Diffraction of a Plane Electromagnetic Wave by a Circular Aperture in a Conducting Screen of Finite Thickness

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Abstract—The paper represents a rigorous solution to the problem of diffraction of a normally incident plane electromagnetic wave by a circular hole in a perfectly conducting screen of arbitrary thickness, obtained using the eigenmode technique with allowance for the presence of a plane dielectric layer on a thick substrate behind the screen, which can play a part of a radiation detector. The main goal of the work is to describe the effect of diffraction lensless focusing in circular apertures and to determine the conditions of its appearance in the near zone of small holes, when its radius, the thickness of a screen, and a dielectric layer are of the order of the wavelength.

1. INTRODUCTION

The classical problem of electromagnetic wave diffraction by a slot or a circular aperture in a conducting screen is of great interest in the context of the recent discovery of two interesting phenomena: the occurrence of an anomalously high local intensity of the transmitted radiation [1–3] and the appearance of lensless focusing of the diffraction slot image, when the effective width of this image turns out to be several times smaller than the transverse dimensions of the aperture [4, 5]. An effective theoretical method for searching the region of existence of the latter effect was proposed in [6, 7][†]. It is obvious that these two phenomena are interconnected, since a high local intensity of the field can take place when it is concentrated in a small region, and these phenomena are noticeably manifested only for microapertures in the near zone, when the aperture dimensions, the screen thickness, and the distance from it to the plane of the diffraction pattern observation have dimensions on the order of one to several wavelengths of radiation. At great distances from the aperture, much greater than its dimensions, the diffraction field turns out to be divergent, and the smaller an aperture is, the stronger divergence is. This result is quite consistent with the conclusions of the traditional theory of optical diffraction by apertures in perfectly conducting screens, which uses the Kirchhoff integral and Green's function [8]. This approximate approach showed itself well in describing diffraction fields for great apertures in the far and middle zones. But in the near zone, directly near the aperture, it can give a distorted pattern of the field, because it comes from unrealistically idealized and simplified boundary conditions on the aperture. It should be noted that the authors [9] attempted to circumvent one of the limitations of the theory, considering the case of not infinite, as usual, but finite conductivity of the screen material. However, they used the single scattering approximation of perturbation theory, and therefore the results of [9] can be applied only to great apertures, when their dimensions are much greater than the diffracting radiation wavelength.

Therefore, in order to elaborate an adequate picture of anomalous phenomena in the near zone of microapertures, it is necessary to utilize a rigorous solution of the Maxwell equations that would completely satisfy all the exact boundary conditions at the aperture boundaries and on the surfaces of the conducting screen, in which this aperture is cut. For a slot aperture of rectangular geometry in a perfectly conducting screen, such a solution by the eigenwave technique was presented in [10, 11], and on its basis, in [7], the study of anomalous high-intensity phenomena and lensless diffraction focusing of radiation near microsized slots was carried out. It turned out that the focusing effect takes place only for TE polarization of the incident plane wave, for which the electric vector is parallel to the edges of the slot, while for orthogonal TM polarization, it practically does not manifest itself. In addition, the magnitude of this effect and its presence in general are very sensitive to changes in the geometric parameters of the diffraction system, and above all, in the width of the slot and the thickness of the conducting screen. In this connection, one is very interested in the case of a circular aperture, for which the existence of the effect of anomalously high intensity of the diffraction field was also confirmed [2, 3], but the phenomenon of lensless focusing has not been studied at all. Until now, a large body of attempts to construct an electromagnetic diffraction theory for circular apertures was made, proceeding from a static approximation suitable for apertures much smaller than the wavelength [12], or, conversely, from the approximation of large apertures using the Green's function method, the Kirchhoff integral, or other similar techniques (for example, [8, 13, 14]), which, strictly speaking, cannot be used for describing the field in the near zone. In this paper, we present an exact theory of diffraction by a circular hole, free from such limitations. Our simulation uses the eigenmode, or mode-matching technique [15, 16]. With its help, a rigorous solution of the Maxwell equations is obtained, and the conditions of the effect of diffraction focusing in a circular aperture are determined.

2. SOLUTION OF DIFFRACTION PROBLEM

Let us consider the normal incidence of a plane electromagnetic wave of unit amplitude on a perfectly conducting screen of finite thickness $2d$ with a circular hole of radius R (Fig. 1). Let this wave be linearly polarized, so that in the three-dimensional Cartesian coordinate system xyz, fixed to the axis

(a)   (b)
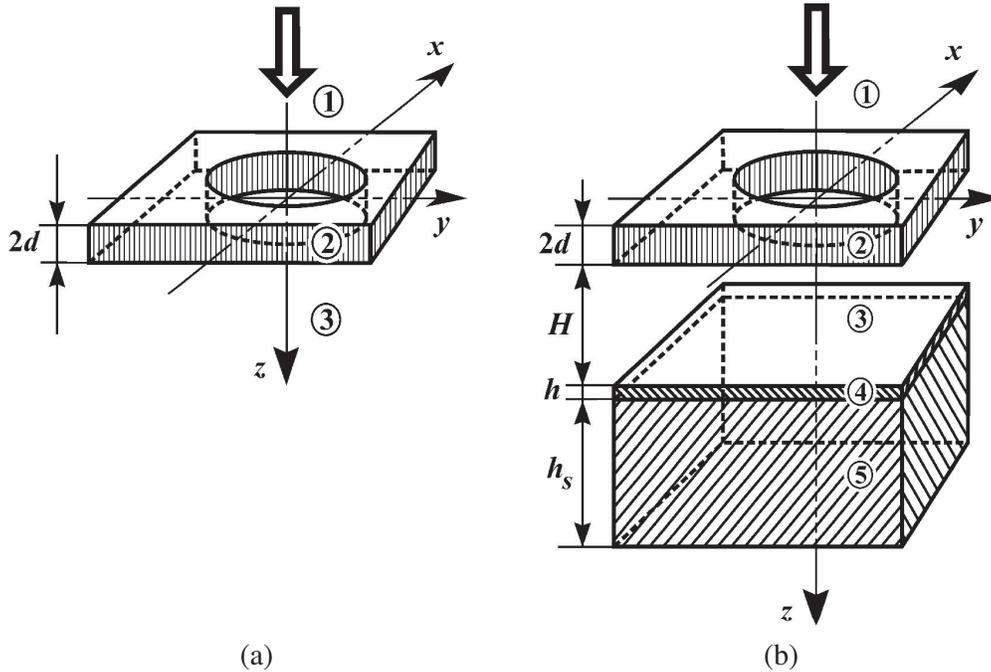
Figure 1. Geometry of the problem of wave diffraction by a circular aperture in a conducting screen, (a) in the absence of dielectrics and (b) in the presence of a thin dielectric layer on the substrate behind the screen.

of the hole, the spatial components of the fields of such a wave can be written as follows:

$$E_y^{(\text{inc})} = e^{ikz}; \quad H_x^{(\text{inc})} = -e^{ikz} \tag{1}$$

and the rest of the components are zero. Here $k = \omega/c$ is the wave number, c the speed of light in vacuum, and the exponential factor $\exp(-i\omega t)$, determining dependence of the field on time t, is omitted. We shall solve the problem of determining the total field in all space as a result of wave incidence onto a screen with a hole, and for this purpose it is convenient to use the cylindrical coordinate system $\rho\varphi z$, for which z is the axis of the symmetry of the problem, coinciding with the axis of the hole (Fig. 1). In this coordinate system, the normal components of the electric and magnetic fields of the incident wave are still zero ($E_z^{(\text{inc})} = 0$, $H_z^{(\text{inc})} = 0$), and the radial and azimuthal components have the form:

$$E_\rho^{(\text{inc})} = e^{ikz}\sin\varphi; \quad E_\varphi^{(\text{inc})} = e^{ikz}\cos\varphi; \quad H_\rho^{(\text{inc})} = -e^{ikz}\cos\varphi; \quad H_\varphi^{(\text{inc})} = e^{ikz}\sin\varphi \tag{2}$$

Our diffraction system (Fig. 1) is symmetrical along the azimuthal coordinate φ, independent of the presence or absence of a parallel plane dielectric behind the screen, so the diffraction field must reproduce exactly the symmetry of the incident field (2) with respect to this coordinate:

$$E_{\rho,z}(\rho,\varphi,z) = \tilde{E}_{\rho,z}(\rho,z)\sin\varphi; \quad E_\varphi(\rho,\varphi,z) = \tilde{E}_\varphi(\rho,z)\cos\varphi; \tag{3a}$$

$$H_{\rho,z}(\rho,\varphi,z) = \tilde{H}_{\rho,z}(\rho,z)\cos\varphi; \quad H_\varphi(\rho,\varphi,z) = \tilde{H}_\varphi(\rho,z)\sin\varphi \tag{3b}$$

As it is known [17, 18], in the cylindrical coordinate system, an azimuthally symmetrical electromagnetic field can be expressed in terms of two independent complex scalar functions $u(\rho,z)$ and $\bar{u}(\rho z)$ of the radial and axial coordinates ρ and z. In our case of single symmetry over the total azimuthal angle 2π, such expressions have the form:

$$\tilde{E}_\rho(\rho,z) = -(ik/\rho)u + \frac{\partial^2 \bar{u}}{\partial\rho\partial z}; \quad \tilde{H}_\rho(\rho,z) = \frac{\partial^2 u}{\partial\rho\partial z} - (ik/\rho)\varepsilon\bar{u}; \tag{4a}$$

$$\tilde{E}_\varphi(\rho,z) = -ik\frac{\partial u}{\partial\rho} + \rho^{-1}\frac{\partial\bar{u}}{\partial z}; \quad \tilde{H}_\varphi(\rho,z) = -\rho^{-1}\frac{\partial u}{\partial z} + ik\varepsilon\frac{\partial\bar{u}}{\partial\rho}; \tag{4b}$$

$$\tilde{E}_z(\rho,z) = \frac{\partial^2 \bar{u}}{\partial\rho\partial z} + k^2\varepsilon\bar{u}; \quad \tilde{H}_z(\rho,z) = \frac{\partial^2 u}{\partial\rho\partial z} + k^2\varepsilon u, \tag{4c}$$

where ε is the complex dielectric permittivity of the medium. Fields (3), (4) represent solutions of the Maxwell equations, if the scalar functions $u(\rho,z)$ and $\bar{u}(\rho z)$ satisfy the cylindrical Helmholtz equation [17, 18]:

$$\left(\frac{\partial^2}{\partial\rho^2} + \rho^{-1}\frac{\partial}{\partial\rho} - \rho^{-2} + \frac{\partial^2}{\partial z^2} + k^2\varepsilon\right)\left\{\begin{array}{c} u \\ \bar{u} \end{array}\right\} = 0. \tag{5}$$

Two scalar functions u and $\bar{u}$ in (4) have the meaning of the normal components of the Hertz vectors [18] and actually describe two various polarizations of an azimuthally symmetrical electromagnetic field. It can be said that the first polarization, determined by the function u, corresponds to H (or TE) polarization of the field in a rectangular coordinate system, and the second one, which is determined by the function $\bar{u}$, is an analogue of E (or TM) polarization, since at the plane interface $z = \text{const}$ between two dielectric media the amplitude reflection and refraction coefficients for the first polarization are the same as for the plane-wave H polarization with an electric vector perpendicular to the plane of incidence, and for the second one they are as for E polarization with a magnetic vector orthogonal to this plane.

In order to obtain Expressions (2) from Equations (3) and (4) for the incident plane wave, we must determine the following scalar functions for that:

$$u^{(\text{inc})} = (i\rho/k)e^{ikz}; \quad \bar{u}^{(\text{inc})} = 0. \tag{6}$$

The field equations must be supplemented with boundary conditions. There are standard vanishing conditions for the tangential components of the electric field on the surface of a perfect conductor and continuity conditions for tangential components of the electric and magnetic fields at the boundaries between dielectrics. In our case, such conditions correspond to the zero values of the components E_ρ and E_φ (3) on the plane surfaces of the screen $z = \pm d$ outside the aperture (at $\rho > R$), and to the

equality of the tangential components E_ρ, E_φ and H_ρ, H_φ on both sides of the upper $(z = -d)$ and lower $(z = +d)$ boundaries of the hole, where the conductor is absent (at $\rho < R$):

$$\tilde{E}_\rho(\rho, \mp d) = 0; \quad \tilde{E}_\varphi(\rho, \mp d) = 0; \quad \text{at} \ \ \rho > R; \qquad (7)$$

$$\left.\begin{array}{l} \tilde{E}_\rho(\rho, -d-0) = \tilde{E}_\rho(\rho, -d+0); \ \ \tilde{E}_\varphi(\rho, -d-0) = \tilde{E}_\varphi(\rho, -d+0) \\ \tilde{H}_\rho(\rho, -d-0) = \tilde{H}_\rho(\rho, -d+0); \ \ \tilde{H}_\varphi(\rho, -d-0) = \tilde{H}_\varphi(\rho, -d+0) \\ \tilde{E}_\rho(\rho, +d-0) = \tilde{E}_\rho(\rho, +d+0); \ \ \tilde{E}_\varphi(\rho, +d-0) = \tilde{E}_\varphi(\rho, +-d+0) \\ \tilde{H}_\rho(\rho, +d-0) = \tilde{H}_\rho(\rho, +d+0); \ \ \tilde{H}_\varphi(\rho, +d-0) = \tilde{H}_\varphi(\rho, +d+0) \end{array}\right\} \quad \text{at} \ \ \rho < R \qquad (8)$$

where the symbol "0" denotes an infinitesimally small positive value. In addition, the finite thickness of the conductive screen dictates the requirement for the tangential components of the electric field E_φ and E_z to be vanish on the inner conducting walls of the circular hole $\rho = R$ (Fig. 1):

$$\tilde{E}_\varphi(R, z) = 0; \quad \tilde{E}_z(R, z) = 0 \ \ \text{at} \ \ -d < z < d. \qquad (9)$$

If a plane dielectric layer on the substrate presents behind the screen (Fig. 1(b)), then we should satisfy the additional continuity conditions for the tangential field components at the plane boundaries of this layer $z = d + H$ and $z = d + H + h$:

$$\tilde{E}_{\rho,\varphi}(\rho, d + H - 0) = \tilde{E}_{\rho,\varphi}(\rho, d + H + 0); \quad \tilde{H}_{\rho,\varphi}(\rho, d + H - 0) = \tilde{H}_{\rho,\varphi}(\rho, d + H + 0); \quad (10\text{a})$$

$$\tilde{E}_{\rho,\varphi}(\rho, d + H + h - 0) = \tilde{E}_{\rho,\varphi}(\rho, d + H + h + 0); \qquad (10\text{b})$$

$$\tilde{H}_{\rho,\varphi}(\rho, d + H + h - 0) = \tilde{H}_{\rho,\varphi}(\rho, d + H + h + 0); \qquad (10\text{c})$$

for any ρ. Furthermore, in diffraction problems for infinite regions of field propagation, additional obvious conditions are imposed usually, which require phase divergence of the total field and non-increase in its total amplitude at propagation to infinity [8, 17, 18].

We shall seek the solution of the diffraction problem for various spatial field components in the form of (3), (4), where the scalar field functions $u(\rho, z)$ and $\bar{u}(\rho z)$ will be presented as superposition of partial solutions of the Helmholtz Equation (5) with indefinite amplitude coefficients, determined then during the solution of the boundary problem. To do this, we shall use the eigenmodes (or mode-matching) technique [15, 16]. It utilizes partition of the entire field propagation region into several subregions of simple geometry filled with an identical medium. In our case, the partition sought is obvious: as such, we should take the half-space in front of the screen $(z \le -d)$, the cylindrical region inside the hole $(-d \le z \le d, \ 0 \le \rho \le R)$, and the half-space behind the screen $(z \ge d)$, which can be partially filled with a plane thin dielectric with thickness h $(d + H \le z \le d + H + h)$ on a substrate with thickness h_s $(d + H + h \le z \le d + H + h + h_s)$. All these regions are numbered in Fig. 1 in the appropriate order. In each of them, a particular solution to the cylindrical Helmholtz Equation (5) is the product of the Bessel function of the first kind of order 1 of the radial coordinate ρ by the imaginary exponent of the normal coordinate z [17, 18]:

$$u(\alpha, \rho, z) = J_1(k\alpha\rho)\exp(\pm ik\beta z), \qquad (11)$$

which is called the cylindrical mode of the field. The total diffraction field will be represented by an infinite sum of such modes with undefined amplitudes. Here the first parameter α is real, and the second parameter β must be related to it by the following equation:

$$\alpha^2 + \beta^2 = \varepsilon. \qquad (12)$$

which ensures the satisfaction of cylindrical Helmholtz Equation (5) for each mode. Both of these parameters characterize the propagation of the mode along the radial coordinate axis ρ and along the normal z axis, respectively.

Then, in infinite Region 1 in front of the screen $(z \le -d)$, two scalar field functions can be represented as follows:

$$u_1(\rho, z) = ik^{-1}\rho\left\{\exp\left[ik(z+d)\right] - \exp\left[ik(z+d)\right]\right\} + ik^{-2}\int_0^{+\infty} A(\alpha)J_1(k\alpha\rho)\exp\left[-ik\beta(z+d)\right]\mathrm{d}\alpha; \quad (13\text{a})$$

$$\bar{u}_1(\rho, z) = ik^{-2}\int_0^{+\infty} \bar{A}(\alpha)J_1(k\alpha\rho)\exp\left[-ik\beta(z+d)\right]\beta^{-1}\mathrm{d}\alpha, \qquad (13\text{b})$$

where the diffraction fields are written in the form of cylindrical Fourier integrals [17–20], which determine the field expansions in the continuous spectrum of cylindrical modes (11),

$$\beta = \sqrt{1 - \alpha^2}, \tag{14}$$

$A(\alpha)$ and $\bar{A}(\alpha)$ are not yet determined amplitudes of these modes, and the coefficients i/k^2 and $i/(k^2\beta)$ are extracted from the integrands in order to simplify the following formulas. For the same purpose, expressions for incident (6) and reflected plane waves in the given region are specially distinguished in (13a).

For diffraction modes (13) of Region 1, among two possible directions of field propagation or attenuation along the z axis, only one was chosen, from the screen to infinity $z = -\infty$. Then the conditions for fields (13) at infinity can be satisfied by a certain choice of the branch of the square root (14) for each of the modes, when the branch with nonnegative real and imaginary parts must be taken into consideration.

Inside the hole (Region 2, Fig. 1), the boundary condition (9) should be additionally taken into account. It considerably restricts the domain of allowable values of the radial propagation parameter α for aperture modes (11) to a discrete set of certain values. From the second condition (9) and Equation (4c), it follows that here the Bessel function itself in a scalar function $\bar{u}(\rho z)$ should vanish, and the first condition (9) and Equation (4b) require its derivative in the other function $u(\rho, z)$ to be zero. Thus, inside the hole, the scalar field functions will no longer be integrals over the continuous spectrum of modes, but they must be represented by infinite sums over modes with a discrete spectrum of propagation parameters. In addition, it must be taken into account that Region 2 is bounded along the z axis, and therefore here one can allow for modes propagation in both the positive and negative directions along this axis:

$$u_2(\rho, z) = ik^{-2} \sum_{n=1}^{+\infty} \{a_n \exp[ik\beta_n(d+z)] + b_n \exp[ik\beta_n(d-z)]\} J_1(k\alpha_n\rho); \tag{15a}$$

$$\bar{u}_2(\rho, z) = -ik^{-2} \sum_{n=1}^{+\infty} \{\bar{a}_n \exp[ik\bar{\beta}_n(d+z)] - \bar{b}_n \exp[ik\bar{\beta}_n(d-z)]\} \bar{\beta}_n^{-1} J_1(k\bar{\alpha}_n\rho), \tag{15b}$$

where a_n and b_n, $\bar{a}_n$ and $\bar{b}_n$ are not yet determined amplitudes of the aperture modes propagating in two opposite directions along the z axis;

$$\alpha_n = \mu_n/(kR); \quad \bar{\alpha}_n = \bar{\mu}_n/(kR); \quad \beta_n = \sqrt{1 - \alpha_n^2}; \quad \bar{\beta}_n = \sqrt{1 - \bar{\alpha}_n^2}, \tag{16}$$

μ_n and $\bar{\mu}_n$ are the zeros of the derivative of the Bessel function and of this function itself:

$$J_1'(\mu_n) = 0; \quad J_1(\bar{\mu}_n) = 0, \tag{17}$$

where the prime denotes the derivative of the Bessel function with respect to its argument.

The choice of the representation of the field behind the screen depends on the presence here of a dielectric layer on the substrate or not. Let us first consider the case when there is no dielectric, and the entire diffraction system consists of a conducting screen with a hole (Fig. 1(a)). In this case, the representation of the field will be similar to representation (13) in the region in front of the screen, and only now the direction from the diffraction system to infinity is the positive direction of the z axis, but not negative. In addition, there is no need to separate any mode as an incident wave from the diffraction integral, and therefore, the scalar field functions in the region behind the screen can be written as:

$$u_3(\rho, z) = ik^{-2} \int_0^{+\infty} B(\alpha) J_1(k\alpha\rho) \exp[ik\beta(z-d)] \, d\alpha; \tag{18a}$$

$$\bar{u}_3(\rho, z) = -ik^{-2} \int_0^{+\infty} \bar{B}(\alpha) J_1(k\alpha\rho) \exp[ik\beta(z-d)] \beta^{-1} d\alpha, \tag{18b}$$

where $B(\alpha)$ and $\bar{B}(\alpha)$ are the unknown mode amplitudes, and β is their normal propagation parameter determined by the previous formula (14) and the above rule of choosing the square root branch.

Expressions for all six spatial components of the electric and magnetic fields in various regions can be obtained by substituting representations (13), (15), or (18) into formulas (4). In turn, substitution of

the resulting expressions into the boundary Equations (7), (8) at $z = \pm d$ makes it possible to determine the unknown amplitudes of all field modes inside and outside the aperture. The obtained boundary equations can be transformed and simplified using the known relations [19]

$$J_1(k\alpha\rho)/(k\alpha\rho) + J_1'(k\alpha\rho) = J_0(k\alpha\rho); \quad J_1(k\alpha\rho)/(k\alpha\rho) - J_1'k\alpha\rho) = J_2(k\alpha\rho),$$

which yields the following form of these equations: on the entire boundary $z = -d$ outside and inside the aperture:

$$\int_0^{+\infty} \left[A(\alpha) + \bar{A}(\alpha)\right] J_0(k\alpha\rho)\alpha d\alpha = \sum_{m=1}^{+\infty} \left[\Phi_m^{(+)} J_0(k\alpha_m\rho) + \bar{\Phi}_m^{(+)} J_0(k\bar{\alpha}_m\rho)\right] \theta(R - \rho); \qquad (19a)$$

$$\int_0^{+\infty} \left[A(\alpha) - \bar{A}(\alpha)\right] J_2(k\alpha\rho)\alpha d\alpha = \sum_{m=1}^{+\infty} \left[\Phi_m^{(+)} J_2(k\alpha_m\rho) - \bar{\Phi}_m^{(+)} J_2(k\bar{\alpha}_m\rho)\right] \theta(R - \rho); \qquad (19b)$$

on the entire boundary $z = d$ outside and inside the aperture:

$$\int_0^{+\infty} \left[B(\alpha) + \bar{B}(\alpha)\right] J_0(k\alpha\rho)\alpha d\alpha = \sum_{m=1}^{+\infty} \left[\Psi_m^{(+)} J_0(k\alpha_m\rho) + \bar{\Psi}_m^{(+)} J_0(k\bar{\alpha}_m\rho)\right] \theta(R - \rho); \qquad (19c)$$

$$\int_0^{+\infty} \left[B(\alpha) - B(\alpha)\right] J_2(k\alpha\rho)\alpha d\alpha = \sum_{m=1}^{+\infty} \left[\Psi_m^{(+)} J_2(k\alpha_m\rho) - \bar{\Psi}_m^{(+)} J_2(k\bar{\alpha}_m\rho)\right] \theta(R - \rho); \qquad (19d)$$

at the boundary $z = -d$ inside the aperture ($\rho < R$):

$$\left\{-4 + \int_0^{+\infty} \left[\beta A(\alpha) + \beta^{-1}\bar{A}(\alpha)\right] J_0(k\alpha\rho)\alpha d\alpha \right.$$
$$\left. + \sum_{m=1}^{+\infty} \left[\Phi_m^{(-)} J_0(k\alpha_m\rho)\beta_m + \bar{\Phi}_m^{(-)} J_0(k\bar{\alpha}_m\rho)\bar{\beta}_m^{-1}\right] \right\} \theta(R - \rho) = 0; \qquad (20a)$$

$$\left\{\int_0^{+\infty} \left[\beta A(\alpha) - \beta^{-1}\bar{A}(\alpha)\right] J_2(k\alpha\rho)\alpha d\alpha \right.$$
$$\left. + \sum_{m=1}^{+\infty} \left[\Phi_m^{(-)} J_2(k\alpha_m\rho)\beta_m - \bar{\Phi}_m^{(-)} J_2(k\bar{\alpha}_m\rho)\bar{\beta}_m^{-1}\right] \right\} \theta(R - \rho) = 0; \qquad (20b)$$

at the boundary $z = d$ inside the aperture ($\rho < R$):

$$\left\{\int_0^{+\infty} \left[\beta B(\alpha) + \beta^{-1}\bar{B}(\alpha)\right] J_0(k\alpha\rho)\alpha d\alpha \right.$$
$$\left. - \sum_{m=1}^{+\infty} \left[\Psi_m^{(-)} J_0(k\alpha_m\rho)\beta_m + \bar{\Psi}_m^{(-)} J_0(k\bar{\alpha}_m\rho)\bar{\beta}_m^{-1}\right] \right\} \theta(R - \rho) = 0; \qquad (20c)$$

$$\left\{\int_0^{+\infty} \left[\beta B(\alpha) - \beta^{-1}\bar{B}(\alpha)\right] J_2(k\alpha\rho)\alpha d\alpha \right.$$
$$\left. - \sum_{m=1}^{+\infty} \left[\Psi_m^{(-)} J_2(k\alpha_m\rho)\beta_m - \bar{\Psi}_m^{(-)} J_2(k\bar{\alpha}_m\rho)\bar{\beta}_m^{-1}\right] \right\} \theta(R - \rho) = 0, \qquad (20d)$$

where

$$\Phi_m^{(\pm)} = \alpha_m \left[a_m \pm b_m \exp(2ik\beta_m d)\right]; \quad \bar{\Phi}_m^{(\pm)} = \bar{\alpha}_m \left[\bar{a}_m \pm \bar{b}_m \exp(2ik\bar{\beta}_m d)\right]; \qquad (21a)$$

$$\Psi_m^{(\pm)} = \alpha_m \left[a_m \exp(2ik\beta_m d) \pm b_m\right]; \quad \bar{\Psi}_m^{(\pm)} = \bar{\alpha}_m \left[\bar{a}_m \exp(2ik\bar{\beta}_m d) \pm \bar{b}_m\right] \qquad (21b)$$

are various linear combinations of in-aperture mode amplitudes, and θ is the Heaviside step function [8]: $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$.

The Fourier-Bessel integrals in the left-hand sides of Equation (19) can be considered as the integral Hankel transform of the amplitude functions $A(\alpha) \pm \bar{A}(\alpha)$ and $B(\alpha) \pm \bar{B}(\alpha)$ [19, 20]. For it, there is also an inverse transform, which allows us to select these functions from under the integral sign. Applying such a transform to integral Equation (19), we can express the amplitudes of the modes of the space above and below the screen in terms of the amplitudes of the hole modes:

$$A(\alpha) = \sum_{m=1}^{+\infty} \Phi_m^{(+)} Q_m^{(+)}(\alpha); \quad \bar{A}(\alpha) = \sum_{m=1}^{+\infty} \left[\Phi_m^{(+)} Q_m^{(-)}(\alpha) + \bar{\Phi}_m^{(+)} \bar{Q}_m^{(+)}(\alpha) \right]; \tag{22a}$$

$$B(\alpha) = \sum_{m=1}^{+\infty} \Psi_m^{(+)} Q_m^{(+)}(\alpha); \quad \bar{B}(\alpha) = \sum_{m=1}^{+\infty} \left[\Psi_m^{(+)} Q_m^{(-)}(\alpha) + \bar{\Psi}_m^{(+)} \bar{Q}_m^{(+)}(\alpha) \right], \tag{22b}$$

where

$$Q_m^{(\pm)}(\alpha) = \left[I_m^{(0)}(\alpha) \pm I_m^{(2)}(\alpha) \right]/2; \quad \bar{Q}_m^{(\pm)}(\alpha) = \left[\bar{I}_m^{(0)}(\alpha) \pm \bar{I}_m^{(2)}(\alpha) \right]/2;$$

$$I_m^{(n)}(\alpha) = k^2 \int_0^R J_n(k\alpha\rho) J_n(k\alpha_m\rho)\rho\mathrm{d}\rho; \quad \bar{I}_m^{(n)}(\alpha) = k^2 \int_0^R J_n(k\alpha\rho) J_n(k\bar{\alpha}_m\rho)\rho\mathrm{d}\rho; \quad n = 0; \; 2.$$

In our case, when the parameters of the radial propagation of in-aperture modes α_m and $\bar{\alpha}_m$ (16) are determined by the zeros of the Bessel function and its derivative, we obtain:

$$Q_m^{(+)}(\alpha) = \begin{cases} k\alpha_m R \left(\alpha_m^2 - \alpha^2 \right)^{-1} J_1(k\alpha_m R) J_1' k\alpha R) & \text{at} \quad \alpha \neq \alpha_m \\ \left(k^2 R^2 - \alpha^{-2} \right) J_1^2(k\alpha_m R)/2 & \text{at} \quad \alpha = \alpha_m \end{cases} \tag{23a}$$

$$Q_m^{(-)}(\alpha) = (\alpha_m \alpha)^{-1} J_1(k\alpha_m R) J_1(k\alpha R); \tag{23b}$$

$$\bar{Q}_m^{(+)}(\alpha) = \begin{cases} k\alpha R \left(\alpha^2 - \bar{\alpha}_m^2 \right)^{-1} J_1'k\bar{\alpha}_m R) J_1' k\alpha R) & \text{at} \quad \alpha \neq \bar{\alpha}_m \\ k^2 R^2 \left[J_1' k\alpha_m R) \right]^2/2 & \text{at} \quad \alpha = \bar{\alpha}_m \end{cases} \tag{23c}$$

$$\bar{Q}_m^{(-)} = 0. \tag{23d}$$

The resulting expressions (22) for the amplitudes of the extra-aperture modes should be substituted into the remaining four Equation (20) for the tangential components of the magnetic field at the aperture boundaries, and then the obtained equations must be solved with respect to the amplitudes of the aperture modes. But these equations should be simplified preliminarily by multiplying them by terms $k^2\rho J_0(k\alpha_n\rho)$, $k^2\rho J_0(k\bar{\alpha}_n\rho)$ or $k^2\rho J_2(k\alpha_n\rho)$, $k^2\rho J_2(k\bar{\alpha}_n\rho)$ with various parameters α_n and $\bar{\alpha}_n$, and then by integrating over the radial coordinate ρ from zero to the edge of the aperture R. Taking into account the relations

$$Q_n^{(+)}(\alpha_m) = 0 \quad \text{and} \quad \bar{Q}_n^{(+)}(\bar{\alpha}_m) = 0 \quad \text{at} \quad m \neq n;$$

$$Q_n^{(-)}(\bar{\alpha}_m) = 0 \quad \text{and} \quad \bar{Q}_n^{(-)}(\alpha_m) = 0 \quad \text{at} \quad \text{any } m \text{ and } n,$$

then we get:

$$\int_0^{+\infty} \left[\beta A(\alpha) Q_n^{(+)}(\alpha) + \beta^{-1} \bar{A}(\alpha) Q_n^{(-)}(\alpha) \right] \alpha\mathrm{d}\alpha + \Phi_n^{(-)} \beta_n Q_n = 2\gamma_n; \tag{24a}$$

$$\int_0^{+\infty} \beta^{-1} \bar{A}(\alpha) \bar{Q}_n^{(+)}(\alpha) \alpha\mathrm{d}\alpha + \bar{\Phi}_n^{(-)} \bar{\beta}_n^{-1} \bar{Q}_n = 0; \tag{24b}$$

$$\int_0^{+\infty} \left[\beta B(\alpha) Q_n^{(+)}(\alpha) + \beta^{-1} \bar{B}(\alpha) Q_n^{(-)}(\alpha) \right] \alpha\mathrm{d}\alpha - \Psi_n^{(-)} \beta_n Q_n = 0; \tag{24c}$$

$$\int_0^{+\infty} \beta^{-1} \bar{B}(\alpha) \bar{Q}_n^{(+)}(\alpha) \alpha\mathrm{d}\alpha - \bar{\Psi}_n^{(-)} \bar{\beta}_n^{-1} \bar{Q}_n = 0, \tag{24d}$$

where

$$Q_n = Q_n^{(+)}(\alpha_n) = \frac{1}{2} \left(k^2 R^2 - \alpha_n^{-2} \right) J_1^2(k\alpha_n R); \quad \bar{Q}_n = \bar{Q}_n^{(+)}(\bar{\alpha}_n) = \frac{1}{2} k^2 R^2 [J_1'(k\bar{\alpha}_n R)]^2; \tag{25}$$

$$\gamma_n = k^2 \int_0^R J_0(k\alpha_n\rho)\rho\mathrm{d}\rho = kR\alpha_n^{-1} J_1(k\alpha_n R).$$

Now, expressions (22) for the amplitudes of extra-aperture modes can be substituted into simplified Equation (24). As a result, we obtain the following system of linear algebraic equations for these amplitudes:

$$\sum_{m=1}^{+\infty} \left(\Phi_m^{(+)} U_{nm} + \bar{\Phi}_m^{(+)} V_{nm} \right) + \Phi_n^{(-)} \beta_n Q_n = 2\gamma_n; \tag{26a}$$

$$\sum_{m=1}^{+\infty} \left(\Phi_m^{(+)} V_{mn} + \bar{\Phi}_m^{(+)} \bar{U}_{nm} \right) + \bar{\Phi}_n^{(-)} \bar{\beta}_n^{-1} \bar{Q}_n = 0; \tag{26b}$$

$$\sum_{m=1}^{+\infty} \left(\Psi_m^{(+)} U_{nm} + \bar{\Psi}_m^{(+)} V_{nm} \right) - \Psi_n^{(-)} \beta_n Q_n = 0; \tag{26c}$$

$$\sum_{m=1}^{+\infty} \left(\Psi_m^{(+)} V_{mn} + \bar{\Psi}_m^{(+)} \bar{U}_{nm} \right) - \bar{\Psi}_n^{(-)} \bar{\beta}_n^{-1} \bar{Q}_n = 0, \tag{26d}$$

where

$$U_{nm} = \int_0^{+\infty} \left[\beta Q_n^{(+)}(\alpha) Q_m^{(+)}(\alpha) + \beta^{-1} Q_n^{(+)}(\alpha) Q_m^{(+)}(\alpha) \right] \alpha d\alpha; \tag{27a}$$

$$V_{nm} = \int_0^{+\infty} Q_n^{(-)}(\alpha) \bar{Q}_m^{(+)}(\alpha) \alpha \beta^{-1} d\alpha; \quad \bar{U}_{nm} = \int_0^{+\infty} \bar{Q}_n^{(+)}(\alpha) \bar{Q}_m^{(+)}(\alpha) \alpha \beta^{-1} d\alpha. \tag{27b}$$

$n = 1, 2, 3, 4, \ldots$ Unknown amplitudes of in-aperture modes enter Equation (26) through linear combinations (21).

The system of Equation (26) is not divided into two independent subsystems for the amplitudes of two different polarizations of the field inside the aperture. This confirms the well-known fact that, in the case of diffraction at the edges of cylindrical surfaces, H and E polarizations of the fields are not independent, but mutually generate each other [17, 18]. They become independent only in one particular case, when the initial diffracting field (2) does not depend at all on the azimuthal coordinate φ. This is the case when an incident wave is a symmetric H type wave H_{0n} [17, 18].

The system of linear Equation (26) is infinite-dimensional with an infinite number of unknowns. In order to directly solve such a system, it is necessary to carry out the procedure of its reduction to a system of finite dimensions. In this respect, the case of a circular aperture is completely analogous to the case of a rectangular slot aperture [10, 11]: the truncation of the infinite series of amplitudes and the infinite dimension of the system itself is carried out on the basis of the same criteria. It requires achieving a certain accuracy in satisfaction of the boundary conditions (7), (8) on screen surfaces $z = \pm d$ and at the aperture boundaries $z = \pm d$, $\rho < R$ for total fields (13), (15), (18). If, when checking the boundary conditions for the fields, it turns out that the accuracy of their fulfillment is sufficient, then we can restrict ourselves to the selected number of aperture and extra-aperture modes and amplitudes. But if the error in these conditions is too great, then we need to increase the number of modes taken into account in the theoretical calculation.

In the presence of a dielectric layer behind a screen with an aperture (Fig. 1(b)), it is necessary to take into account the different forms of the field in different dielectric media and take into account additional boundary conditions (9) on plane boundaries between these media. Usually, the thickness of the dielectric layer h, which plays a part of a detector of optical radiation, is very small, being on the order of the wavelength of light, but the thickness of the substrate h_s, supporting the layer, is quite great to ensure mechanical stability and fixation of a thin layer in space. Therefore, it will not be a mistake to consider the thickness of the substrate to be infinitely great and to assume that it entirely occupies the half-space $z \geq d + H + h$. Such an assumption does not strongly distort the field pattern in the dielectric layer and in front of it, but significantly simplifies the solution of the diffraction problem.

So, let the field behind the screen ($z \geq d$) fills three regions (Fig. 1(b)): Region 3 between the conducting screen and the dielectric layer ($d \leq z \leq d + H$), Region 4 inside the dielectric layer ($d + H \leq z \leq d + H + h$), and Region 5 inside the substrate ($z \geq d + H + h$). By analogy with (18),

the general expression for scalar field functions in these regions can be written as follows:

$$u_M(\rho, z) = ik^{-2} \int_0^{+\infty} B(\alpha) J_1(k\alpha\rho) \left[f_M^+(\alpha, z)/D(\alpha) \right] d\alpha; \tag{28a}$$

$$\bar{u}_M(\rho, z) = -ik^{-2} \int_0^{+\infty} \bar{B}(\alpha) J_1(k\alpha\rho) \beta_3^{-1} \left[\bar{f}_M^+(\alpha, z)/\bar{D}(\alpha) \right] d\alpha, \tag{28b}$$

where $M = 3$; 4; 5 is the medium number,

$$f_3^\pm(\alpha, z) = P(\alpha) \exp\left[ik\beta_3(z - d)\right] \pm R(\alpha) \exp\left[ik\beta_3(2H + d - z)\right]; \tag{29a}$$

$$f_4^\pm(\alpha, z) = T_{34} \left\{ \exp\left[ik\beta_4(z - d - H)\right] \pm R_{45} \exp\left[ik\beta_4(2h + H + d - z)\right] \right\}; \tag{29b}$$

$$f_5^\pm(\alpha, z) = T_{34}T_{45} \exp\left[ik\beta_5(z - d - H - h)\right] \exp\left[ik(\beta_3 H + \beta_4 h)\right]; \tag{29c}$$

$$P(\alpha) = 1 + R_{34}R_{45}\exp(2ik\beta_4 h); \quad R(\alpha) = R_{34} + R_{45}\exp(2ik\beta_4 h), \tag{30}$$

and similarly for the same quantities of another polarization, which are indicated by a bar above,

$$\beta_M = \sqrt{\varepsilon_M - \alpha^2} \tag{31}$$

is the parameter of normal propagation of cylindrical modes along the z axis in various dielectric media, which must satisfy condition (12) in each medium, and ε_M is the permittivity of the medium filling given region ($\varepsilon_3 = 1$, $\beta_3 = \beta$ (14));

$$R_{34} = \frac{\varepsilon_4^\nu \beta_3 - \varepsilon_3^\nu \beta_4}{\varepsilon_4^\nu \beta_3 + \varepsilon_3^\nu \beta_4}; \quad T_{34} = \frac{2\varepsilon_3^\nu \beta_3}{\varepsilon_4^\nu \beta_3 + \varepsilon_3^\nu \beta_4}; \quad R_{45} = \frac{\varepsilon_5^\nu \beta_4 - \varepsilon_4^\nu \beta_5}{\varepsilon_5^\nu \beta_4 + \varepsilon_4^\nu \beta_5}; \quad T_{44} = \frac{2\varepsilon_4^\nu \beta_4}{\varepsilon_5^\nu \beta_4 + \varepsilon_4^\nu \beta_5}$$

are the amplitude reflection and transmission coefficients [8, 16] on the plane boundaries $z = d + H$ and $z = d + H + h$ between media 3–4 and 4–5; $\nu = 0$ for our conventional H polarization and $\nu = 1$ for E polarization,

$$D(\alpha) = P(\alpha) + R(\alpha)\exp(2ik\beta_3 H); \quad \bar{D}(\alpha) = \bar{P}(\alpha) - \bar{R}(\alpha)\exp(2ik\beta_3 H). \tag{32}$$

To determine the spatial components of the field in various dielectric media behind the screen, expressions (28) must be substituted into Equation (4). In this case, the fields in front of the screen and inside the aperture will be determined by the same expressions (13), and (15), (16), as in the case of the absence of a dielectrics behind the screen.

Scalar functions (28) differ from those in (18) in the case of the absence of dielectrics by a more complex form of functions (29), which determine the dependence of the mode field on the normal z-coordinate. Since here each dielectric layer has two boundaries, there are terms in these functions, which describe two waves, propagating in the forward and backward directions from the aperture (in the z-axis), except, of course, for the substrate with no lower boundary, generating opposite wave upon reflection. The coefficients of these terms have been selected in such a way that for each mode, conditions (9) of the continuity of the tangential components of the electric and magnetic fields are satisfied at both interfaces of various dielectrics $z = d + H$ and $z = d + H + h$. Thus, for fields (28) as a whole, the conditions at the boundaries of dielectrics are satisfied automatically, and for them it remains to require the fulfillment of boundary conditions (7), (8) on the screen surface $z = d$ and on the lower boundary of the aperture.

The transformation of the corresponding boundary equations and their reduction to equations for the amplitudes of the aperture modes is carried out in exactly the same way as in the case of the absence of dielectrics behind the screen. The only difference from the case with dielectrics is that additional factors appear in the integrals over the modes of the space behind the screen, due to the presence of terms $f_3^\pm(\alpha, d)/D(\alpha)$ in (28) instead of simple exponentials in (18). But their constant coefficients are chosen so that $f_3^+(\alpha d)/D(\alpha) = 1$; $\bar{f}_3^-(\alpha d)/\bar{D}(\alpha) = 1$, then the amplitudes of the continuous spectrum modes $B(\alpha)$ and $\bar{B}(\alpha)$ will be determined by the same formulas (22b) as in the case of free space behind the screen. The formal difference of the case with dielectrics will manifest itself only in the fact that in Equations (26c), (26d) the coefficients (27) of the system of amplitude equations in the integrands will have additional factors

$$f_3^-(\alpha, d)/D(\alpha) = S(\alpha)/D(\alpha); \quad \bar{f}_3^+(\alpha, d)/\bar{D}(\alpha) = \bar{S}(\alpha)/\bar{D}(\alpha),$$

where $S(\alpha) = P(\alpha) - R(\alpha)\exp(2ik\beta_3 H)$; $\bar{S}(\alpha) = \bar{P}(\alpha) + \bar{R}(\alpha)\exp(2ik\beta_3 H)$, moreover, the functions $P(\alpha)$ and $D(\alpha)$ with an overscribed bar and without a bar are defined here by formulas (30), (32). As a result, instead of system (26) for aperture mode amplitudes, we get a new system, where the first two equations are exactly written as (26a), (26b), and the other two equations will have the following form:

$$\sum_{m=1}^{+\infty} \left(\Psi_m^{(+)}W_{nm} + \bar{\Psi}_m^{(+)}K_{nm} \right) - \Psi_n^{(-)}\beta_n Q_n = 0; \tag{33a}$$

$$\sum_{m=1}^{+\infty} \left(\Psi_m^{(+)}K_{mn} + \bar{\Psi}_m^{(+)}\bar{W}_{nm} \right) - \bar{\Psi}_n^{(-)}\bar{\beta}_n^{-1}\bar{Q}_n = 0, \tag{33b}$$

where

$$W_{nm} = \int_0^{+\infty} \left\{ \beta_3 \left[P(\alpha)/D(\alpha) \right] Q_n^{(+)}(\alpha)Q_m^{(+)}(\alpha) + \varepsilon_3\beta_3^{-1} \left[\bar{P}(\alpha)/\bar{D}(\alpha) \right] Q_n^{(-)}(\alpha)Q_m^{(-)}(\alpha) \right\} \alpha\mathrm{d}\alpha; \tag{34a}$$

$$K_{nm} = \int_0^{+\infty} \left[\bar{P}(\alpha)/\bar{D}(\alpha) \right] Q_n^{(-)}(\alpha)\bar{Q}_m^{(+)}(\alpha)\varepsilon_3\alpha\beta_3^{-1}\mathrm{d}\alpha; \tag{34b}$$

$$\bar{W}_{nm} = \int_0^{+\infty} \left[\bar{P}(\alpha)/\bar{D}(\alpha) \right] \bar{Q}_n^{(+)}(\alpha)\bar{Q}_m^{(+)}(\alpha)\varepsilon_3\alpha\beta_3^{-1}\mathrm{d}\alpha. \tag{34c}$$

The remaining quantities are given here by the same expressions (23), (25), (27) as in the case of diffraction without dielectrics.

Just as in the case of slot apertures [11], the quantities $D(\alpha)$ and $\bar{D}(\alpha)$ (32), included in expressions for fields (28) and in formulas (34) for the coefficients of the system of amplitude equations (33), can become very small in magnitude or vanish altogether. Their appearance in these formulas is due to the reflection and refraction of diffraction radiation at the boundaries of the dielectric layer placed behind the aperture, and the vanishing of these quantities corresponds physically to the excitation of waveguide modes in the plane dielectric layer at the corresponding spatial frequencies [11]. As a result, field integrals (28) and integrals of coefficients (34) acquire poles — singular points, where the integrands become infinite. The calculation of such integrals requires a special approach, and one of the simplest and most effective of them is the separation of a singularity as an isolated term [11].

But for the calculation of the singular integrals, we can apply another simpler regularization method without selecting isolated waveguide components. Namely, it is sufficient to assume that the dielectric layer has a weak fictitious absorption (on the order of 10^{-5}–10^{-6}), the appearance of which practically does not distort the diffraction field pattern. Then the zeros of the functions $D(\alpha)$ and $\bar{D}(\alpha)$ (32) will shift so far enough from the real axis of the argument α that we can neglect the singularity of the integrands. Physically, this corresponds to the situation when the field of waveguide modes at infinity is negligibly small, and the region of its existence lies entirely within the region of excitation of the nonresonant diffraction field. Indeed, the amplitudes of the waveguide components of the fields will be proportional to the Bessel function and its derivative, and therefore, they decrease with increasing argument (the radial coordinate ρ). Hence, in contrast to the rectangular two-dimensional geometry of slot diffraction [11], here the waveguide modes, diverging in all directions from the circular aperture, will be characterized by decreasing amplitudes, even if the dielectric layer and substrate are transparent, with the decay being approximately the same as for the reminder of the diffraction field. This physically confirms the possibility of formally taking into account waveguide modes in the total diffraction field without special separation of them as isolated field components.

As an illustration, Fig. 2 shows the results of calculation of the spatial distribution of the amplitudes of two components of the total electric field at the diffraction of a plane electromagnetic wave of unit amplitude on a circular hole with the radius $R = 1.2\lambda$ in a perfectly conducting screen with the thickness of $2d = 1.6\lambda$, behind which at a distance $H = 0.6\lambda$ there is a plane dielectric with the thickness $h = 0.8\lambda$ having the refractive index $n = 1.62$, fixed on an infinitely thick substrate with refractive index $n_s = 1.46$. This figure demonstrates effectiveness of the presented method for calculation of the field pattern in the near zone. However, below we will be interested in the electric field only in a thin dielectric film, which plays a part of a radiation detector or an object under study in optical spectroscopy. For evaluation of the focusing properties of a circular aperture, we will use the same technique, as for the case of a rectangular slot [7].
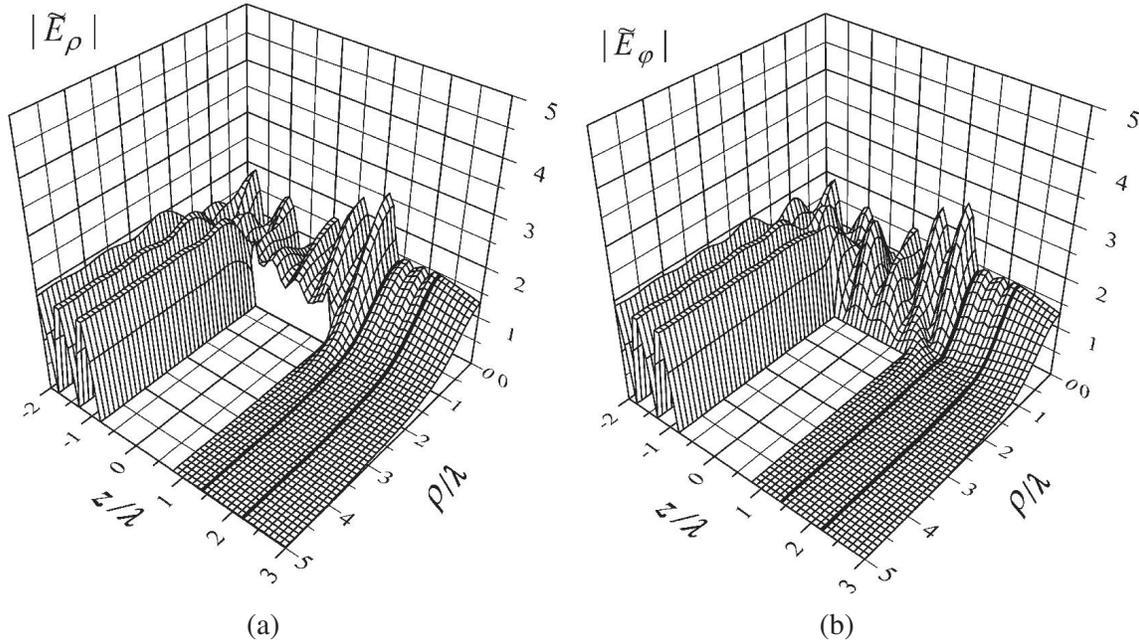
Figure 2. The magnitude of the amplitude of (a) the radial and (b) azimuthal components of the total electric field of diffraction in various points of space. Thicker amplitude lines indicate the boundaries of the dielectric layer.

3. FOCUSING PROPERTIES OF A CIRCULAR MICROHOLE

The time-averaged energy density of the electric field (3a) at each point inside the dielectric layer (film) is determined by the expression [17]:

$$w(\rho, \varphi, z) = \frac{\varepsilon}{16\pi} \left\{ \left[\left| \tilde{E}_\rho\left(\rho, z\right) \right|^2 + \left| \tilde{E}_z\left(\rho, z\right) \right|^2 \right] \sin^2\varphi + \left| \tilde{E}_\varphi\left(\rho, z\right) \right|^2 \cos^2\varphi \right\} \tag{35}$$

(if the dielectric permittivity of the film is complex, then one can take its real part as the quantity ε, which is usually much greater in magnitude than the imaginary one). The energy density (35) in the near zone weakly depends on the normal coordinate z inside the layer (see Fig. 2), so it makes sense to average expression (35) over the thickness of the dielectric from $z = d + H$ to $z = d + H + h$. We will consider the relative density of the electric energy of the field

$$\bar{W}(\rho, \varphi) = \frac{\varepsilon}{hw_0} \int_{d+H}^{d+H+h} \left\{ \left[\left| \tilde{E}_\rho\left(\rho, z\right) \right|^2 + \left| \tilde{E}_z\left(\rho, z\right) \right|^2 \right] \sin^2\varphi + \left| \tilde{E}_\varphi\left(\rho, z\right) \right|^2 \cos^2\varphi \right\} dz,$$

where w_0 is the average electric energy density of the incident plane wave (1). Since the fields are symmetrical along the azimuthal angle φ, it is also reasonable to average over this spatial coordinate as well. Then the relative intensity of the electric field inside the dielectric at a distance ρ from the aperture axis is

$$W(\rho) = \frac{1}{2\pi} \int_0^{2\pi} \bar{W}(\rho, \varphi) d\varphi = \frac{\varepsilon}{2hw_0} \int_{d+H}^{d+H+h} \left[\left| \tilde{E}_\rho\left(\rho, z\right) \right|^2 + \left| \tilde{E}_\varphi\left(\rho, z\right) \right|^2 + \left| \tilde{E}_z\left(\rho, z\right) \right|^2 \right] dz, \tag{36}$$

The value (36) can be calculated for all values of ρ using the solution of the diffraction problem (28), (29) for the region $M = 4$. The distribution of this value along the radial coordinate ρ, of course, will be nonuniform. Let us determine the effective amplitude A of the uniform intensity distribution along the radius, which is equivalent to the inhomogeneous distribution (36) according to the equality:

$$\int_0^{+\infty} A\Delta(\rho)\rho d\rho = \int_0^{+\infty} W(\rho)\Delta(\rho)\rho d\rho,$$

where $\Delta(\rho)$ is the weight function. By analogy with the case of a simple slot [7], we set $\Delta(\rho) = W^2(\rho)$ for it. Then the effective amplitude is equal to:

$$A = W_3/W_2, \quad \text{where } W_n = \int_0^{+\infty} W^n(\rho)\rho d\rho.$$

It corresponds to effective aperture radius R_a, namely: the product of the effective amplitude A by the area of a circle with such an effective radius gives the total field energy contained in the dielectric:

$$A \cdot \pi R_a^2 = \int_0^{+\infty} d\rho \int_0^{2\pi} \bar{W}(\rho, \varphi)\rho d\varphi.$$

And since the integral in the right hand side is equal to $2\pi W_1$, then $\pi R_a^2 = 2\pi W_1/A = 2\pi W_1 W_2/W_3$. This expression approximately characterizes the effective area of the diffraction image of a circular hole in a dielectric film. Then the index of focusing, or the index of the relative reduction of the aperture image, introduced in [7], can be defined as the ratio of the real aperture area to the effective area of the aperture image:

$$F = \pi R^2/(\pi R_a^2) = R^2 W_3/(2W_1 W_2) \tag{37}$$

This index makes it possible to evaluate the quality of the aperture image formed by the diffraction field in a dielectric film, depending on the parameters of the problem: the aperture radius R, the thickness of the conducting screen $2d$, the distance from the screen to the dielectric layer H, and its thickness h. For a slot aperture of rectangular geometry, the last two parameters have little effect on the quality of diffraction image, and the main factors influencing that are the aperture size and screen thickness [7]. Therefore, we will study the quality of diffraction images of a circular aperture by analogy with the case of a slot aperture [7], calculating the value of the focusing parameter F (37) depending on the radius R and the half-thickness of the screen d. To make it convenient for the comparison of these two cases of diffraction, a rectangular slot and a circular aperture, we use the same values of the remaining parameters of the system (H, h, and refractive indices) in calculations as in [7] ($n = 1.62$, $n_s = 1.46$).
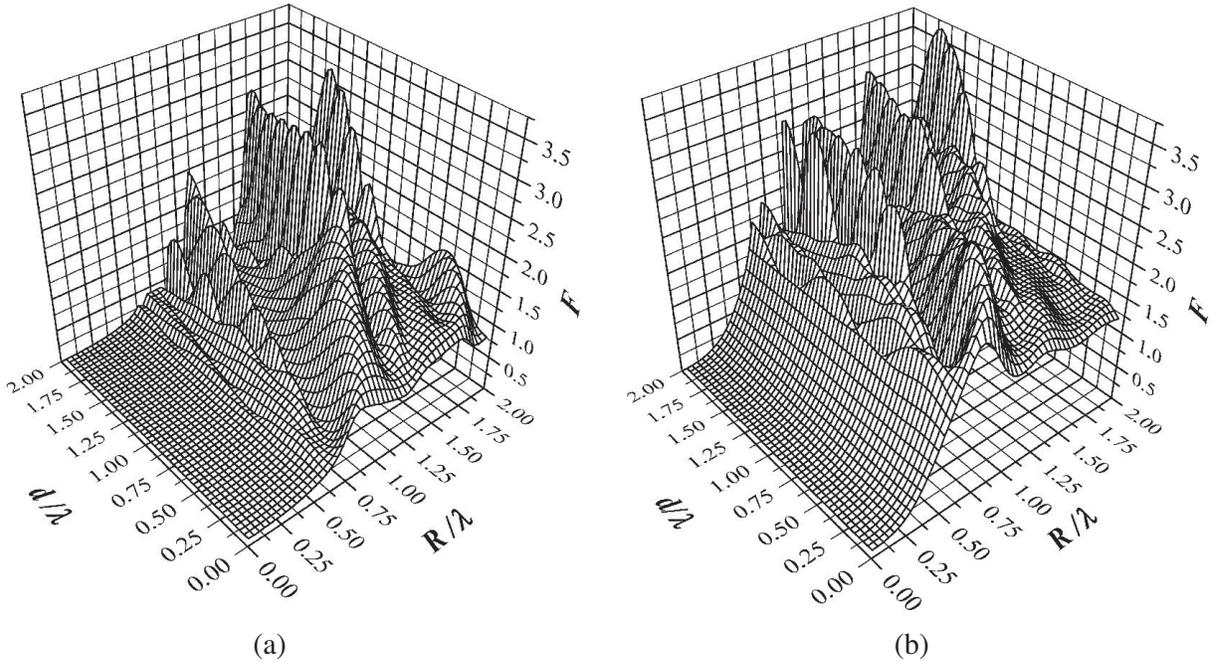

Figure 3. Focusing parameter F (37) of the image of a circular hole as a function of its radius R and half-thickness d of a perfectly conducting screen d at diffraction of a linearly polarized plane wave on a screen with a hole, (a) in empty space without dielectrics and (b) in a dielectric film with the thickness $h = 1.2\lambda$, located at the distance $H = 0.2\lambda$ behind the screen.

Figure 3 shows the results of calculation of the focusing parameter F (37) depending on parameters R and d of the diffraction system, calculated using the above obtained solution of the diffraction problem in the absence and presence of a thin dielectric film on the substrate behind the aperture. The comparison of Figs. 3(a) and 3(b) shows that the positions of the focusing parameter maxima on the plane of parameters R and d inside the dielectric layer are slightly shifted from their position for the case of empty space, and the magnitude of these maxima in the dielectric is slightly greater. Therefore, when a dielectric layer appears in the near zone behind the aperture, one should expect an increase in the intensity of the diffraction image and an improvement in its quality.

Figures 4 and 5 show two examples of the radial distribution of the relative electrical energy density (36) of the total diffraction field in the dielectric layer. Fig. 6(a) shows the results of calculation
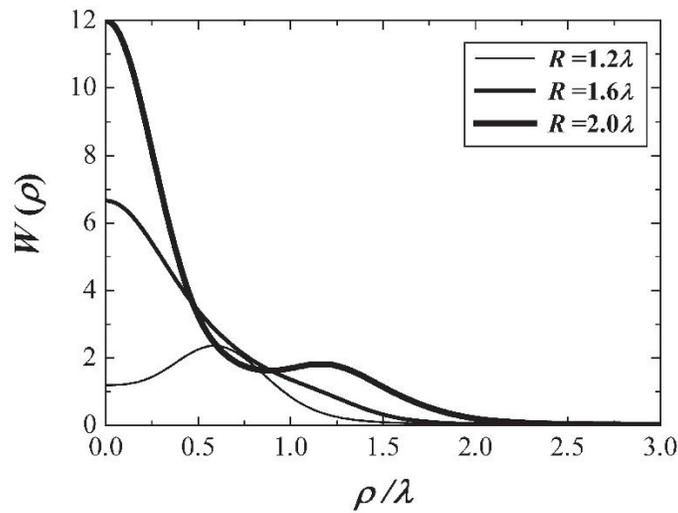
Figure 4. Distribution of the relative energy density of the electric field inside the dielectric film with the thickness of $h = 1.2\lambda$, located at the distance of $H = 0.2\lambda$ from the circular aperture with the radius R in a conducting screen having the half-thickness $d = 1.72\lambda$.
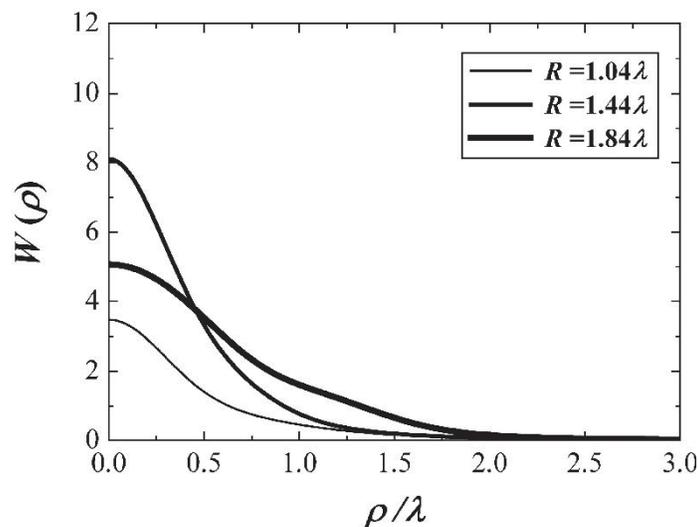
Figure 5. Distribution of the relative energy density of the electric field inside the dielectric film with the thickness of $h = 1.2\lambda$, located at the distance of $H = 0.6\lambda$ from the circular aperture with the radius R in the conducting screen having the half-thickness $d = 1.2\lambda$.
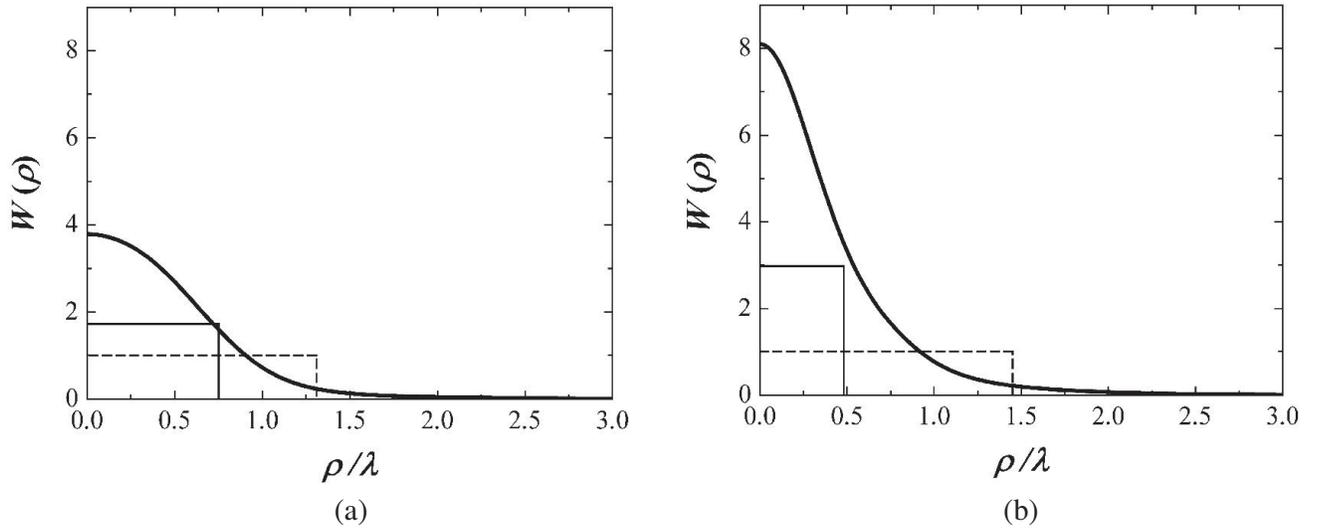
Figure 6. Distribution of the relative energy density of the electric field inside a dielectric film with the thickness of $h = 1.2\lambda$, located at the distance $H = 0.6\lambda$ from a circular aperture of the radius of (a) $R = 1.32\lambda$ and (b) $R = 1.44\lambda$ in a conducting screen having the half-thickness $d = 1.2\lambda$. The thin line shows the ideal image of the aperture in the form of an approximating step of height F, the dotted line reproduces real dimension of the hole.

of the image intensity as a function of the radial coordinate ρ for the same parameters of the diffraction system as in the case of TE polarization diffraction by a slot [7] (here, the role of the half-width of the slot l is played by the aperture radius R). The comparison of Fig. 6(a) with the similar Fig. 4(a) of the work [7] shows that the circular aperture gives a more diffuse image (its intensity drops to zero at a distance of 1.75λ from the center, which is greater than the radius of the hole, while the slot aperture of the same size of 1.2λ provides a narrower image focus area, which is less than one wavelength). However, a circular aperture can give a noticeably greater total intensity of the diffraction spot, and the maxima of its intensity, determined by the maxima of the focusing parameter F, are achieved at other values of the aperture size and screen thickness than for a slot aperture (see Fig. 6(b)).

4. CONCLUSION

Thus, in the near zone of a small circular aperture, excited by a normally incident linearly polarized plane wave, the same anomalous phenomena can occur as for a slot aperture, excited by the TE-polarized wave, whose electric vector is parallel to the slot edges. This is the phenomenon of a sharp increase in the intensity of the total diffraction field and the phenomenon of lensless focusing of this field into a small region smaller than the aperture area. In both cases, these phenomena manifest themselves at aperture sizes of the order of the radiation wavelength and more, but not less, for various thicknesses of the conducting screen. The conditions for the simultaneous existence of these phenomena can be determined by calculation and estimation of the focusing parameter (37), and in both cases these conditions are very sensitive to changes in the size of the aperture (the radius of the circular hole R, the half-width of the slot l) and the thickness of the conducting screen d, but for that and other apertures they do not agree. The effect of increasing the total intensity of the diffraction image can be stronger by several times for a circular aperture, but the effect of lensless focusing is noticeably weaker for that: under similar conditions, with offset from the axis, the image intensity of a circular hole decreases more slowly than the intensity of the diffraction pattern for the slot.

The developed theory, together with the theory of diffraction by slot apertures [7, 10], is quite rigorous, since it does not allow any mathematical approximations and any limitations on aperture and dielectric film dimensions. However, it should be borne in mind that the field of its application,

strictly speaking, is limited by the range of sufficiently long electromagnetic waves, up to the far infrared region, and for the optical and near infrared radiation it should be used with caution. The fact is that for these ranges there are no solid materials that could be considered as perfectly conducting, except for alkali metals, but for ordinary metals, the conductivity value is not high enough to consider them as perfect conductors [8, 21]. Nevertheless, the authors of [5] experimentally corroborated the presence of the phenomenon of lensless focusing for laser radiation with the wavelength of $\lambda = 488$ nm for a slot aperture in a plane aluminum screen. They clearly observed diffracted beam narrowing in 3–4 times less than the slot width at the distance of approximately 3–4 µm from the screen, and, accordingly, corresponding sharp increase in the local intensity of this beam.

ACKNOWLEDGMENT

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