An alternative procedure for deriving vector-valued solutions has been provided by Hillion [65]. Let the scalar function $\Phi(\vec{r},t)$ be a solution of the Klein-Godon equation (86). Then, a magnetic vector Hertz potential is defined as $\vec{\Pi}_m(\vec{r},t) = \Phi(\vec{r},t)\vec{a}_z$. The corresponding transverse electric (TE) fields are defined as

$$\begin{aligned}\vec{E}_m(\vec{r},t) &= -\frac{\partial}{\partial t}\mu_0\nabla\times\vec{\Pi}_m(\vec{r},t),\\ \vec{H}_m(\vec{r},t) &= \nabla\times\nabla\times\vec{\Pi}_m(\vec{r},t).\end{aligned} \tag{89}$$

Similarly, an electric vector Hertz potential is defined as $\vec{\Pi}_e(\vec{r},t) = \partial\Phi(\vec{r},t)/\partial t\vec{a}_z$. Then, transverse magnetic (TM) fields are given as follows:

$$\begin{aligned}\vec{E}_m(\vec{r},t) &= \nabla\times\nabla\times\partial\vec{\Pi}_m(\vec{r},t)/t,\\ \vec{H}_m(\vec{r},t) &= \varepsilon_0\left(\frac{\partial^2}{\partial t^2}+\omega_p^2\right)\vec{\Pi}_m(\vec{r},t).\end{aligned} \tag{90}$$

ε_0 and μ_0 denote the electric permittivity and magnetic permeability of vacuum. Both the TE and TM fields obey the modified Maxwell equations

$$\begin{aligned}\nabla\times\vec{E}_{e,m} &= -\frac{\partial}{\partial t}\mu_0\vec{H}_{e,m},\\ \nabla\times\vec{H}_{e,m} &= \varepsilon_0\frac{\partial}{\partial t}\vec{E}_{e,m} - \varepsilon_0\omega_p^2\frac{\partial}{\partial t}\vec{E}_{e,m},\\ \nabla\cdot\vec{E}_{e,m} &= 0,\\ \nabla\cdot\vec{E}_{e,m} &= 0.\end{aligned} \tag{91}$$

The fields $\vec{E}_{e,m}$ and $\vec{H}_{e,m}$ individually obey the Klein-Gordon equation.

Finally, the restriction of the discussion in this article to transparent (lossless) dispersive media must be mentioned. Lifting this limitation for nonparaxial localized solutions requires more thorough treatment. However, lossy dispersive media have been discussed in the literature for pulsed beam propagation within the framework of the paraxial approximation (see, e.g., [50, 63, 64]).

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