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Some Selected Unsolved Problems in Classical and Quantum Electromagnetics

Weng Cho Chew[1, 2, *], Boyuan Zhang[1], and Jie Zhu[1]

[1] *Elmore Family School of Electrical and Computer Engineering, Purdue University, West Lafayette, Indiana 47906, USA*
[2] *Previously: Department of Electrical Engineering, University of Illinois Urbana-Champaign, Urbana, Illinois 61801, USA*

ABSTRACT: In this paper, we propose some suggestions for unsolved problems in classical and quantum electromagnetics. We aim to explain these problems in the simplest way possible. Some issues like the quantum computer may need a lot more work. The subject matter is interdisciplinary needing international collaboration in many different areas such as physics, math, engineering, and material science.

1. INTRODUCTION

We approach the writing of this paper with some trepidation as what is unsolved is very much in the eyes of the beholder. There must be umpteen unsolved problems in the history of science and technology. What we are covering is a tiny subset that is confined to the field of electromagnetics. Electromagnetics is important as it has an enduring legacy that has impacted our lives for close to 200 years. Modern electromagnetics began with the works of Faraday, Ampere, Gauss, Coulomb, plus the works of many unsung heroes [1, 2]. In the early days, it was driven by our communication needs, e.g., in the quest for better telegraphy systems. The simple invention of the voltage cell [3] allowed one to communicate through long distances using two wires. By closing and disrupting the connection of the two pieces of wire to a voltage source, one could send signals along the wires using Morse code [1]. For instance, Queen Victoria of England was able to send a signal from England to the far reaches of the British Empire as far as Hong Kong via submarine cables made of copper wires [4].

However, the propagation of electrical signals on these submarine cables was not well understood. In particular, it was noted that loss on the cables due to the finite conductivity of the copper wire distorted the electrical signals. The need for a quantitative way of describing these signals was obvious: a mere heuristic understanding of the signals was insufficient. In the early days, these signals were described by telegrapher's equations. The electromagnetic field was not well understood, and researchers did a humongous amount of experiments to quantify these fields as laws. In those days, electromagnetics was quantified in terms of Ampere's law, Faraday's law, Gauss' law, and Coulomb's law.[1]

It was James Clerk Maxwell, in 1865, who quantified these laws in terms of mathematics and augmented Ampere's law with the concept of displacement current [6]. Subsequently, the four main equations of electromagnetics have been known as Maxwell's equations.

2. LOW-FREQUENCY BREAKDOWN PROBLEM[2]

The well-known Maxwell's equations, reported in many textbooks, in the time domain are [7, p. 81]

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}, \tag{1}$$

$$\nabla \times \mathbf{H} = \frac{\partial}{\partial t}\mathbf{D} + \mathbf{J}_{im}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = \varrho_{im}, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

where $\mathbf{J}_{im}$ and ϱ_{im} are impressed current and charge sources. The above four equations are not all independent. By taking the divergence of the first two equations, it can be shown that the third and the fourth equations can be arrived at after using the current continuity equation;

$$\nabla \cdot \mathbf{J}_{im} = -\frac{\partial}{\partial t}\varrho_{im}. \tag{5}$$

It has the physical meaning of charge conservation.

By assuming time-harmonic dependence of the form $\exp(-i\omega t)$, Maxwell's equations can be written in the frequency domain as

$$\nabla \times \mathbf{E} = i\omega\mathbf{B}, \tag{6}$$

$$\nabla \times \mathbf{H} = -i\omega\mathbf{D} + \mathbf{J}_{im}, \tag{7}$$

* Corresponding author: Weng Cho Chew (wcchew@purdue.edu).

[1] These laws, though motivated by experimental observations in the early days, can actually be derived from Hamiltonian theory and energy conservation arguments [5].

[2] This problem is largely solved. But we will report this problem for completeness. We shall explain this problem in the simplest way possible using Einstein's dictum.

$$\nabla \cdot \mathbf{D} = \varrho_{im}, \tag{8}$$
$$\nabla \cdot \mathbf{B} = 0, \tag{9}$$

where the charge conservation (the continuity equation) can be derived by taking the divergence of (7) and using (8), viz.,

$$\nabla \cdot \mathbf{J}_{im} = i\omega \varrho_{im}. \tag{10}$$

Again, the third and the fourth equations can be derived from the first two by taking their divergences, and then invoking the current continuity equation or charge conservation.

The above equations are not independent, since the third and the fourth equations are derivable from the first two, oftentimes, only the first two equations are solved. In mathematics, the sources in these equations, $\mathbf{J}_{im}$ and ϱ_{im} need to be stipulated or are known. Together, they form the driving terms for the equations of motion. But there are four unknowns in Maxwell's equations **E**, **B**, **H**, and **D**. Since the above has only two independent equations with four unknowns, two more equations are needed. For linear problems, these additional equations are the constitutive relations, viz.,

$$\mathbf{D} = \varepsilon \mathbf{E} \tag{11}$$
$$\mathbf{B} = \mu \mathbf{H}. \tag{12}$$

Now, the above system, (6) to (9) can be solved. To this end, we can proceed to eliminate **B** and **H** from the first two Maxwell's equations using the constitutive relations to arrive at

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} - \omega^2 \varepsilon \mathbf{E} = i\omega \mathbf{J}_{im}. \tag{13}$$

In the low-frequency case, the second term on the left-hand side of (13) is less important than the rest of the equation because when $\omega \to 0$, $O(\omega^2) \ll O(\omega)$. In the spirit perturbation analysis [8], the leading order approximation to the above equation is

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{E} \approx i\omega \mathbf{J}_{im}. \tag{14}$$

The curl operator does not have a unique inverse because it has a null space. In other words,

$$\nabla \times \nabla \Psi \equiv 0. \tag{15}$$

The above equation, (14), cannot be solved uniquely for this reason. If the above is converted to a matrix equation by the subspace projection method [9, Chap. 36, Sect. 36.1.1], or by finite-difference [9, Chap. 37], the corresponding matrix equation is also singular and not invertible.

3. REMEDY TO LOW-FREQUENCY BREAKDOWN

The remedy to low-frequency breakdown is to seek a solution that satisfies all four Maxwell's equations. This can be achieved by using the vector-scalar potential formulation or the **A**-Φ formulation [10]. To this end, we have

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{16}$$
$$\mathbf{E} = -\partial_t \mathbf{A} - \nabla \Phi. \tag{17}$$

Consequently, it can be shown that, for the homogeneous medium case, after using the Lorenz gauge that $\nabla \cdot \mathbf{A} = -\mu\varepsilon \frac{\partial \Phi}{\partial t}$, Φ and **A** satisfy the following equations [7, p. 66]

$$\nabla^2 \Phi - \mu\varepsilon \partial_t^2 \Phi = -\varrho/\varepsilon \tag{18}$$
$$\nabla^2 \mathbf{A} - \mu\varepsilon \partial_t^2 \mathbf{A} = -\mu \mathbf{J}. \tag{19}$$

When the frequency $\omega \to 0$, $\partial_t = 0$, and the above becomes

$$\nabla^2 \Phi = -\varrho/\varepsilon \tag{20}$$
$$\nabla^2 \mathbf{A} = -\mu \mathbf{J}. \tag{21}$$

The above is the Poisson's equation, and the vector Poisson's equation, respectively [11]. The important point is that the Laplacian operator ∇^2 is a negative definite operator. In a word, if the Laplacian operator is converted to become a matrix operator, it has eigenvalues that are always less than zero, but never identically equal to zero. Or the Laplacian operator does not have a null space or the **A**-Φ formulation removes the null space.

Even though the Laplace problem is solvable, it is still ill-conditioned as the discretization density of the mesh increases [12, 13]. Iterative solvers are in vogue these days because they can be made matrix-free greatly reducing memory requirements [14, p. 59]. The challenge here is to develop an iterative solver that has a uniform convergence rate from statics to optics.

4. ELECTROMAGNETIC THEORY AND GEOMETRY

Even though electromagnetics fields live in the 3D space, it is not a simple extension of the 1D space. In some communities of scholars, the electromagnetics fields that live in this space can be divided into curl conforming type as well as divergence conforming type [15, 16]. In another community, electromagnetic theory can be intimately related to geometry. This has been noted by a number of researchers. Rumor has it that Maxwell himself was aware of it [17]. For instance, Yang-Mills theory is known also as a generalized electromagnetic theory [18]. Electromagnetic theory can also be expressed using differential geometry language as in differential forms [19]. De Rham complex and Whitney forms are concepts that grow out of this community of scholars [20].

At this juncture, it may be of interest to introduce differential forms, concepts derived from differential geometry to describe electromagnetic theory [21]. According to this theory, space has texture (or structure) that are often ignored in old calculus that assumes space is infinitely divisible. But in differential geometry, space has texture: In this space, different kinds of vector fields live in it. Some fields live on a line or a contour, and they form mutually exclusive field lines, called one-forms, that are intertwined.

The fluxes B are two-forms that are associated with an incremental surface. In a textured space, B and H are two different forms, and they cannot be connected point-wise as in the old continuum calculus. They live in different parts of the textured space. Some B field lives as fluxes which are measured by the number of field lines that passes through an incremental area.

There have been lots written about this subject, and we will not delve into the details here [19, 21–24]. Consequently, Maxwell's equations can be represented in differential forms

as [21, 25, 26]:

$$dE = i\omega B, \tag{22}$$
$$dH = -i\omega D + J, \tag{23}$$
$$dD = \varrho, \tag{24}$$
$$dB = 0, \tag{25}$$

where E and H are 1-forms representing the electric field and magnetic field, respectively; D and B are 2-forms representing the generalized electric flux density and magnetic flux density, respectively; the impressed current density J is a 2-form and the impressed charge density ϱ is a 3-form. Note that ϱ_{im} in (8) is the impressed charge density in classical vector calculus, while ϱ in (24) is the impressed charge density in differential geometry. They are related quantities.

We can imagine 1-forms to be field lines that live on line contours in a 3D space; 2-forms to be fluxes associated with incremental areas in a 3D space where flux field are linked. The scalar quantity like charge density is associated with incremental volumes in a 3D space. (The scalar potential is not used in this section, and it is associated with a point in space, and is considered to be a 0-form.) Hence, we can visualize a 3D space as more richly textured with different kinds of fields living in it: scalar fields, vector fields, and flux fields, and charge density. Associated with these fields, there are sources that are like line currents J which are the 2-forms and sources that are like charges ρ which are the 3-forms. More discussion can be found in [27, 28].

With the above picture in mind, we see that field lines live in a different "space" compared to fluxes. Hence, fluxes cannot be directly related to field lines locally: they have to be related by Hodge-star operators:

$$D = \star_{\epsilon}^{(1)} E, \tag{26}$$
$$H = \star_{\mu^{-1}}^{(2)} B, \tag{27}$$

where $\star_{\epsilon}^{(1)}$ and $\star_{\mu^{-1}}^{(2)}$ are called the Hodge star operators. They can be thought of as an operator that maps fields that live on curved lines to fields that link incremental surfaces.

5. DISCRETE EXTERIOR CALCULUS

In the discrete, computational world, the field lines are approximated by their average values on incremental line segments, and the fluxes are replaced by their average values at cross-sectional incremental surface segments. Since they live in different spaces, a primal mesh and a dual mesh have to support them in the discrete world (see Figure 1). More discussions can be found in [25, 26, 29–31].

It has been shown that the **A**-Φ formulation does not have low-frequency breakdown [32–36], and hence, we will use this formulation here. For inhomogeneous media, the **A**-Φ formulation gives the following partial differential equation to be solved [10]:

$$\nabla \times \frac{1}{\mu} \nabla \times \mathbf{A} - \omega^2 \tilde{\epsilon} \mathbf{A} - \tilde{\epsilon} \nabla [\chi^{-1} \nabla \cdot (\tilde{\epsilon} \mathbf{A})] = \mathbf{J}_{im} \tag{28}$$

$$\nabla \cdot (\tilde{\epsilon} \nabla \Phi) + \omega^2 \chi \Phi = -\varrho_{im}, \tag{29}$$

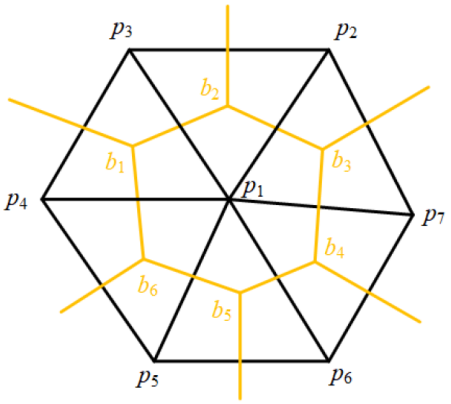

FIGURE 1. The primal and dual mesh based on the centroids of the primal mesh in the 2D case. More details are discussed in [28].

where $\chi = \alpha\mu\tilde{\epsilon}^2$. Here, α is arbitrary and nonzero.

Upon discretization using discrete exterior calculus [26, 29], we arrive at

$$\left(\overline{\mathbf{d}}^{(1)}\right)^T \star_{\mu^{-1}}^{(2)} \overline{\mathbf{d}}^{(1)} \boldsymbol{A} - \omega^2 \star_{\epsilon}^{(1)} \boldsymbol{A}$$
$$+ \star_{\epsilon}^{(1)} \overline{\mathbf{d}}^{(0)} \star_{\chi^{-1}}^{(3)} \left(\overline{\mathbf{d}}^{(0)}\right)^T \star_{\epsilon}^{(1)} \boldsymbol{A} = \boldsymbol{J}, \tag{30}$$

$$- \left(\overline{\mathbf{d}}^{(0)}\right)^T \star_{\epsilon}^{(1)} \overline{\mathbf{d}}^{(0)} \boldsymbol{\Phi} + \omega^2 \star_{\chi}^{(0)} \boldsymbol{\Phi} = -\boldsymbol{\varrho}. \tag{31}$$

In the above, $\boldsymbol{\Phi}$ and $\boldsymbol{A}$ are the cochain vectors defined in [25, 28], respectively; $\boldsymbol{\varrho}$ is the cochain vector in [25, 28]; $\boldsymbol{J}$ is dual 2-cochain vector for the impressed current density.

The subspace projection method [9, Chap. 36, Sect. 36.1.1] can be used to find the matrix representation of the Hodge-star operators. In other words,

$$\left[\star_{\epsilon}^{(1)}\right]_{i,j} = \left\langle \mathbf{W}_i^1, \tilde{\epsilon} \cdot \mathbf{W}_j^1 \right\rangle, \tag{32}$$

$$\left[\star_{\mu^{-1}}^{(2)}\right]_{i,j} = \left\langle \mathbf{W}_i^2, \mu^{-1} \cdot \mathbf{W}_j^2 \right\rangle, \tag{33}$$

$$\left[\star_{\chi}^{(0)}\right]_{i,j} = \left\langle W_i^0, \chi \cdot W_j^0 \right\rangle, \tag{34}$$

where $\mathbf{W}_i^1$ is the Whitney 1-form associated with primal edge l_i; $\mathbf{W}_i^2$ is the Whitney 2-form associated with primal face S_i. ϵ and μ are piecewise constant parameters within each tetrahedron. Note that the Galerkin Hodge star operators are highly sparse, but not diagonal. Moreover, W_i^0 is the Whitney 0-form associated with primal node p_i defined in χ; it can be considered as a piecewise constant parameter within each tetrahedron.

The holy grail is to construct Hodge-star operators that are diagonal. Given the overlapping nature of the basis functions, the above operators are not diagonal. If we think of FDTD as a special case of DEC, the corresponding Hodge-star operator in FDTD is diagonal. Desbrun's group has a way to make the Hodge-star operators diagonal [37]. The remaining question is if a simpler approach exists.

This problem has plagued the CSE community. Often, these operators are called mass operators, and the lumped-mass approximations have been used to seek their diagonal form [38]. The diagonal forms of these operators will make their inversion easy. It is to be noted that when FDTD [39] is used to find the matrix representation of Maxwell's equations, the matrices are all diagonal, including the analog of the Hodge-star operators.

6. THE ELLIPTIC PDE PROBLEMS

The Laplace equation is important in many technologies. It is encountered in low-frequency electromagnetics as the wavelength becomes very long. Following the classification of PDEs from eons ago [40], these are classified as elliptic PDEs. Very simply put, the Laplace equation is

$$\nabla^2\Phi(\mathbf{r}) = S(\mathbf{r}) \tag{35}$$

where $\Phi(\mathbf{r})$ is the scalar field, while $S(\mathbf{r})$ is a scalar source term. We compare the above with the Helmholtz equation which is

$$\left(\nabla^2 + k^2\right)\Phi(\mathbf{r}) = S(\mathbf{r}) \tag{36}$$

where $k = \omega/c$ where c is the velocity of the wave. The Helmholtz equation is termed the hyperbolic partial differential equation [40] because, in the time domain, it can support the propagation of singular fields [41, p. 528].

One sees that the Helmholtz equation reduces to the Laplace equation when $c \to \infty$. This implies that in the Laplace equation, the information travels at infinite speed across the simulation domain. This fact can be used to speed up the convergence of iterative solvers for elliptic equations. This means that if a coarse grid is used to solve these equations, the approximate solution is aware of the entire simulation domain. To obtain a refined solution, the mesh is refined, but using the coarse grid solution as an initial guess for the fine-mesh solution. If an iterative solver is used to solve the matrix equation, a good initial guess can speed up its convergence.

This is the spirit of the multi-level multi-grid method where coarse grid solutions are used to initial-guess the fine grid solutions [42, 43]. This allows the solution to converge quickly irrespective of the size of the simulation domain. In a word, the iterative solution converges in C steps where C is independent of domain size. Such iterative solvers are known to converge in $O(1)$ steps.

Another characteristic of the Laplace solution is that it is infinitely smooth away from source regions. This can be appreciated by observing the Green's function of the Laplace solution [44]. In three dimensions for Laplace's equation, Green's function (also called the fundamental solution by the math community) is given by

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{1}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{37}$$

Observing this Green's function on the x-axis, or letting $|\boldsymbol{r}-\boldsymbol{r}'| = x$, and taking its derivatives, we have

$$\frac{\partial}{\partial x}\frac{1}{|x|} = -\frac{1}{|x|^2} \quad \to 0,\ x\to\infty \tag{38}$$

$$\frac{\partial^2}{\partial x^2}\frac{1}{|x|} = -\frac{\partial}{\partial x}\frac{1}{|x|^2} = \frac{2}{|x|^3} \quad \to \mathcal{O}\left(\frac{1}{x^3}\right),\ x\to\infty \tag{39}$$

$$\frac{\partial^3}{\partial x^3}\frac{1}{|x|} = \frac{\partial}{\partial x}\frac{2}{|x|^3} = -\frac{6}{|x|^4} \quad \to \mathcal{O}\left(\frac{1}{x^4}\right),\ x\to\infty. \tag{40}$$

From the above, notice that the higher-order derivatives of the kernel[3] actually become smaller the further one is from the source point. This implies that the field is becoming smoother as one gets away from the source point. Because of the smoothness of the field, elliptic equations are noted for their inability to propagate singularity (or information) in the time domain.

In contrast, we look at the wave (Helmholtz) equation next which is a hyperbolic equation [40].

$$\left(\nabla^2 + k^2\right)G(\boldsymbol{r},\boldsymbol{r}') = -\delta(\boldsymbol{r}-\boldsymbol{r}'). \tag{41}$$

For the point source located at $\boldsymbol{r}'$, the Green's function in 3D is

$$G(\boldsymbol{r},\boldsymbol{r}') = \frac{e^{ik_0|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}. \tag{42}$$

Observing the Green's function on the x-axis as before, the higher-order derivatives of an oscillatory kernel behave as follows:

$$\frac{\partial}{\partial x}\frac{e^{ik|x|}}{|x|} = -\frac{e^{ik|x|}}{|x|^2} + ik\frac{e^{ik|x|}}{|x|} \quad \to \mathcal{O}\left(\frac{1}{x}\right),\ x\to\infty \tag{43}$$

$$\frac{\partial^2}{\partial x^2}\frac{e^{ik|x|}}{|x|}:\ -k^2\frac{e^{ik|x|}}{|x|} + H.O.T. \ \to \mathcal{O}\left(\frac{1}{x}\right),\ x\to\infty \tag{44}$$

$$\frac{\partial^3}{\partial x^3}\frac{e^{ik|x|}}{|x|}:\ -ik^3\frac{e^{ik|x|}}{|x|} + H.O.T. \ \to \mathcal{O}\left(\frac{1}{x}\right),\ x\to\infty \tag{45}$$

where $H.O.T.$ stands for 'higher-order terms'. Notice that irrespective of the number of derivatives, there is always a residual $\mathcal{O}(1/x)$ term regardless of how far we are from the source point. Moreover, the higher-order derivatives become larger with increasing frequency. Hyperbolic equations are noted for their ability to propagate information over long distance. Electromagnetic waves, manifesting themselves as photons, can send information across the galaxy.

The unsolved problem in wave physics is that there a computational, error-controllable method to solve these problems in $\mathcal{O}(1)$ iterations. One way is to have a quantum computer where the math operations can be executed in parallel using quantum parallelism. But we have yet to build a quantum computer.

At this point, it is prudent to mention that fast integral equation solvers (IES) have been actively researched. The advantage of IES is that information is send across a computational domain using the Green's function. The information can be sent across the domain per every matrix-vector product. Using stationary-phase method, oscillatory integrals can be evaluated efficiently. In this manner, $\mathcal{O}(1)$ iterations can achieve convergence. This idea has been promulgated by Bruno and his team [45, 46].

[3] A term to mean the Green's function in the math community.

7. LEAST ACTION OR ENERGY CONSERVATION?

7.1. Least Action Principle

The Lagrangian formulation is based on the least action. Taking a simple pendulum as an example, the Lagrangian of the system is [47, p. 8]:

$$L(\dot{q}, q) = T - V \tag{46}$$

where $\dot{q}$ is the velocity. For a fixed t, q and $\dot{q}$ are independent variables, since $\dot{q}$ cannot be derived from q if it is only known at one given t. The equations of motion are derived from the principle of least action which says that $q(t)$ that satisfies the equations of motion between two times t_1 and t_2 should minimize the action integral

$$S = \int_{t_1}^{t_2} L(\dot{q}(t), q(t))dt. \tag{47}$$

Assuming that $q(t_1)$ and $q(t_2)$ are fixed, then the function $q(t)$ between t_1 and t_2 should minimize S, the action. In other words, a first-order perturbation in q from the optimal answer that minimizes S should give rise to a second-order error in S. Hence, taking the first variation of (47) should be zero. Thus, we have [47, 48]

$$\begin{aligned}\delta S &= \delta \int_{t_1}^{t_2} L(\dot{q}, q)dt = \int_{t_1}^{t_2} L(\dot{q}+\delta\dot{q}, q+\delta q)dt - \int_{t_1}^{t_2} L(\dot{q}, q)dt \\ &= \int_{t_1}^{t_2} \delta L(\dot{q}, q)dt = \int_{t_1}^{t_2} \left(\delta\dot{q}\frac{\partial L}{\partial \dot{q}} + \delta q \frac{\partial L}{\partial q}\right) dt = 0.\end{aligned} \tag{48}$$

We can show that for the above to vanish, after using integration by parts on the term that contains $\delta\dot{q}$, it is necessary that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0. \tag{49}$$

The above is the Lagrange equation from which the equation of motion of a pendulum can be derived. For this, we let the kinetic energy $T = \frac{1}{2}m\dot{q}^2$ and the potential energy $V = \frac{1}{2}\kappa q^2$ where κ is the spring constant. Using the Lagrange equation (49), we can show that

$$m\ddot{q} = -\kappa q. \tag{50}$$

The above is the equation of motion of a pendulum.

7.2. Hamiltonian Theory Based on Energy Conservation

Hamiltonian theory, developed by William R. Hamilton (1805–1865), is motivated by energy conservation [49]. The Hamiltonian H of a dynamical system is given by its total energy, namely that

$$H = T + V = \frac{p^2}{2m} + \frac{1}{2}\kappa q^2 \tag{51}$$

where T is the kinetic energy, and V is the potential energy of the system. For a simple pendulum, the kinetic energy T and the potential energy V are given by

$$T = \frac{mv^2}{2} = \frac{m^2v^2}{2m} = \frac{p^2}{2m}, \qquad V = \frac{1}{2}\kappa q^2. \tag{52}$$

In the above, $p = mv$ is the momentum of the pendulum weight, and q, as before, is its position. Here, p and q are termed independent conjugate variables.

But $p(t)$ and $q(t)$ time evolve together to conserve energy or to keep H, the total energy, constant or independent of time. In other words,

$$\frac{d}{dt}H\left(p(t), q(t)\right) = 0 = \frac{dp}{dt}\frac{\partial H}{\partial p} + \frac{dq}{dt}\frac{\partial H}{\partial q}. \tag{53}$$

Therefore, the Hamilton equations of motion are derived to be[4]

$$\frac{dp}{dt} = -\frac{\partial H}{\partial q}, \qquad \frac{dq}{dt} = \frac{\partial H}{\partial p}. \tag{54}$$

Combining the two equations in above, we have

$$m\ddot{q} = -\kappa q \tag{55}$$

The above is the same equation from least action and, which is also derivable by Newton's law.

8. THE 3 GHZ BOTTLENECK IN THE COMPUTER CHIP

The 3 GHz cap (see Figure 2) is due to joule heating in the microchip [9, p. 192]. A MOSFET (transistor), driven by a power delivery network via the interconnects can be modeled simply as a gate capacitance (see Figure 3). The metal loss in the interconnects can be modeled as a resistor R in series with a voltage source V. As the frequency increases (due to increased clock rate), more current flows through the capacitance, and hence through the resistor increasing the I^2R loss giving rise to joule heating. This causes the microchip to heat up, forcing the clock rate to saturate at 3 GHz.[5]

One way to overcome joule heating is to use microwave and optical interconnects. They permit the increase in data rate without the corresponding joule heating [50–52]. The deterrent has been the cost of such interconnects.

9. WHY HAVE WE NOT BUILD A WORKING QUANTUM COMPUTER YET?

The promise of quantum computers is tremendous: With quantum parallelism (also called advantage or supremacy), a fast Fourier transform has a computational complexity of $O(n \log n)$ on a classical computer, but with quantum computer, it has a complexity of $O((\log n)^2)$ [53]. Despite this, we have not been able to build a working quantum computer yet. The answer is simple: we do not have yet the knowledge base to build a quantum computer. We are now stagnating in the era of NISQC (noisy intermediate-scale quantum computer) [54, 55].

[4] Note that the Hamilton equations are determined to within a multiplicative constant, because one has not stipulated the connection between space and time, or we have not calibrated our clock [49].

[5] Some chips are operating from 5 to 7 GHz in the booster mode unsustainably.

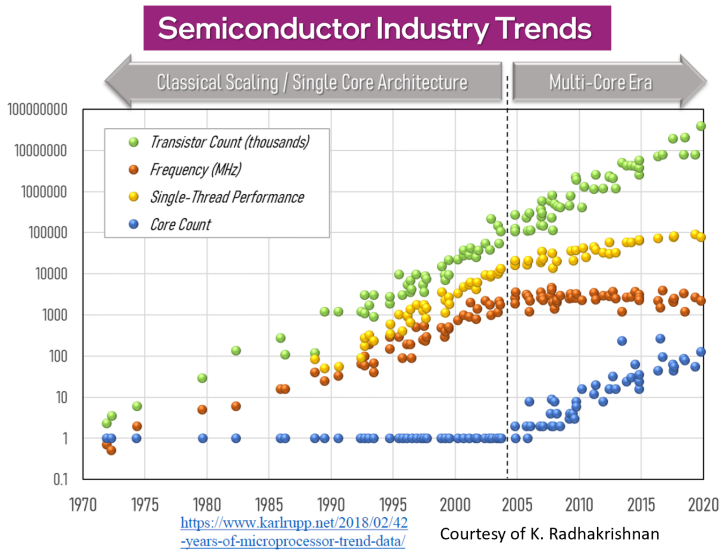

FIGURE 2. The semiconductor trends reveal the self-fulfilling prophesy of Moore's law. Ingenious engineering and 3D architecture of the microchip design allow the transistor and core counts to improve the performance of the microchip. Also, the clock rate (frequency) has not increased in the last two decades due to joule heating.

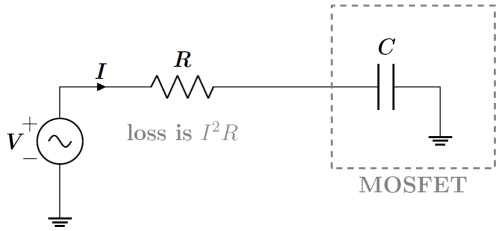

FIGURE 3. MOSFET driven by a power delivery network via the interconnects can be modeled simply as a gate capacitance that models the MOSFET transistor (more in the text). (This physical picture was explained lucidly to the author (WCC) by Paul Y. S. Cheung of HKU.).

9.1. Need to Mitigate the Noise Sources and Improve the S/N Ratio

Due to the maturity of microwave technology, most of the NISQC relies on using microwave photons via superconducting qubits [56–58] (and references therein). Microwave photons are about 1 million times weaker than optical photons. Superconducting qubits work with Cooper pairs and there are three major sources of noise for these qubits: thermal photons [59] from the environment, microwave photons from the phonon baths [60], and breaking of the Cooper pairs [61–64]. Hence, a microwave photon is very susceptible to the aforementioned microwave noise pollution. However, an optical photon will give us a better signal-to-noise ratio. This gives us a strong reason to move away from using microwave photons.

9.2. Use Natural Bosons-Photons

The reason for Cooper pairs used in superconducting qubit and many of the NISQC is that they are quasi bosons exhibiting cooperative behaviours, and hence superconductivity. However, photons are also quantum particles which are natural bosons. For optical photons with high frequencies, they are easily detectable. They also show the cooperative behaviour of bosons as in optical fiber: they can propagate with loss as low as 0.14 dB per kilometer [65].

In a laser cavity, they work cooperatively to produce coherent radiations [66]. Furthermore, in a mode-locked laser, which supports a short pulse which is broadband, different frequency components lock their phases together to form a short pulse [67, 68].

Further, we could use a number of noise-reduction technique such as multi-shot averaging to improve the signal-to-noise of the quantum system [69, 70]. We could create signal phase and amplitude stability with mode-locked signals [67, 68].

9.3. Using Dielectric Resonators as Quantum Qubits

Given the shorter wavelengths of optical frequency, it is possible to use dielectric particles as resonators [71, 72]. They can replace the superconducting transmons where the Josephson junction and a capacitor act as nonlinear LC resonator circuit. Here, nonlinear dielectrics or crystals can be investigated to build resonators resembling the transmons but at a much higher frequency without the need for cryogenics since these photons have higher energy.

With these qubits operating at room temperature, they follow the path of technology development of masers and semiconductor lasers. Masers [73] first operated with microwave photons requiring cryogenics but was subsequently replaced by lasers without needing cryogenics [74, 75] (see Wiki also).

Another technology that follows this pathway is the semiconductor laser [76, 77]. In the beginning, it needed cryogenics for it to lase, but with clever design where both photons and electron-hole pairs are confined together in a tight space, it could operate (or lase) at room temperature. There were many

unsung heroes and Nick Hononyak made the first visible light laser diode (see Wiki also).

9.4. Quantum Measurements

Another challenge problem is on quantum measurements. Quantum observables are random, and many measurements are needed to reach a stable solution. For instance, quantum phases are important in ascertaining the eigenvalues of unitary operators. To be able to measure the phases accurately is important in quantum Fourier transforms, which could be important in electrical engineering [53]. This is an active field of research.

10. CONCLUSIONS

The above are some incomplete suggestions for unsolved problems in classical and quantum electromagnetics. They are meant to motivate and stimulate the thinking of young researchers in this field. Hopefully, these researchers will rise to these challenges. Moreover, some problems in the quantum computer may need a lot more work. The subject matter is interdisciplinary needing international collaboration in many different areas such as physics, math, engineering, and material science.

With the advent of AI, whole sleuth of problems open up for future research in electromagnetics. We can exploit AI in the synthesis and analysis of modern electromagnetics technologies. An excellent book has been written by S. D. Campbell, and D. H. Werner on this topic [78].

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