Progress In Electromagnetics Research C, Vol. 133, 1–13, 2023

Nonuniform Structured Waveguides. WKB Approach

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Abstract—The results of the development of an approximate approach for describing structured waveguides, which can be considered as an analogue of the WKB method, are presented. This approach gives possibility to divide the electromagnetic field in structured waveguides with slow varying geometry into forward and backward components and simplify the analysis of the field characteristics, especially the phase distribution. The accuracy of this method was estimated by comparing the solution of the approximate system of equations with the solution of the general system of equations. For this, a special code was written that combines the proposed approach with the more accurate one developed earlier. For the case of fast damping of evanescent waves, a simple solution of the matrix equations is obtained. Based on this approach, the possibility of correcting the phase distribution in a chain of coupled resonators has been studied.

1. INTRODUCTION

Structured waveguides[†] based on coupled resonators play an important role in many applications. Their applications in active devices have several distinctive features. First, transverse magnetic (TM) modes with a large longitudinal component of electric field are used. These modes also have a radial electric field component. Such configuration of the electric field leads to a significant increase in the electric field near some boundary regions. This makes it difficult to describe such distributions analytically by using simple functions. Second, in accelerators and high-frequency electronic devices, the distribution of longitudinal electromagnetic fields (especially their phases) over the structure plays a decisive role. Therefore, we need to have fast and versatile models that can be used for the calculation of electric field distribution with high accuracy. This demand restricts the usage of many approximate approaches that are effective in optics and photonics (see, for example, [1]).

There are powerful programs that can be used to calculate the RF characteristics of structured waveguides, but they are difficult to use for analysis and preliminary design. For these purposes, simpler models are needed. It is desirable that these models not only allow numerical analysis, but also are based on equations whose solutions can be analyzed in simple cases. In many cases, at the first stage of the study, the exact geometry of the waveguide can be replaced with a simpler, but similar one.

Electromagnetic fields in the homogeneously (periodically) structured waveguides can be effectively described by the Floquet-Bloch's theory and electrodynamic approach based on the field expansion in forward and backward waves which constitute a complete orthogonal set of vector functions and have strict physical foundation. In this case, the problem is one-dimensional as the amplitudes of the waves depend only on one coordinate and can be found by solving the ordinary differential equations.

In the general case, when the coupled resonators are different, there are no physical concepts that could simplify the understanding of the electromagnetic process. If the parameters of resonators change slowly, we can expect that the electromagnetic fields will have some features of forward and backward

Received 26 March 2023, Accepted 2 May 2023, Scheduled 8 May 2023
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[†] Waveguides that consist of similar (but not always identical) cells.

waves. In the case of inhomogeneous smooth waveguides, approximate approaches are a powerful tool for studying its properties [2–5]. The development of asymptotic approaches for structured waveguides is at an early stage. A formal method based on a generalization of the theory of coupled modes is proposed [6, 7]. Since the calculation of the modes of periodic waveguides is not a simple task, the matrix approach, which provides a procedure for direct calculation of the approximate distribution of the electric field, seems more attractive [8].

For inhomogeneously structured waveguides, the electromagnetic problem becomes two-dimensional, since the radial distribution of electromagnetic fields varies from one cell to another. As is customary in electrodynamics, using the surface integral equation approach can reduce the dimension of the problem. The method of Coupled Integral Equations (CIE) has been developed to calculate the characteristics of structured waveguides used to create high-frequency filters [9]. Recently, it has been proposed to use it to calculate the characteristics of accelerating structures [10–12]. Since the surfaces in the CIE method are flat, an efficient method for solving the CIE system is to expand the unknown field distributions in an orthonormal basis. In this case, we obtain the matrix difference equation or the system of the matrix difference equations. In the general case, the vectors to find have an infinitive dimension. Usually, we reduce them to some finite dimensions. The dimension of the reduced vectors determines the number of eigenwaves that are involved into consideration. Indeed, let consider the homogeneous structured waveguide. From the matrix difference equation we can get a transfer matrix T that maps V_k to V_{k+1}: $V_{k+1} = TV_k$, where $V_k \in C^n$ is the vector of coefficients of electric field expansion. The transfer matrix must have dimension $(n \times n)$ $(T \in C^{n\times n})$. It has n eigenvalues λ_s and n eigenvectors U_s. If $V_{k_0} = U_s$, then $V_k = \lambda^{k-k_0} U_s$, $k \geq k_0$. In the general case, $V_k = \sum\limits_{s=1}^{n} \lambda_s^{k-k_0} U_s C_s$, where C_s are constant complex numbers. It follows that n eigenvectors (waves) constitute solution. If only the main (or several first) wave propagates, then $(n-1)$ eigenwaves are evanescent, which decay (or increase) along the waveguide.

So, if we want to increase n (increase the accuracy of calculations), we need to be ready to deal with a lot of evanescent solutions. This can be a serious problem because the matrix determinant equals the product of all eigenvalues and will greatly decrease (increase) as n grows. There is also the problem of calculating eigenvalues for large n. The CIE method is based on the procedure of dividing complex volumes into simpler ones. The integral equations are formulated on the surfaces that connect these simple regions. In many cases, electric fields have singularities at the boundaries of these surfaces, and for a correct description of the field distribution, one has to use a lot of basis functions. To avoid the above-mentioned problems with bad matrices, it is desirable to choose surfaces where the fields have no singularities, and we can use a small number of basis functions. In the developed approaches [9], the transverse electrical fields on the surfaces are unknowns. It is desirable to develop an approach in which the longitudinal electric fields are unknown as this component plays the main role in the process of beam-wave interaction.

Moreover, there is a question about the accuracy of approximate models which are widely used for calculating the characteristics of structured waveguides with a sufficiently slow change in their dimensions. This is especially important when studying the beam energy spread in accelerators and its minimum, which can be achieved. Thus, in addition to complex computer models, approximate approaches with a known accuracy and possibility of field characteristic calculations are needed.

In this paper we present results of investigation that solve some of the described above problems.

2. BASIC EQUATIONS

Consider a segment of a cylindrical structured waveguide (circular corrugated waveguide), whose geometry is shown in Figure 1. The right and left ends of segment are connected to semi-infinite circular waveguides. All segment volumes are filled with dielectric $\varepsilon = \varepsilon' + i\varepsilon''$, $\varepsilon'' > 0$. We divide the segment into subregions, each of which is a circular waveguide. Unlike earlier works [9], we divide each volume with large cross-section into two equal subvolumes (in general, they can be different) [11]. Volumes with large cross section will be numbered by the index $k\,(1 \leq k \leq N_R)$, subvolumes — by k_1 and k_2 $(k_1 = k_2 = k,\ b_{k_1} = b_{k_2} = b_k)$. A small cross-sectional volume placed to the left of a large cross-sectional volume with an index k will be numbered by the index $k'\,(1' \leq k' \leq (N_R + 1)')$. We will

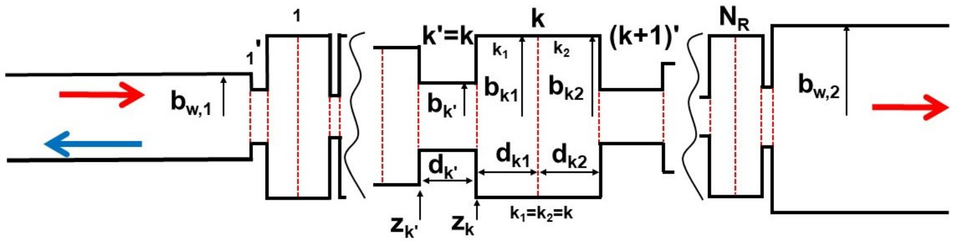


Figure 1. Chain of pieces of cylindrical waveguides that is connected with semi-infinite cylindrical waveguides.

consider only axially symmetric fields with E_z, E_r, H_φ components (TM fields). Time dependence is $\exp(-i\omega t)$.

Longitudinal components of electric field on the new surfaces are represented as

$$E_z^{(k)}\left(r, z = z_k + d_{k_1}\right) = \sum_m Q_m^{(k)} J_0\left(\frac{\lambda_m}{b_k} r\right), \tag{1}$$

where $J_0(\lambda_m) = 0$.

Using in each volume the standard expansion of the field into simple cylindrical waves and equating the fields on the boundary surfaces, we obtain the matrix difference equation [11]

$$T^{(k)} Q^{(k)} = T^{+(k)} Q^{(k+1)} + T^{-(k)} Q^{(k-1)}, \quad k = 2, ..., N_R - 1, \tag{2}$$

with such boundary conditions

$$\begin{aligned} &T^{(Q_1)} Q^{(1)} + T^{(Q_2)} Q^{(2)} = Z^{Q_1}, \\ &T^{\left(Q_{N_R}-1\right)} Q^{(N_R-1)} + T^{\left(Q_{N_R}\right)} Q^{(N_R)} = 0, \end{aligned} \tag{3}$$

where $T^{(k)}$, $T^{+(k)}$, $T^{-(k)}$, $T^{(Q_1)}$, $T^{(Q_2)}$, $T^{(Q_{N_R}-1)}$, $T^{(Q_{N_R})}$ are the matrices (see their definition in [11]), and $Q^{(k)} = (Q_1^{(k)}, Q_2^{(k)}, Q_3^{(k)}, ...)^T$ are vectors of the coefficients of longitudinal field expansion in the cross section inside the resonators. Z^{Q_1} is proportional to the amplitude of the incident wave in the left semi-infinite cylindrical waveguide.

Equation (2) is very similar in structure to the equations of coupled resonators. The coupled resonators equations are approximate and relate the amplitudes of only the main eigenmodes (or several different modes) in three neighboring resonators. The matrix difference equation is exact and takes into account the coupling of all possible resonator eigenmodes (for infinite size of the vector $Q^{(k)}$). Thus, the matrix equation (2) can be considered as an improved model of coupled resonators.

We will use the possibilities of the CASCIE code [12] for computing $T^{(k)}$, $T^{+(k)}$, $T^{-(k)}$ matrices and to compare the approximate results with the results obtained from solving Equation (2).

Before proceeding with the transformation of Equation (2), we need to characterize the properties of matrices $T^{(k)}$, $T^{+(k)}$, $T^{-(k)}$. The matrices $T^{(k)}$ do not have any peculiarities (the determinants are not small and the condition numbers are not large), and we can rewrite Equation (2) as

$$Q^{(k)} = \tilde{T}^{+(k)} Q^{(k+1)} + \tilde{T}^{-(k)} Q^{(k-1)} \tag{4}$$

where $\tilde{T}^{\pm(k)} = T^{(k)-1} T^{\pm(k)}$.

One of the important properties of the matrices $\tilde{T}^{\pm(k)}$ is its non-defectivity. Equation (4) describes the electric field at the given frequency ω. It is known that for a homogeneous periodic waveguide ($\tilde{T}^{+(k)} = \tilde{T}^{-(k)} = \tilde{T}$) the electric field is the sum of an infinite number of eigenwaves, which can differ greatly from each other. There are the propagation waves and evanescent fields. Floquet multiplies λ_s of the evanescent fields take both very small and very large values. If we designate the eigenvalues of the matrix $\tilde{T}$ as θ_s, then it follows from (4) that the Floquet multipliers λ_s of the electromagnetic field are determined by the characteristic equation

$$\lambda_s^2 - \theta_s^{-1} \lambda_s + 1 = 0 \tag{5}$$

from which we get

$$\lambda_s = 0.5\theta_s^{-1} \mp \sqrt{0.25\theta_s^{-2} - 1} \tag{6}$$

To get a set of increasing and decreasing values of λ_s, the values of θ_s must decrease as the index s increases. The same is true for inhomogeneous waveguides. Therefore, the determinants of matrices $\tilde{T}^{\pm(k)}$ will decrease if we increase their sizes and can become very small. It can lead to large errors when performing matrix operations. All of the above is confirmed by the results of calculations which are presented in Table 1. These data refer to a homogeneous disk-loaded waveguide with $b_{k'} = 1.381\,\text{cm}$, $b_k = 4.16186\,\text{cm}$, $d_k = 2.9147\,\text{cm}$, $d_{k'} = 0.5842\,\text{cm}$, phase shift per cell $\varphi = 2\pi/3$, $f = 2.856\,\text{GHz}$.

Table 1. Determinant and eigen values of the matrix $\tilde{T}$.

Size of matrix $\tilde{T}$	2×2	3×3	4×4	5×5	6×6
Determinant	−1.97E−05	−4.41E−13	−1.36E−23	−5.97E−37	−3.96E−53
θ_1	−1.01E+00	−1.00E+00	−1.00E+00	−1.00E+00	−1.00E+00
θ_2	1.95E−05	3.98E−05	4.67E−05	4.78E−05	4.79E−05
θ_3		1.11E−08	4.44E−08	7.59E−08	8.91E−08
θ_4			6.55E−12	4.22E−11	1.10E−10
θ_5				3.89E−15	3.49E−14
θ_6					3.06E−18

3. TRANSFORMATION OF THE BASIC EQUATIONS

As Equation (4) is of the second order, we can represent the solution of this matrix difference equation as the sum of two new vectors [13]

$$Q^{(k)} = Q^{(k,1)} + Q^{(k,2)}. \tag{7}$$

By introducing new unknowns $Q^{(k,i)}$ instead of the one $Q^{(k)}$, we can impose an additional condition. We write this condition in the form

$$Q^{(k+1)} = M^{(k,1)}Q^{(k,1)} + M^{(k,2)}Q^{(k,2)}, \tag{8}$$

where $M^{(k,i)}$ are arbitrary nondefecting matrices. Using (7) and (8) we can rewrite (4) as

$$\begin{aligned}
&\tilde{T}^{+(k+1)}\left(M^{(k+1,1)} - M^{(k+1,2)}\right)Q^{(k+1,1)} \\
&= -\left(\tilde{T}^{-(k+1)} + \left(\tilde{T}^{+(k+1)}M^{(k+1,2)} - I\right)M^{(k,1)}\right)Q^{(k,1)} - \left(\tilde{T}^{-(k+1)} + \left(\tilde{T}^{+(k+1)}M^{(k+1,2)} - I\right)M^{(k+1,2)}\right. \\
&\quad \left. + \left(\tilde{T}^{+(k+1)}M^{(k+1,2)} - I\right)\left(M^{(k,2)} - M^{(k+1,2)}\right)\right)Q^{(k,2)},
\end{aligned} \tag{9}$$

$$\begin{aligned}
&\tilde{T}^{+(k+1)}\left(M^{(k+1,1)} - M^{(k+1,2)}\right)Q^{(k+1,2)} \\
&= \left(\tilde{T}^{-(k+1)} + \left(\tilde{T}^{+(k+1)}M^{(k+1,1)} - I\right)M^{(k,2)}\right)Q^{(k,2)} + \left(\tilde{T}^{-(k+1)} + \left(\tilde{T}^{+(k+1)}M^{(k+1,1)} - I\right)M^{(k+1,1)}\right. \\
&\quad \left. + \left(\tilde{T}^{+(k+1)}M^{(k+1,1)} - I\right)\left(M^{(k,1)} - M^{(k+1,1)}\right)\right)Q^{(k,1)}.
\end{aligned} \tag{10}$$

We choose the matrices $M^{(k,i)}$ $(i = 1, 2)$ so that they satisfy the quadratic matrix equations

$$\tilde{T}^{-(k+1)} + \left(\tilde{T}^{+(k+1)}M^{(k+1,i)} - I\right)M^{(k+1,i)} = 0. \tag{11}$$

Then the system (9) is transformed into

$$\begin{aligned}&\tilde{T}^{+(k+1)}\left(M^{(k+1,1)}-M^{(k+1,2)}\right)Q^{(k+1,1)}\\=&\left\{\tilde{T}^{-(k+1)}M^{(k+1,2)-1}\left(M^{(k+1,1)}-M^{(k+1,2)}\right)+\tilde{T}^{-(k+1)}M^{(k+1,2)-1}\left(M^{(k,1)}-M^{(k+1,1)}\right)\right\}Q^{(k,1)}\\&+\tilde{T}^{-(k+1)}M^{(k+1,2)-1}\left(M^{(k,2)}-M^{(k+1,2)}\right)Q^{(k,2)},\end{aligned}\tag{12}$$

$$\begin{aligned}&\tilde{T}^{+(k+1)}\left(M^{(k+1,1)}-M^{(k+1,2)}\right)Q^{(k+1,2)}\\=&-\left\{\tilde{T}^{-(k+1)}M^{(k+1,1)-1}\left(M^{(k+1,2)}-M^{(k+1,1)}\right)+\tilde{T}^{-(k+1)}M^{(k+1,1)-1}\left(M^{(k,2)}-M^{(k+1,2)}\right)\right\}Q^{(k,2)}\\&-\tilde{T}^{-(k+1)}M^{(k+1,1)-1}\left(M^{(k,1)}-M^{(k+1,1)}\right)Q^{(k,1)}.\end{aligned}\tag{13}$$

If elements of matrices $M^{(k,i)}$ vary sufficiently slowly with k, then the differences $|M_{s,m}^{(k+1,i)}-M_{s,m}^{(k,i)}|$ are small, and we can get approximate equations by neglecting the second terms in the right side of Equations (12) and (13) [13].

The matrices $M^{(k,i)}$ are the solutions of the same quadratic matrix equations

$$\tilde{T}^{-(k)}+\left(\tilde{T}^{+(k)}M^{(k,i)}-I\right)M^{(k,i)}=0.\tag{14}$$

For homogeneous waveguide $\tilde{T}^{-(k)}=\tilde{T}^{+(k)}$, if the waveguide structure varies sufficiently slowly with k, then the differences between entries of the matrices $\tilde{T}^{-(k)}$ and $\tilde{T}^{+(k)}$ are small.

The matrices $\tilde{T}^{-(k)}$ and $\tilde{T}^{+(k)}$ are non-defective, and their eigenvalue expansion can be written as

$$\tilde{T}^{+(k)}=U^{+(k)}\Theta^{+(k)}U^{+(k)-1},\tag{15}$$

$$\tilde{T}^{-(k)}=U^{-(k)}\Theta^{-(k)}U^{-(k)-1}=\left(U^{+(k)}+\Delta U^{-(k)}\right)\left(\Theta^{+(k)}+\delta\Theta^{-(k)}\right)\left(U^{+(k)}+\Delta U^{-(k)}\right)^{-1}.\tag{16}$$

Since the matrices $\Delta U^{-(k)}$ and $\delta\Theta^{-(k)}$ are small, we can neglect the quadratic terms

$$\tilde{T}^{-(k)}=U^{-(k)}\Theta^{-(k)}U^{-(k)-1}=\tilde{T}^{+(k)}+\Delta_1\tilde{T}^{-(k)}+\Delta_2\tilde{T}^{-(k)},\tag{17}$$

where

$$\Delta_1\tilde{T}^{-(k)}=U^{+(k)}\delta\Theta^{-(k)}U^{+(k)-1},\tag{18}$$

$$\Delta_2\tilde{T}^{-(k)}=\Delta U^{-(k)}\Theta^{+(k)}U^{+(k)-1}-U^{+(k)}\Theta^{+(k)}U^{+(k)-2}\Delta U^{-(k)}.\tag{19}$$

It can be shown that $\|\Delta_2\tilde{T}^{-(k)}\|\ll\|\Delta_1\tilde{T}^{-(k)}\|$[‡], and it is natural to make the assumption that we can neglect $\Delta_2\tilde{T}^{-(k)}$, then

$$\tilde{T}^{-(k)}\approx\tilde{T}^{+(k)}+\Delta_1\tilde{T}^{-(k)}=U^{+(k)}\Theta^{-(k)}U^{+(k)-1}.\tag{20}$$

With this approximation we can find the solutions of the quadratic matrix equations (14). Taking into account the decompositions (15) and (20), we can find that

$$M^{(k,i)}=U^{+(k)}\Lambda^{(k,i)}U^{+(k)-1},\tag{21}$$

where $i=1,2$, $\Lambda^{(k,i)}=\mathrm{diag}(\lambda_1^{(k,i)},\lambda_2^{(k,i)},\ldots)$. From (14) we get characteristic equations for $\lambda_s^{(k,i)}$

$$\theta_s^{+(k)}\lambda_s^{(k,i)2}-\lambda_s^{(k,i)}+\theta_s^{-(k)}=0,\tag{22}$$

whose solutions are

$$\lambda_s^{(k,i)}=\frac{1}{2\theta_s^{+(k)}}\mp\sqrt{\left(\frac{1}{2\theta_s^{+(k)}}\right)^2-\frac{\theta_s^{-(k)}}{\theta_s^{+(k)}}}.\tag{23}$$

‡ We will use the Frobenius norm $\|M\|_F=\sqrt{\sum\limits_{i,j}M_{i,j}^2}$.

We can also transform Equations (12) and (13)

$$Q^{(k+1,1)} = M_{Q_1}^{r(k)} Q^{(k,1)} = \left\{ M^{(k+1,1)} + \tilde{M}^{(k+1,1)} \left(M^{(k,1)} - M^{(k+1,1)} \right) \right\} Q^{(k,1)}, \tag{24}$$

$$Q^{(k+1,2)} = M_{Q_2}^{r(k)} Q^{(k,2)} = \left\{ M^{(k+1,2)} - \tilde{M}^{(k+1,2)} \left(M^{(k,2)} - M^{(k+1,2)} \right) \right\} Q^{(k,2)}, \tag{25}$$

where $\tilde{M}^{(k,i)} = U^{+(k)} \tilde{\Lambda}^{(k,i)} U^{+(k)-1}$, $\tilde{\Lambda}^{(k,i)} = \mathrm{diag}\left(\tilde{\lambda}_1^{(k,i)}, \tilde{\lambda}_2^{(k,i)}, ... \right)$,

$$\tilde{\lambda}_s^{(i)} = \lambda_s^{(k,i)} \left(\lambda_s^{(k,1)} - \lambda_s^{(k,2)} \right)^{-1}. \tag{26}$$

From (26) it follows that Equations (24) and (25) are not valid for $\lambda_s^{(k,1)} \approx \lambda_s^{(k,2)}$ that occur at the edges of passbands. This case requires special consideration.

We will call $Q^{(k,1)}$ the forward solution ($|\lambda_s^{(k,1)}| \to 0$, $s \to +\infty$) and $Q^{(k,2)}$ as the backward ($|\lambda_s^{(k,2)}| \to 0$, $s \to -\infty$).

Equation (25) gives growing solutions when moving in the positive direction of the waveguide axis; therefore, it is necessary to calculate the vectors by moving in the negative direction. For this, we need to find a matrix inverse to $M_{Q_2}^{r(k)}$

$$Q^{(k,2)} = M_{Q_2}^{r(k)-1} Q^{(k+1,2)} = M_{Q_2}^{l(k)} Q^{(k+1,2)}. \tag{27}$$

Matrix $M_{Q_2}^{r(k)}$ contains the difference $(M^{(k,2)} - M^{(k+1,2)})$, and its inverse cannot be found directly by using the properties of eigenvalue decomposition. As $(M^{(k,2)} - M^{(k+1,2)})$ is small, we can find an approximate decomposition for $M^{(k,2)}$. We use the same procedure which we used to get the solutions of the quadratic matrix equations (14). This procedure gives

$$M_{Q_2}^{l(k)} = U^{+(k+1)} \Lambda^{l(k)} U^{+(k+1)-1}, \tag{28}$$

where $\Lambda^{l(k)} = \mathrm{diag}\left(\lambda_1^{l(k)}, \lambda_2^{l(k)}, ... \right)$, $\lambda_s^{l(k)} = \left(\lambda_s^{(k+1,1)} - \lambda_s^{(k+1,2)} \right) \left[\lambda_s^{(k+1,2)} \left(\lambda_s^{(k+1,1)} - \lambda_s^{(k,2)} \right) \right]^{-1}$.

Finally, our Wentzel-Kramers-Brillouin (WKB) system of equations take the form

$$Q^{(k+1,1)} = M_{Q_1}^{r(k)} Q^{(k,1)}, \tag{29}$$

$$Q^{(k,2)} = M_{Q_2}^{l(k)} Q^{(k+1,2)}. \tag{30}$$

Equations (29) and (30) have such solutions

$$\begin{aligned} Q^{(k+1,1)} &= M_{Q_1}^{r(k)} \times ... \times M_{Q_1}^{r(N_C)} Q^{(N_C,1)} = \prod_{k'=N_C}^{k} M_{Q_1}^{r(k')} Q^{(N_C,1)} \\ Q^{(k,2)} &= M_{Q_2}^{l(k)} \times ... \times M_{Q_2}^{l(N_R-N_C)} Q^{(N_R-N_C+1,2)} = \prod_{k=N_R-N_C}^{k} M_{Q_2}^{l(k')} Q^{(N_R-N_C+1,2)}, \end{aligned} \tag{31}$$

where $\prod\limits_{k'=N_R-N_C}^{k} X_{k'} = X_k \times ... \times X_{N_R-N_C}$. Decomposition (7)–(8) and the solutions (31) were used for $k = N_C, ..., N_R - N_C + 1$ as the approximate method is not applicable to the ending cells. We used $N_C = 5$. Using solutions (31), the system of linear equations (2), (3) (size $(N_Z \times N_R) \times (N_Z \times N_R)$) that describe the chain of N_R resonators ($Q \in C^{N_Z}$) can be transformed into a new system with much smaller size $(N_Z \times 2N_C) \times (N_Z \times 2N_C)$ as $N_C \ll N_R$.

4. ACCURACY ESTIMATION

We will investigate the characteristics of a smooth transition between two disk loaded waveguides with parameters: $b_{k',I} = 1.3810\,\text{cm}$, $b_{k,I} = 4.1618\,\text{cm}$, $\beta_{g,I} = v_{g,I}/c = 0.02$, $b_{k',II} = 1.0200\,\text{cm}$, $b_{k,II} = 4.0785\,\text{cm}$, $\beta_{g,II} = 0.0062$, $d_{k,I} = d_{k,II} = 2.9147\,\text{cm}$, $d_{k',I} = d_{k',II} = 0.5842\,\text{cm}$, $D_k = d_{k,I} + d_{k',I} = \lambda_0/3$, $\omega_0 = 2\pi 2.856\,\text{GHz}$ and the phase shift per cell $\varphi_I = \varphi_{II} = 2\pi/3$[§] The aperture and resonator radii change according the formulas (see [12])

$$b_{k'} = 0.6745 + \sqrt{-0.0506 + 27.4745\beta_{g,k}}, \tag{32}$$

$$b_k = 0.1576 b_{k'}^2 - 0.1476 b_{k'} + 4.0652, \tag{33}$$

where

$$\beta_{g,k} = \frac{\beta_{g,I} + \beta_{g,II}}{2} + \frac{\beta_{g,I} - \beta_{g,II}}{2} \frac{\arctan\{\alpha(k - k_0)\}}{\arctan\{\alpha(3 - k_0)\}}, \tag{34}$$

and $k = 3, \ldots, N_R - 2$, $N_R = 201$, $k_0 = 101$. Formulas (32) and (33) define the sizes of homogeneous disk-loaded waveguide which has a phase shift $\varphi = 2\pi/3$ and a group velocity $\beta_{g,k}$ [12]. Dependences of the geometric dimensions of three transitions with $\alpha = 0.03$, $\alpha = 0.1$, and $\alpha = 0.5$ (denoted as 1, 2 and 3) are shown in Figure 2.

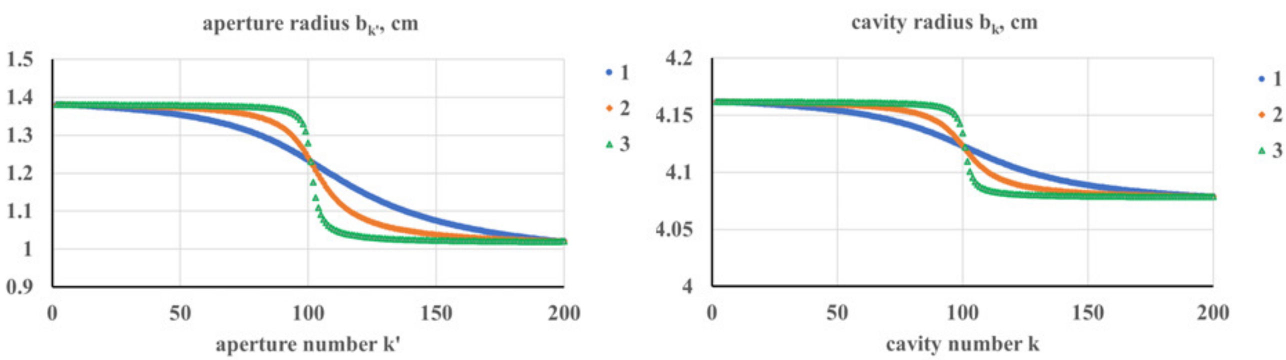


Figure 2. Dependences of aperture radius $b_{k'}$ and resonator radius b_k on the cavity number for three transitions: 1 — $\alpha = 0.03$, 2 — $\alpha = 0.1$, 3 — $\alpha = 0.5$.

The distributions of the longitudinal component of the electric field for the three transitions, calculated with using the CASCIE code, are presented in Figure 3. It can be seen that for a sharp transition ($\alpha = 0.5$), a noticeable reflected wave (reflection coefficient $R \approx 0.1$) is generated in the region of the largest gradient.

The accuracy of the considered approximate method can be estimated by comparing the solution of the approximate system of equations, which uses expressions (31), with the solution of general system (2). Results of such a comparison are shown in Figure 4. For the homogeneous waveguide there is no difference between the solutions of the approximate system and system (2) (an error is less than 10^{-8}). It can be seen that for small gradients of geometric dimensions, the approximate approach is a good method, giving small differences from the general solution.

The above results were obtained under the assumption that the chain under consideration is connected with cylindrical waveguides through tuned couplers at its ends. If the right end of the chain is shortened, then a reflected wave should appear, whose amplitude is equal to the amplitude of the forward wave. The calculation results show that Equation (30) correctly describes the backward solution for $\alpha \leq 0.1$.

The results of all calculations show that the assumptions made do not give permanent differences between the solutions of the approximate system of Equations (31) and the solutions of the general system (2). These differences tend to zero as α decreases. This confirms that our approach gives correct asymptotic results.

§ These parameters correspond to the initial and final cells of the Stanford Linear Accelerator Center (SLAC) 3 m section [14].

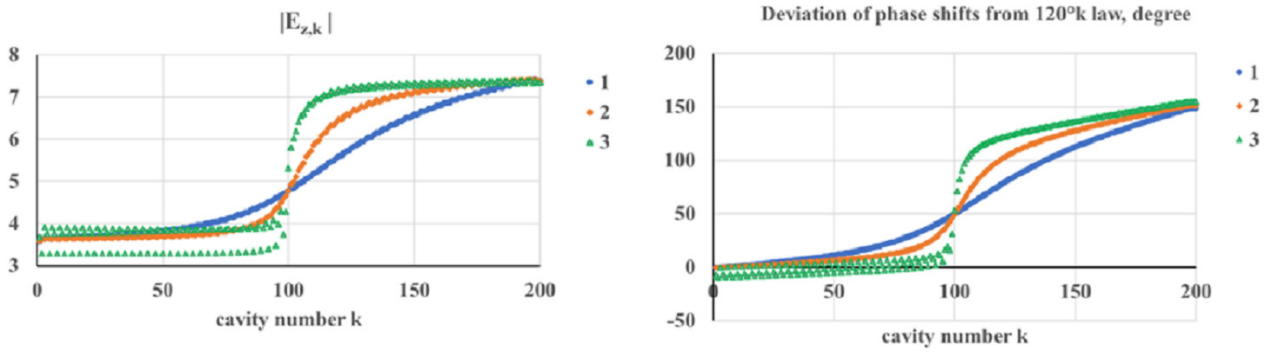


Figure 3. Amplitude and phase of the longitudinal electric field at the centers of resonators for three transitions: 1 — $\alpha = 0.03$, 2 — $\alpha = 0.1$, 3 — $\alpha = 0.5$. Reflection coefficients for these geometries are: 1 — $R = 1.1\text{E}{-03}$, 2 — $R = 3.7\text{E}{-03}$, 3 — $R = 9.9\text{E}{-02}$.

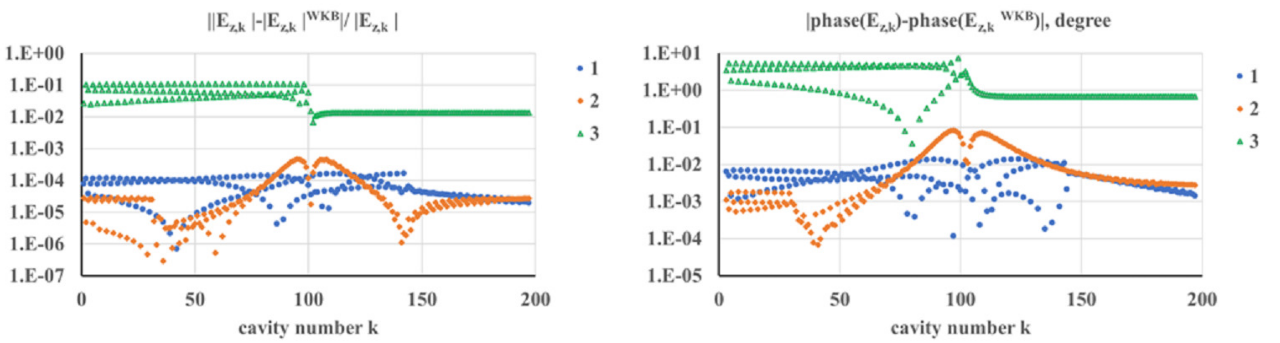


Figure 4. Difference between the solutions of Equations (2) (CASCIE) and the solutions of the approximate system of equations, which uses the expressions (31), for three transitions: 1 — $\alpha = 0.03$, 2 — $\alpha = 0.1$, 3 — $\alpha = 0.5$.

5. APPROXIMATE SOLUTION OF THE BASIC EQUATIONS

Let us find out the role of evanescent modes in creating the field distribution in an inhomogeneous waveguide. First of all, consider the homogeneous waveguide. In the passband the forward solution is described by Equation (29) with constant matrix $M_{Q_1}^{r(k)} = M^{(1,1)} = U^{+(1)}\Lambda^{(1,1)}U^{+(1)-1}$, in which eigenvalues $\Lambda^{(1,1)} = \text{diag}(\lambda_1^{(1,1)}, \lambda_2^{(1,1)}, \ldots)$ and eigenvectors are determined by the matrix $\tilde{T}^{+(k)}$. If the initial vector $Q^{(1,1)}$ equals the sum of eigenvectors $U_s^{+(1)}$: $Q^{(1,1)} = \sum\limits_s q_s U_s^{+(1)}$, then $Q^{(k,1)} = \sum\limits_s \lambda_s^{(1,1)k-1} q_s U_s^{+(1)}$, $k \geq 2$. If $\left|\lambda_s^{(1,1)}\right| \ll 1$ $(s \geq 2)$, then all evanescent modes decay rapidly, and only a propagating wave will determine the structure of the field. The situation is more complicated for an inhomogeneous waveguide. Difference equation (29) contains slightly different matrices $M_{Q_1}^{r(k)} = U^{r(k)}\Lambda^{r(k)}U^{r(k)-1}$

$$Q^{(k+1,1)} = M_{Q_1}^{r(k)} Q^{(k,1)}. \tag{35}$$

As $M_{Q_1}^{r(k)}$ are nondefective, eigenvectors $U_s^{r(k)}$ can be considered as a basis, and we can represent vector $Q^{(k,1)}$ as $Q^{(k,1)} = \sum\limits_s q_s^{(k)} U_s^{r(k)}$. Then from (41) we get $Q^{(k+1,1)} = \sum\limits_s \lambda_s^{r(k)} q_s^{(k)} U_s^{r(k)}$. Changing the basis $U_s^{r(k)} = \sum_{s'} u_{s,s'}^{(k+1)} U_{s'}^{r(k+1)}$ gives

$$Q^{(k+1,1)} = \sum_s \lambda_s^{r(k)} q_s^{(k)} U_s^{r(k)} = \sum_{s'} U_{s'}^{r(k+1)} \sum_s \lambda_s^{r(k)} q_s^{(k)} u_{s,s'}^{(k+1)} = \sum_s q_s^{(k+1)} U_s^{r(k+1)}, \tag{36}$$

where $q_s^{(k+1)} = \sum_{s'} \lambda_{s'}^{r(k)} q_{s'}^{(k)} u_{s',s}^{(k+1)}$.

Therefore, the solution of the matrix difference equation of the first order (35) can be written as the sum of the "local" vectors, which define the radial field distribution. Indeed, for the electric field we have

$$E_z^{(k)}(r, z = z_k + d_{k_1}) = \sum_p Q_p^{(k)} J_0\left(\frac{\lambda_p}{b_k} r\right) = \sum_s q_s^{(k)} \sum_p U_{s,p}^{r(k)} J_0\left(\frac{\lambda_p}{b_k} r\right). \tag{37}$$

In many cases $\lambda_s^{r(k)}$ for evanescent modes have small values. For example, from Table 1 and formula (6) we find $\lambda_2^r = 4,79\text{E}-05$, $\lambda_3^r = 8.94\text{E}-08$. This circumstance gives possibility to simplify the described above procedure. We can suppose that $\lambda_s^{r(k)} = 0$ for $s > 1$ and get simple expressions for vectors $Q^{(k,1)}$ ($|\, u_{1,s}^{(k)} \,| \ll 1$ for $s \geq 2$)

$$Q^{(k,1)} = q_1^{(k)} U_1^{r(k)} + \sum_{s\geq 2} \lambda_1^{r(k-1)} q_1^{(k-1)} u_{1,s}^{(k)} U_s^{r(k)} \approx q_1^{(k)} U_1^{r(k)} \tag{38}$$

and the longitudinal electrical field

$$E_z^{(k)}(r, z = z_k + d_{k_1}) = q_1^{(k)} \sum_p U_{1,p}^{r(k)} J_0\left(\frac{\lambda_p}{b_k} r\right). \tag{39}$$

The solution to the first order difference equation

$$q_1^{(k+1)} = \lambda_1^{r(k)} u_{1,1}^{(k+1)} q_1^{(k)}, \tag{40}$$

has a simple form

$$q_1^{(k+1)} = q_1^{(1)} \prod_{s=1}^{k} \lambda_1^{r(s)} u_{1,1}^{(s+1)}. \tag{41}$$

The factor $q_1^{(1)}$ is determined by the initial vector $Q^{(1,1)}$ ($Q^{(1,1)} = \sum_s q_s^{(1)} U_{1,s}^{r(1)}$).

For a homogeneous waveguide $\lambda_1^{r(s)} = \lambda_1^r$, $u_{1,1}^{(s+1)} = 1$ and the longitudinal electrical field takes the expected form

$$E_z^{(k)}(r, z = z_k + d_{k_1}) = q_1^{(1)} (\lambda_1^r)^k \sum_p U_{1,p}^r J_0\left(\frac{\lambda_p}{b} r\right). \tag{42}$$

It follows from (39) that we have separated the "radial" variation from the "longitudinal" one. Indeed, $q_1^{(k)}$ is a solution to difference equation (40) and depends on the previous values of $q_1^{(s)}$, $s < k$, while the vector $U_1^{r(k)}$ is determined by the local matrix $M_{Q_1}^{r(k)}$ (the first eigenvector of this matrix). For inhomogeneity (34) the "radial" factors $\sum_{p=1}^{4} U_{1,p}^{r(k)}$ are real and give changes in electric field up to 10% (see Figure 5).

Calculations show that for structured waveguides with an inhomogeneity (34) and small gradients ($\alpha \leq 0.3$) the approximate solution (38) gives good amplitude and phase accuracy (see Figure 6). Since matrices $M_{Q_1}^{r(k)}$ are a slow function of the cell number k, the factors $u_{1,1}^{(k)}$ ($k \geq 2$) have absolute values close to 1, but setting them equal to 1 gives a few percent error.

It should be noted that the simplifications used above do not reduce the requirements for the accuracy of calculating the matrices $M_{Q_1}^{r(k)}$ and their eigen characteristics. Analysis shows that for the

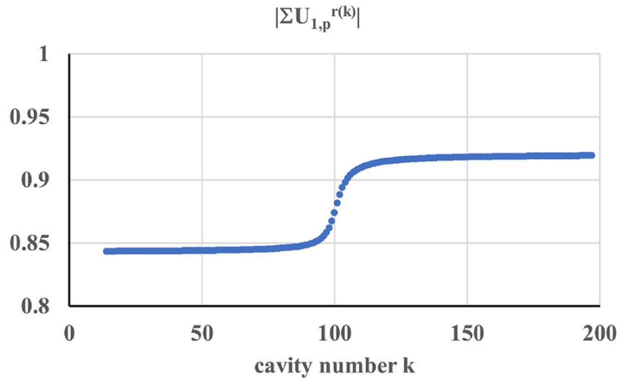


Figure 5. "Radial" factors $\sum_{p=1}^{4} U_{1,p}^{r(k)}$ at the centers of resonators ($r = 0$) for inhomogeneity (34) with $\alpha = 0.1$.

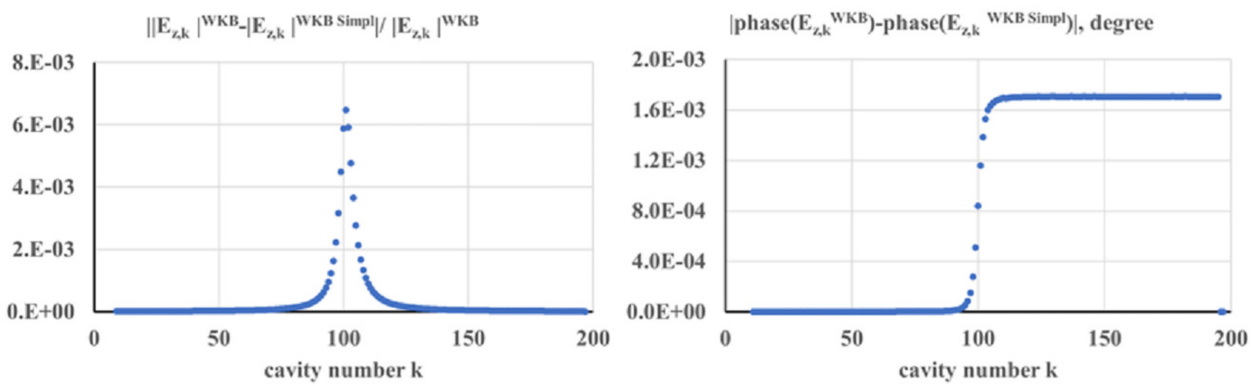


Figure 6. Difference between the solution of the approximate system of equations, which uses the expressions (31), $|E_{z,k}|^{\text{WKB}}$ and the solution (39) $|E_{z,k}|^{\text{WKBSimpl}}$ at the centers of resonators for inhomogeneity (40) with $\alpha = 0.3$.

waveguides under consideration the necessary accuracy is achieved for the sizes of matrices $M_{Q_1}^{r(k)}$ $N_z \geq 4$. It means that the presence of evanescent modes plays an important role in calculating $\lambda_1^{r(k)}$, $u_{1,1}^{k}$, $U_1^{r(k)}$.

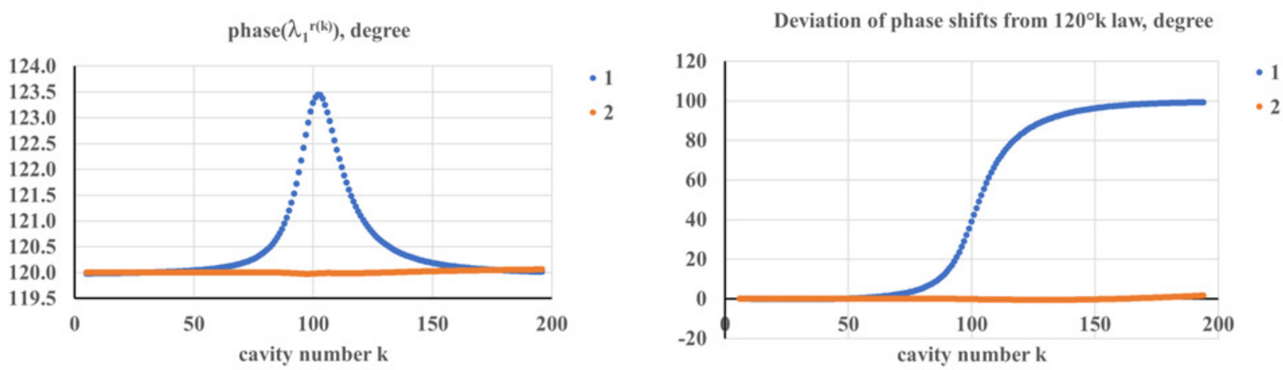
It follows from the above results that the phase distribution of the longitudinal component of the electric field in considered conditions is determined only by phases of the first eigen values $\lambda_1^{r(k)}$ of the matrices $M_{Q_1}^{r(k)}$. It is important to note that the module of $\lambda_1^{r(k)}$ is not equal to one in our lossless, but inhomogeneous case. Its value is determined by the inhomogeneity gradient.

6. PHASE DISTRIBUTION CORRECTION

It was said above that the distribution of longitudinal electromagnetic fields over the structure plays a decisive role in the accelerators. To ensure synchronism between an electromagnetic wave and particles, the phase of the longitudinal electromagnetic field must change according to the law determined by the speed of the particles. For ultra-relativistic particles, the phase shift along one cell $\Delta\varphi^{(k)}$ must equal $\frac{\omega D_k}{c} = 2\pi\frac{D_k}{\lambda}$, where $\omega = 2\pi f$, $f = c/\lambda$ — the working frequency, D_k — the cell length, c — the velocity of light. If $D_k = \lambda/3$, then $\Delta\varphi^{(k)} = 2\pi/3$ and the ideal phase distribution has the form $\varphi^{(k)} = 2\pi/3 \times k$. Phase shift between cells in real structure can differ from $\Delta\varphi^{(k)}$ on the small value, but the value $\sum_{i}^{k}\Delta\varphi^{(i)}$ has to be close to the ideal phase distribution. This circumstance leads to the need to tune the section after manufacturing (see, for example, [14–16]).

In previous sections, we considered an inhomogeneous waveguide, each cell of which has the radius equals the radius of a cell of a homogeneous waveguide with $\Delta\varphi^{(k)} = 2\pi/3$ (33). Since the aperture size changes, the phase distribution differs from $2\pi/3 \times k$ law. It can be seen from Figure 3 that the full phase deviation is significant, and the acceleration process will change to the deceleration process. Therefore, correction of the phase distribution is necessary. Typically, such a correction is carried out by successively changing the natural frequencies of the cells (usually by indenting the cell walls). So, at the intermediate stage of tuning there can be inner reflections in the waveguide. Approximate equation (35), in contrast to the exact system of Equations (2), does not correctly describe such geometrical configurations. It was shown that its solutions become unstable for small, but sharp changes in the parameters [10]. Nevertheless, the approximate equation (35) can confirm (or disprove) the possibility of realizing the desired phase distributions.

To correct the phase distribution (Figure 7(a)), we can try to change the cell radii determined by formula (33) so that the phases of $\lambda_1^{r(k)}$ take equal values $2\pi/3$. We can add to the right hand of (33) a new function. Since this function must be equal to zero for a homogeneous waveguide, it must be the function of the gradient. We used correction function $b_k = b_{0,k} + \Delta b_k = b_{0,k} + \eta_1(b_{0,k} - b_{0,k-1}) + \eta_2(b_{0,k-1} - 2b_{0,k} + b_{0,k+1})$, where $b_{0,k}$ are determined by (33). Such correction significantly improves the phase distribution. The maximum deviation from the $2\pi/3 \times k$ law is 1.8° (see Figure 7(b)).



(a) (b)

Figure 7. Phase distributions without (a) and with (b) correction for $\alpha = 0.1$.

Better result is obtained by using a correction for the case of a linear change in the group velocity with the cell number (constant gradient). This case plays an important role in accelerator technique as on the base of such distributions the section with constant accelerating field can be created. The most known section of this type is the SLAC 3-meter section [14]. Formula (34) with $\alpha \to 0$ gives values that are very close to the values of group velocity in the SLAC section. With the help of the correction $b_k = b_{0,k} + \eta_1(b_{0,k+1} - b_{0,k})$, a maximum deviation of 0.45° from the $2\pi/3 \times k$ law can be obtained (losses in the structure are also taken into account by introducing the imaginary part of the permittivity $\varepsilon = 1 + 0.000073i$).

The results obtained show that for considered inhomogeneities there are "smooth" geometric distributions of cell parameters in which the phase distributions differ from those required by small values. Since the procedure for tuning the waveguide based on a successive change in the eigen frequencies of the cells can lead to a "biperiodic" frequency distribution [10], but there are "smooth" desired distributions, it is necessary to implement a multi-stage tuning procedure to avoid biperiodicity [15, 16].

7. CONCLUSIONS

The results of the development of an approximate approach, which can be considered as an analogue of the WKB method, are presented. This approach gives possibility to divide the electromagnetic field into forward and backward components and simplify the analysis of the field characteristics. The accuracy of this method was estimated by comparing the solution of the approximate system of equations with the solution of the general system of equations. For this, a special code was written that combines the proposed approach with the more accurate one developed earlier. The main attractive feature of the proposed approximate approach is the possibility of studying the characteristic of electromagnetic waves in structured waveguides without taking into account the influence of the end cells (couplers), which are frequency sensitive. We can also use the resulting equations and its solutions to study the propagation of waves with different frequencies (within the propagation band), which is important for taking into account certain characteristics and transient effects. Based on the proposed approach, a system of equations can be constructed that describes the excitation of a waveguide by an external current, taking into account the structure of the waveguide.

ACKNOWLEDGMENT

This work was partly supported by the Stanford Global Studies' Scholar Rescue Fund, in partnership with the Institute for International Education's Scholar Rescue Fund program.

The author would like to thank David Reis and Valery Dolgashev for their support and interest in the work.

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