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Underdetermined Equation Model Combined with Improved Krylov Subspace Basis for Solving Electromagnetic Scattering Problems

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ABSTRACT: To accelerate the solution of electromagnetic scattering problems, compressive sensing (CS) has been introduced into the method of moments (MoM). Consequently, a computational model based on underdetermined equations has been proposed, which effectively reduces the computational complexity compared with the traditional MoM. However, while solving surface-integral formulations for three-dimensional targets by MoM, due to the severe oscillation of current signals, commonly used sparse bases become inapplicable, which renders the application of the underdetermined equation model quite challenging. To address this issue, this paper puts forward a scheme that employs Krylov subspace, which is constructed with low complexity by meticulously designing a group of non-orthogonal basis vectors, to replace the sparse transforms in the algorithmic framework. The principle of the method is elaborated in detail, and its effectiveness is validated through numerical experiments.

1. INTRODUCTION

Method of moments (MoM) [1] is one of the most used numerical methods for solving electromagnetic scattering problems due to its high computational accuracy and the use of Green's functions that automatically satisfy the radiation boundary conditions. In recent years, compressive sensing (CS) theory [2] has been successfully introduced into MoM, and two fast solutions have been proposed.

One is the computational model based on underdetermined equations [3, 4], which extracts partial rows of the impedance matrix as the measurement matrix and constructs a suitable sparse transform to sparsely represent the unknown current coefficients, and finally applies recovery algorithms to obtain an accurate result of unknowns. The other is a rapid multiple right-hand sides (MRHS) solver employing new excitation sources [5, 6], which takes into account the independence of the impedance matrix and incident angle, and designs several new excitations with rich angle information to obtain the measurements of the induced current vectors over multiple incident angles by solving a few times of the matrix equations, and finally reconstructs all the current vectors by recovery algorithms and sparse transforms. Furthermore, a dual CS solution combining the two computational models has also been proposed recently [7].

Moreover, a lot of improvements on these two fast solutions are put forward successively, e.g., fixed step-size extraction from the original matrix equation of MoM is used to eliminate the instability of the underdetermined equation model [8]; discrete wavelet transform is performed to save the computational costs of matrix-vector multiplication [9]; linear basis function is introduced to further improve the calculation efficiency of solving the problems of complex linear structures [10]; various sparse transformation methods are employed to reduce the required number of measurements for the fast MRHS solver [11–13]; the principle of on-surface discretized boundary equation (OS-DBE) is applied as a priori knowledge to predefine the number of measurements for wide-angle electromagnetic scattering problems [14]; NURBS patches are introduced to reduce the number of unknowns [15]; the acceleration algorithms, such as fast multipole method (FMM), multilevel fast multipole method (MLFMM), adaptive integration method (AIM), are utilized to reduce the storage and computational requirements of matrices during the iterative process [16].

However, there remain some challenging issues, one of which is to construct suitable sparse bases while using the underdetermined equation model to solve three-dimensional electromagnetic scattering problems. Since the current coefficient sequence becomes violently oscillated, the common sparse bases, e.g., Fourier basis, discrete cosine transform (DCT), basis, wavelet basis, could be hardly effective. Up to the present, only a few approaches have been reported to deal with this problem, most of which center around the strategy of using characteristic basis functions (CBFs) or characteristic mode basis functions (CMBFs) [17–19]. Besides, we used to put forward another strategy which employs Krylov subspace bases to re-

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place the sparse transform matrix so as to simplify the recovery process to one-time least-square calculation [20]. In order to further reduce the cost of Krylov subspace generation and thus to improve this strategy, this paper constructs a kind of non-orthogonal bases we used to design for GMRES solver [21] to span the Krylov subspace, and uses the improved strategy to deal with 3-D PEC targets.

The improvement proposed in this paper can effectively reduce the computational complexity of the solution based on the underdetermined equations in conjunction with the Krylov subspace while not affecting its accuracy of the computational results. The principle is discussed in detail, and its effectiveness is verified through numerical examples.

2. PRINCIPLE

2.1. Underdetermined Equation Model for Solving MoM

It is generally known that the electromagnetic field integral equations, e.g., electric field integral equation (EFIE), magnetic field integral equation (MFIE), combined field integral equation (CFIE), can be transformed into the matrix equation form by using MoM:

$$\mathbf{ZI} = \mathbf{V} \tag{1}$$

where $\mathbf{Z}$ represents an $N \times N$ impedance matrix; $\mathbf{I}$ is an N dimensional column vector which represents the unknown current coefficients; and $\mathbf{V}$ represents the N-dimensional excitation vector.

Based on (1) the underdetermined equation can be expressed as:

$$\mathbf{Z}_{ud}\mathbf{I} = \mathbf{V}_{ud} \tag{2}$$

where $\mathbf{Z}_{ud}$ represents an $M \times N$ matrix which is constructed by randomly selecting M rows from $\mathbf{Z}$, while $\mathbf{V}_{ud}$ is composed of the corresponding M elements selected from $\mathbf{V}$, and typically, $M \ll N$. Viewed from the theory of CS, $\mathbf{Z}_{ud}$ and $\mathbf{V}_{ud}$ can be regarded as the measurement matrix and measurement result.

Then, we could construct an $N \times N$ sparse transform basis noted by $\mathbf{\Psi}$ to sparsely represent the unknown current coefficients:

$$\mathbf{I} = \mathbf{\Psi}\boldsymbol{\alpha} \tag{3}$$

where $\boldsymbol{\alpha}$ is an N-dimensional column vector which represents the sparse transformation coefficients.

Thus, (2) can be rewritten as:

$$\mathbf{A}\boldsymbol{\alpha} = \mathbf{V}_{ud} \tag{4}$$

where $\mathbf{A} = \mathbf{Z}_{ud}\mathbf{\Psi}$.

The final step involves selecting an appropriate recovery algorithm to solve

$$\hat{\boldsymbol{\alpha}} = \arg\min \|\boldsymbol{\alpha}\|_L \ \text{ s.t. } \mathbf{A}\boldsymbol{\alpha} = \mathbf{V}_{ud} \tag{5}$$

and the unknown current coefficients can be obtained from

$$\hat{\mathbf{I}} = \mathbf{\Psi}\hat{\boldsymbol{\alpha}} \tag{6}$$

Obviously, with the introduction of CS, the computational resources required to solve the matrix equation originated from MoM have been significantly reduced. If the number of basis functions in the calculation domain is N, the computational complexity of filling the impedance matrix is $O(N^2)$. By using iterative methods to solve the matrix equation, the calculation complexity is $O(pN^2)$, where p is the number of iteration steps. While employing the acceleration algorithms, e.g., FMM, adaptive cross approximation (ACA), this complexity can be reduced to $O(pN \log N)$. If M rows are extracted from $\mathbf{Z}$ and $\mathbf{V}$ to establish an underdetermined equation, the complexity of filling the new impedance matrix, i.e., $\mathbf{Z}_{ud}$ (as shown in (2)), is $O(MN)$. If the sparse representation coefficients of the current signal are K-sparse, the computational complexity of solving the underdetermined equations using orthogonal matching pursuit (OMP) [22] algorithm is $O(KMN)$. By exploiting the low-rank characteristic of the far-field terms in the impedance matrix, this complexity can be reduced to $O(KM \log N)$ using ACA. Therefore, the underdetermined equation model can reduce the computational complexity to KM/pN compared with the classical iterative method.

However, while three-dimensional problems are addressed, the current coefficients become violently oscillated, and the conventional sparse bases such as discrete Fourier transform (DFT), DCT, and discrete wavelet transformation (DWT), have become ineffective in sparsely representing the unknowns which makes the algorithm difficult to be applied to the resolution of 3-D objectives.

2.2. Proposed Method

As discussed above, although the computational model based on undetermined equations can greatly reduce the complexity for solving (1) with the help of CS technology, constructing efficient sparse transformation matrices for three-dimensional objects remains a challenge. One of the effective approaches is to replace the sparse transformation basis constituting a complete space with a set of basis vectors that form a Krylov subspace, thus the CS reconstruction process using recovery algorithms could be simplified to a one-time least squares computation. The Krylov subspace generated by $\mathbf{Z}$ and $\mathbf{V}$ can be obtained by

$$K_n(\mathbf{Z}, \mathbf{V}) = \text{span}\left\{\mathbf{V}, \mathbf{ZV}, \mathbf{Z}^2\mathbf{V}, \cdots, \mathbf{Z}^{n-1}\mathbf{V}\right\} \tag{7}$$

with an Arnoldi orthogonalization process, by which a set of orthogonal basis vectors

$$\mathbf{Q}_n = \begin{bmatrix} \mathbf{q}_1 & \mathbf{q}_2 & \cdots & \mathbf{q}_n \end{bmatrix} \tag{8}$$

is constructed.

Substituting (8) into (2), we can obtain

$$\mathbf{Z}_{ud}\mathbf{Q}_n\mathbf{y}_n = \mathbf{V}_{ud} \tag{9}$$

As the least squares solution for $\mathbf{y}_n$ (denoted by $\hat{\mathbf{y}}_n$) is derived, the solution of (1) can be determined by

$$\hat{\mathbf{I}} = \mathbf{Q}_n\hat{\mathbf{y}}_n \tag{10}$$

As analyzed above, with the integration of Krylov subspace, the undetermined equations (shown as (2)) can be solved as a standard least squares problem, rather than a sparse reconstruction problem. However, the construction process of Krylov subspace often needs long recursive relationships which mainly involves two parts: matrix-vector multiplication operations and vector orthogonalization. In order to decrease the computational cost of generating the Krylov basis vectors, the accelerate algorithms, e.g., FMM, multi-level fast multipole method (MLFMA), ACA [23], could be introduced to significantly accelerate the computation of matrix-vector multiplication. At this juncture, the process of vector orthogonalization dominates the computational effort. To further reduce the computational complexity of this part, one solution is to use some fixed short-term recurrence [24, 25] to replace the long recursive process, and another approach is to adopt some strategies to decrease the computational cost in each iterative step. In previous work, we have drawn upon the idea of incomplete orthogonalization from quasi generalized minimum residual (QGMRES) and based on it, proposed a strategy of random extraction for the orthogonalization process [20]. In this paper, another much more efficient strategy is put forward, whereby the Krylov basis vectors constructed will no longer be orthogonal, but they will remain linearly independent. The specific steps are as follows:

First, based on (1) and (2), the 1st basis vector of the non-orthogonal Krylov bases is determined by

$$\mathbf{q}'_1 = \mathbf{V}/\left\|\mathbf{V}_{ud}\right\|_2 \tag{11}$$

Then, like the conventional Arnoldi algorithm, in order to generate the subsequent basis vectors, matrix-vector multiplications also need to be executed. Specifically, to generate the $(j+1)$th basis vector (denoted by $\mathbf{q}'_{j+1}$), $\mathbf{Z}\mathbf{q}'_j$ needs to be calculated at first. Fortunately, by using the accelerate algorithms for MoM (such as FMM and ACA), the amount of computation can be reduced greatly. Subsequently, unlike the traditional Arnoldi process, we do not seek to obtain the orthogonal projection of $\mathbf{Z}\mathbf{q}'_j$ on the group of the basis vectors that have already been generated (i.e., $\mathbf{q}'_1, \mathbf{q}'_2, ..., \mathbf{q}'_j$). Instead, the vector composed of the projection coefficients of $\mathbf{Z}\mathbf{q}'_j$ onto $\mathbf{q}'_1, \mathbf{q}'_2, ..., \mathbf{q}'_j$ is determined by

$$\mathbf{H}_j = \begin{bmatrix} \mathbf{q}'_{1ud} & \cdots & \mathbf{q}'_{jud} \end{bmatrix}^T \cdot \left(\mathbf{Z}_{ud}\mathbf{q}'_j\right) \tag{12}$$

where $\mathbf{q}'_{iud}\,(i = 1, 2, ..., j)$ is generated by extracting the corresponding rows from $\mathbf{q}'_i$. Just like $\mathbf{Z}_{ud}$ and $\mathbf{V}_{ud}$ (as shown in (2)) do. It is easy to see that, compared with calculating the orthogonal projection, which could be represented as $[\mathbf{q}_1, ..., \mathbf{q}_j]^T \cdot (\mathbf{Z}\mathbf{q}_j)$, the computation amount of (12) significantly decreases, especially in the situation when j continues to increase.

Based on (12), the residual vector between $\mathbf{Z}\mathbf{q}'_j$ and its corresponding oblique projection on span $\left\{\mathbf{q}'_1, \mathbf{q}'_2, ..., \mathbf{q}'_j\right\}$ can be obtained by

$$\mathbf{w}'_{j+1} = \mathbf{Z}\mathbf{q}'_j - \left[\mathbf{q}'_1, ..., \mathbf{q}'_j\right] \cdot \mathbf{H}_j \tag{13}$$

Thus, the $(j+1)$th basis vector is acquired by

$$\mathbf{q}'_{j+1} = \mathbf{w}'_{j+1}/\left\|\mathbf{w}'_{(j+1)ud}\right\|_2 \tag{14}$$

where $\mathbf{w}'_{(j+1)ud}$ indicates that corresponding rows in $\mathbf{w}'_{j+1}$ are extracted.

As the Krylov subspace gradually expands, it can be observed that, by repeating the iterations defined by (12)–(14), the generated basis vectors, i.e., $\mathbf{q}'_1, \mathbf{q}'_2, ..., \mathbf{q}'_{j+1}$, are not orthogonal. However, we can derive that $\mathbf{q}'_{1ud} \perp \mathbf{q}'_{2ud} \perp ... \mathbf{q}'_{(j+1)ud}$, and $\mathbf{q}'_1, \mathbf{q}'_2, ..., \mathbf{q}'_{j+1}$ are linearly independent.

Finally, by denoting $\mathbf{Q}'_n = [\mathbf{q}'_1\ \mathbf{q}'_2 \dots \mathbf{q}'_n]$ and $\mathbf{A}_n = \mathbf{Z}_{ud}\mathbf{Q}'_n$, (2) will be transformed into

$$\mathbf{A}_n\mathbf{y}'_n = \mathbf{V}_{ud} \tag{15}$$

where $\mathbf{y}'_n$ represents the projection coefficient vector of $\mathbf{I}$ on $\mathbf{Q}'_n$. By solving (15), which is a single least squares problem, the unknown current coefficients are obtained from

$$\hat{\mathbf{I}} = \mathbf{Q}'_n\hat{\mathbf{y}}_n \tag{16}$$

where $\hat{\mathbf{y}}'_n$ represents the least squares solution for $\mathbf{y}'_n$.

Based on the above description, incorporating Krylov subspace into the underdetermined equation model effectively avoids the problem that the sparse basis of the current signal is difficult to construct for 3-D objects and simplifies the CS recovery process to only one least-square calculation. The strategy of constructing non-orthogonal bases considerably reduces the computational cost required to build the Krylov subspace. Assume that the number of unknowns in the undetermined equation is N, and M rows are extracted from $\mathbf{Z}, \mathbf{V}$ to construct $\mathbf{Z}_{ud}$ and $\mathbf{V}_{ud}$. The dimension of the Krylov subspace is n. Since the matrix-vector multiplication is accelerated dramatically by MLFMM, ACA, etc., the vector orthogonalization operation, whose computational complexity could be represented by $O(Nn^2)$, gradually commands the computational resources within the conventional construction process of the Krylov subspace. By utilizing the proposed scheme of generating non-orthogonal Krylov basis vectors, this complexity decreases to $O(Mn^2)$. In general, M is much smaller than N, thus the efficiency of constructing Krylov subspace is greatly improved.

3. NUMERICAL RESULT

In order to verify the effectiveness of the proposed method, three numerical examples are provided, among which ACA is chosen as a representative of fast methods to accelerate the computation of matrix-vector multiplication. The program was run on an Inter Core i5-6200U CPU at 2.3 GHz with an internal memory capacity of 8 GB in double precision.

3.1. Numerical Example 1

A plane wave with a frequency of 3 GHz is incident on a prefect electric conductor (PEC) sphere with a radius of 0.1 m. 1920 Rao-Wilton-Glisson (RWG) basis functions are established on the surface of the sphere to solve EFIE using MoM. In order to determine the Krylov subspace dimension required to solve this problem by the underdetermined equation in conjunction with Krylov subspace, the number of rows randomly selected from the original matrix equation of MoM is assumed to always be one more than the number of generated Krylov basis vectors in this numerical experiment. The computational errors varying with the subspace dimension by using the non-orthogonal bases and the ones with the conventional orthogonal bases are calculated and compared, as shown in Figure 1, in which the relative root mean square error (R-RMSE) is obtained by

$$\text{R-RMSE} = \frac{\left\|\hat{\mathbf{I}} - \mathbf{I}\right\|_2}{\left\|\mathbf{I}\right\|_2} \tag{17}$$

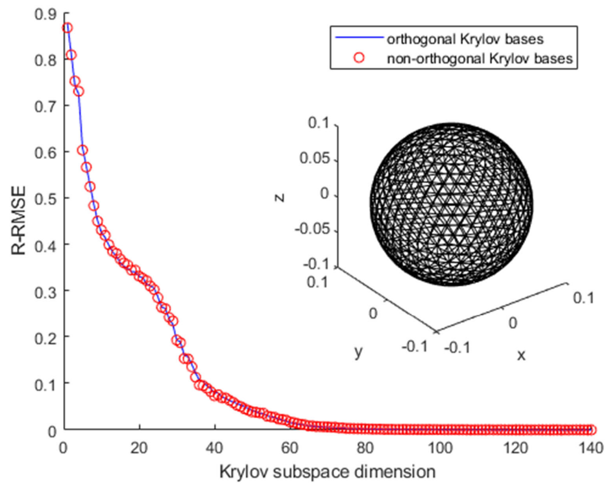

FIGURE 1. Comparison of R-RMSE curves before and after the implementation of the proposed strategy for numerical example 1.

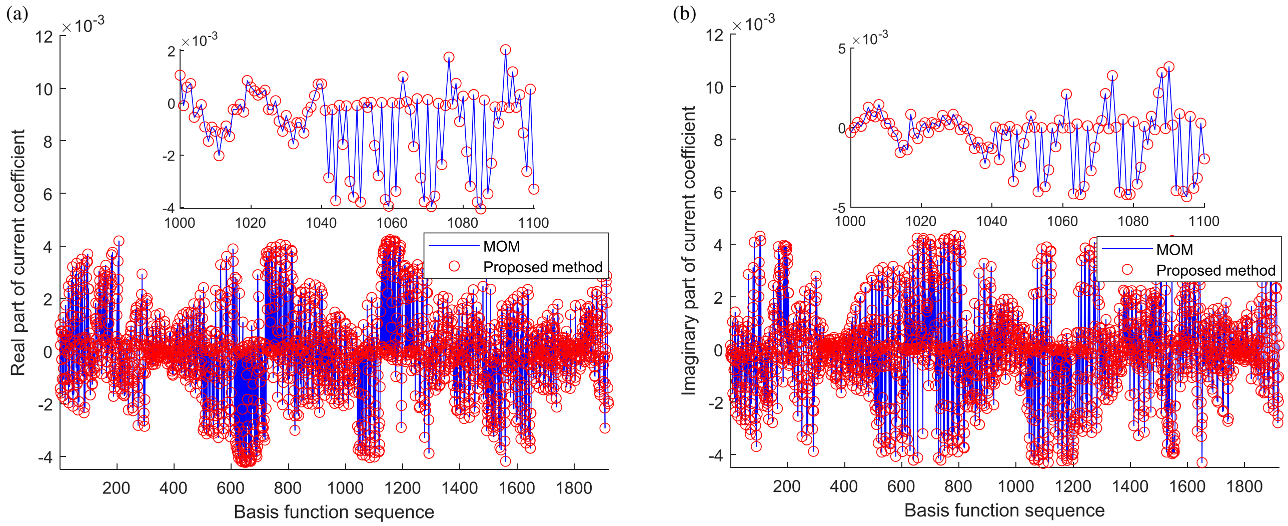

FIGURE 2. Comparison of the calculated current coefficient for numerical example 1. (a) Real part. (b) Imaginary part.

where $\hat{\mathbf{I}}$ represents the solution obtained from the underdetermined equation model, and $\mathbf{I}$ represents the solution from traditional MoM.

The error curves match well according to Figure 1, indicating that the computational accuracy using non-orthogonal bases is the same as that using orthogonal Krylov bases.

Then, selecting the subspace dimension as 122, where the error reaches 10^{-4}, we compare the computational result of the proposed method with the solution of the traditional MoM in Figure 2.

In numerical example 1, when the subspace dimension is 122, the total computation time using the traditional orthogonal Krylov bases is 4.42 s, while the total computation time using the proposed non-orthogonal Krylov bases in this paper is 1.39 s. From comparison of the total computation time, it can be seen that the proposed method can save a significant amount of computation time without affecting the calculation accuracy.

3.2. Numerical Example 2

The scatterer is set to an aircraft model with 1486 triangle surface meshes, which is illuminated by incident waves at a frequency of 3 GHz. With the establishment of RWG basis functions, a matrix equation with 2212 unknowns is obtained by solving MFIE using MoM. In this numerical experiment, we fix the extraction about $1/3$ of the impedance matrix rows, which amounts to 700 rows, to construct an underdetermined equation. As the Krylov subspace expands, the error curves calculated from the underdetermined equation model, using orthogonal and non-orthogonal basis vectors respectively, are shown in Figure 3.

From Figure 3, we found that beyond the subspace dimension of 678, the error remains consistently below 10^{-4}, e.g., when the dimension is 679, and the error is 9.7×10^{-5}.

Subsequently, taking the generating of 680 Krylov basis vectors as an example, the radar cross-section (RCS) is calculated

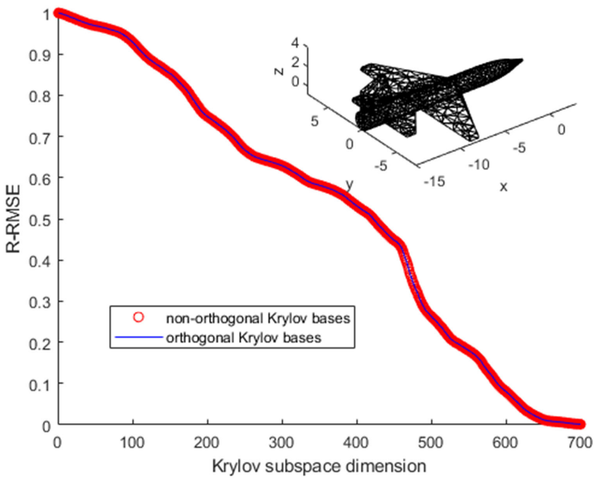

FIGURE 3. Comparison of R-RMSE curves before and after the implementation of the proposed strategy for numerical example 2.

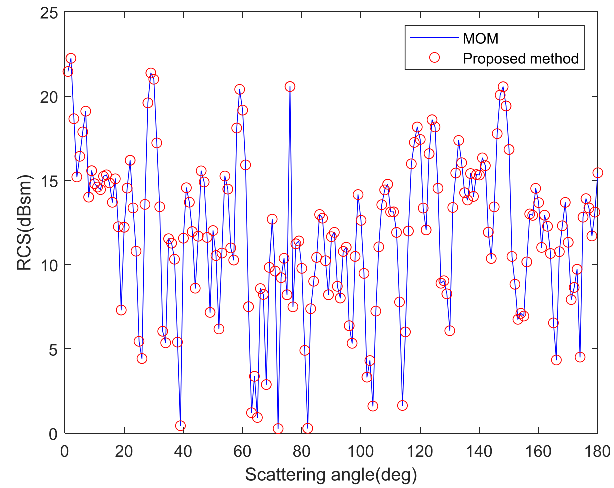

FIGURE 4. Comparison of RCS curves of the scatterer for numerical example 2.

from the proposed method and compared with the results obtained from the traditional MoM, as shown in Figure 4.

In numerical example 2, when the subspace dimension is 168, the total computation time using the traditional orthogonal Krylov bases is 28.24 s, while the total computation time using the proposed non-orthogonal Krylov bases in this paper is 12.33 s. It can be observed that the proposed method can yield accurate RCS results while also managing to save more time.

3.3. Numerical Example 3

A large electrical PEC cylinder with a base radius of 50 meters and a height of 70 meters illuminated by a plane wave with a frequency of 10 MHz is considered. 12168 RWG basis functions are established on the surface of the target to solve the CFIE using MoM, and $1/4$ of the rows are extracted from the original matrix equation to form the underdetermined equations. Figure 5 displays the error curves for the current coefficients calculated by the underdetermined equation model with orthogonal subspace bases and non-orthogonal bases.

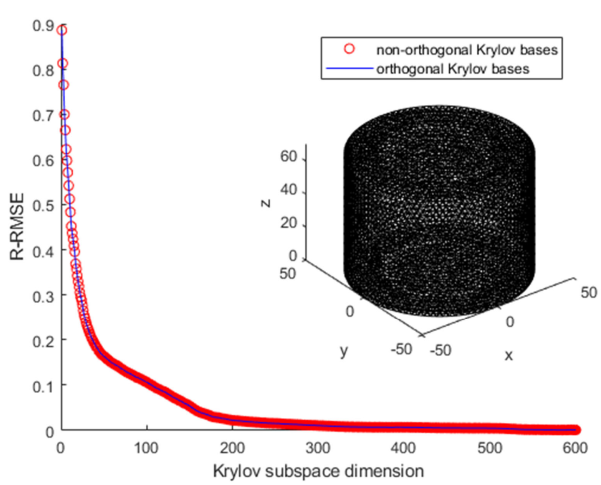

FIGURE 5. Comparison of R-RMSE curves for numerical example 3.

While the subspace dimension reaches 514, the two error curves both remain below 10^{-4}. Then, taking the generation of the 515-dimension Krylov subspace as an example, the total computation time using the traditional orthogonal Krylov bases is 1000.52 s, and the total computation time using the proposed non-orthogonal Krylov bases is 369.93 s.

4. CONCLUSION

To address the challenge of constructing a suitable sparse basis of current signals for solving three-dimensional electromagnetic scattering problems by a MoM-based underdetermined equation model, a scheme that employs Krylov subspace to replace the conventional sparse transform has been proposed. In order to further reduce the computational complexity of this approach, this paper presents a strategy of constructing non-orthogonal Krylov subspace bases, which can effectively decrease the computation cost arising from the process of vector orthogonalization while the computation of matrix-vector multiplication is accelerated by techniques such as FMM and ACA. The numerical experiments demonstrate the excellent performance of the proposed method.

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