Progress In Electromagnetics Research B, Vol. 101, 85–100, 2023

Quasi-Stationary Approximation of Dynamic Inductive Wireless Power Transfer

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Abstract—Dynamic Inductive Wireless Power Transfer (DIWPT), used for charging and powering electric vehicles (EVs), has been presented lately as a solution for increasing the distance range of electric vehicles and reducing the utilization of heavy and bulky battery systems. In most DIWPT designs, the voltage induced by the movement of the receiving coil over a time-varying magnetic field is neglected and never quantified. In this work, a simplified phasor expression for the total induced voltage on a coil that is moving in a sinusoidal time-variant magnetic vector field is developed. If no rotation is observed in the coil, a 90° out of phase voltage component proportional to the speed of the coil is added to the induced voltage that would be calculated if the coil was stationary. The phase of this voltage component is delayed or advanced with respect to the stationary induced voltage, according to whether the coil is moving into or out of a region of higher magnetic flux. Then, under some assumptions on the geometry of inductive coil configurations, it is possible to estimate the minimum induction frequency for which the quasi-stationary approximation can be considered. The resulting frequency value for a representative geometry is calculated, indicating that, for automotive applications, the relative error in the induced voltage is actually negligible, except in the vicinity of the points of zero-crossing in the magnetic flux, where the absolute value of the induced voltage is low anyway.

1. INTRODUCTION

Inductive Wireless Power Transfer (IWPT) has been under extensive research for the past years. Used for several types of applications, IWPT has proven to be an interesting solution for charging and powering electric vehicles (EVs) [1–3]. By using stationary charging in parking spaces, stop signs, red lights, energy can be transferred to the vehicle without the need for electrical contact or user action, thus increasing the range of EVs and reducing the necessity of large and bulky energy storage units. However, newer applications consider the charging of the vehicle on the move, which is called Dynamic Inductive Wireless Power Transfer (DIWPT) [4–6]. In DIWPT applications, the power is transferred between primary and secondary coils while they are in relative movement to each other [7–10]. For the analysis of both stationary and dynamic IWPT applications, complete theoretical support was provided in 1861 by Maxwell's Theory of Electromagnetism and equations [11], as popularized in Heaviside's vector notation in 1894 [12], which consistently and precisely model all purely electromagnetic macroscopic terrestrial phenomena so far observed [13]. In DIWPT works, the voltage induced on the secondary coil due to the movement of the vehicle is usually neglected, only being considered the stationary coupling of the two coils for each instant of the vehicle movement [4, 5, 14]. Although this dynamically induced voltage is expected to be negligible at the high frequencies and the relatively low speeds involved in vehicular applications, this component of the total induced voltage is not known to have been assessed and quantified. In this paper, the validity of the quasi-stationary approximation to this dynamically

Received 20 February 2023, Accepted 29 May 2023, Scheduled 13 July 2023

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induced voltage is studied and assessed. Simple and easy-to-use expressions that quantify the error of using this approximation are developed. These expressions are expected to be of practical assistance in the design of more general DIWPT systems.

2. VOLTAGE INDUCED IN A MOVING COIL BY A MAGNETIC FIELD VECTOR FIELD B

The voltage induced in a receiving coil is the time derivative of the total magnetic flux crossing that coil. It will depend not only on the variation of the magnetic flux due to the time-varying generated magnetic field B, but also on the variation of the flux caused by the displacement of that coil in space. Therefore, in general, the voltage induced in a moving coil, $V^D(t)$, will be potentially different from the voltage $V^0(t)$ induced on the same coil, by the same magnetic field B, at the same position in space, if the coil was stationary (velocity $v = 0$) with respect to all the sources creating the magnetic field B, as illustrated in Figure 1.

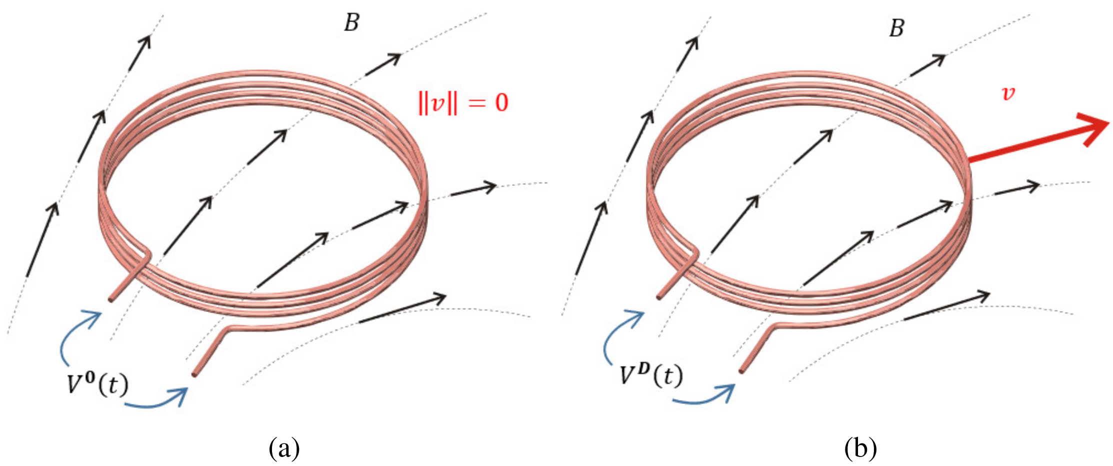

Figure 1. Voltage induced on a coil subjected to the same magnetic field in stationary and dynamic conditions.

Consider that the magnetic field will be uniform and invariant in space, in the reference frame of the coil, thus resulting in equal voltages $V^0(t)$ and $V^D(t)$, when all entities generating the vector field B keep their relative positions to the coil, that is, when these sources move (and rotate) solidarily to the coil in its reference frame. Intuitively, the rate of change of the magnetic field is expected to be approximately invariant in the coil space, causing these two induced voltages to be approximately equal, either when the sources of magnetic field are very far from the coil or, equivalently, when the dimensions of the coil are very small, with respect to the coil speed, and the coil is keeping the angular orientation relative to the (average) local magnetic field, i.e., it is not rotating.

However, as it will be shown in the next section, for most electric mobility application cases, the commonly high range of induction frequencies used in inductive WPT and the relatively low speed of vehicles with respect to the wireless transfer ground structures make it possible to approximate $V^D(t)$ by $V^0(t)$. For this reason, in most cases, a DIWPT system powering an EV can be designed or analyzed as a quasi-stationary IWPT, that is, considering that the power transfer to a vehicle can be calculated and averaged as if the vehicle's pick-up coil in the EV was at rest at each point along all of its path.

3. DYNAMICALLY INDUCED VOLTAGE

Let us assume that a primary coil consists of a wire loop that can be modeled as a closed continuous curve K that is continuously differentiable at all except a finite number of points — which is always a reasonable assumption for real relatively thin wire loops that are composed of a finite number of "smooth" curved wire segments. If K is displaced in a region G of the three-dimensional Euclidean space where there is a time-varying magnetic flux vector field B, then the integral of the time-varying electric vector field E on the closed path K is well defined, and according to the Maxwell-Faraday equation, the dynamically induced voltage V_K^D on the wire loop can be expressed as:

$$V_K^D(t) = \oint_K E \cdot dK = -\frac{d\Phi_S}{dt} = -\frac{d}{dt}\int_S B \cdot dS, \tag{1}$$

where S is any orientable surface delimited by the curve K, such that $K = \partial S$. This expression was experimentally found truth and holds independently of the sources of the magnetic field B, which can be permanent magnets, other loops with variable current along the path of the wire, either fixed or moving in G, or an arbitrary combination of them. A geometrical view of this modeling of the wire loop moving in the magnetic field is shown in Figure 2, where two different states of the wire loop, at times t and $t + \Delta t$, are represented.

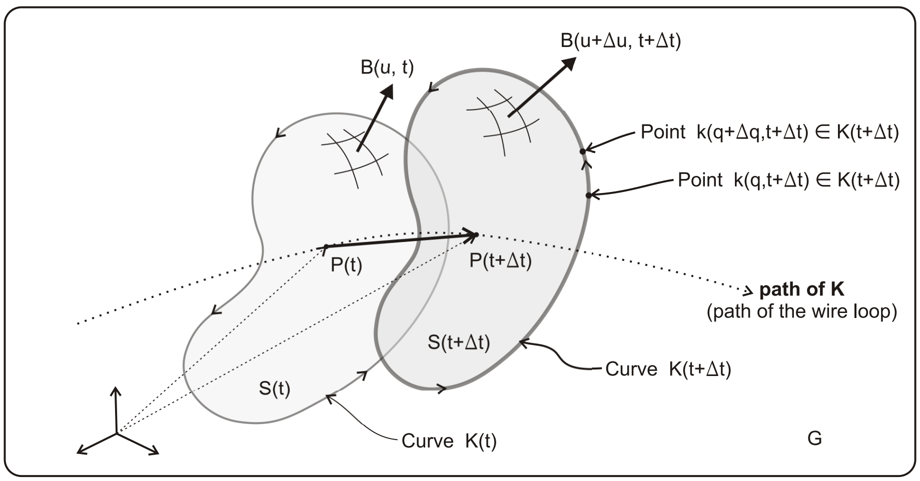

Figure 2. Modeling of the displacement of a wire loop in a magnetic flux vector field.

At each instant of time t, the wire loop has its centroid at $P(t) \in G$ and defines the position of curve $K(t)$, for which a surface $S(t)$ is considered. The total magnetic flux, due to B, crossing the oriented surface S, given by Φ_S, is also a function of the time and point $P(t)$ where the primary coil is located, and does not depend on the actual surface S, but rather on the contour curve K of S. The time derivative of the flux of a vector field through a moving element of area, as expressed in (1), can be calculated by applying a two-dimensional variant of the Leibniz Integral Rule theorem [15], which is also presented in a more general form in [16], resulting in the expression:

$$V_K^D(t) = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint_K (v \times B) \cdot dK - \int_S (\nabla \cdot B) v \cdot dS, \tag{2}$$

However, according to Maxwell-Gauss equation, $\nabla \cdot B = 0$, then (2) becomes:

$$V_K^D(t) = -\int_S \frac{\partial B}{\partial t} \cdot dS + \oint_K (v \times B) \cdot dK, \tag{3}$$

The first term of the right side of Equation (3) can be promptly identified as the voltage V_K^0 induced on the wire loop as if B would depend only on t, that is, as if the wire loop $K(t)$ were permanently "at rest" (stationary) at position $P(t)$:

$$V_K^0(t) = -\int_S \frac{\partial B}{\partial t} \cdot dS, \tag{4}$$

so, we can rewrite (3) as:

$$V_K^D(t) = V_K^0(t) + \oint_{K(t)} (v \times B) \cdot dK, \tag{5}$$

The second term on the right, in expressions (3) and (5), that we will name $V_K^v(t)$, is a component of the induced voltage that simultaneously depends both on the velocity field v of the wire loop and on the magnetic field B at time t along the curve $K(t)$, so we write:

$$V_K^D(t) = V_K^0(t) + V_K^v(t) \tag{6}$$

This new component $V_K^v(t)$ is not observed when the wire loop is stationary in G, and it does not directly depend on the time-varying pattern of the magnetic field B, but rather on the instantaneous spatial distribution of B at time t, as if it were a constant field in time with the incidental value B_t. Noticeably, the assumptions made on the curve $K(t)$ are general enough not to restrict the validity of (6) if the wire loop is accelerating, rotating, bending, expanding, or contracting, or even changing its shape with time, as long as these changes are "smooth", and there is always a surface comprised by $K(t)$ that remains orientable. In order to better explain this movement-dependent term of the induced voltage and how it will affect DIWPT analysis and design, a few special cases are considered in the following subsections.

3.1. Dynamically Induced Voltage Due to DC Components of the Magnetic Field

Considering that B can be divided in a time-invariant component B_{DC} and a time-variant component B_{AC} $(u \in G)$:

$$B(u,t) = B_{DC}(u) + B_{AC}(u,t), \tag{7}$$

B_{DC} will have no effect in the "stationary" term $V_K^0(t)$, as $\partial B/\partial t = \partial B_{AC}/\partial t$. However, it will cause the "dynamically" induced component $V_K^v(t)$ of $V_K^D(t)$, in (6), to split in two potentially non-zero terms, $V_K^{v,B_{DC}}(t)$ and $V_K^{v,B_{AC}}(t)$, which will depend on the combination of the velocity field v and the magnetic field B itself, as derived from (5):

$$V_K^D(t) = V_K^0(t) + \oint_{K(t)} (v \times B_{DC}) \cdot dK + \oint_{K(t)} (v \times B_{AC}) \cdot dK, \tag{8}$$

or equivalently:

$$V_K^D(t) = V_K^0(t) + V_K^{v,B_{DC}}(t) + V_K^{v,B_{AC}}(t), \tag{9}$$

where

$$V_K^{v,B_{DC}}(t) = \oint_{K(t)} (v \times B_{DC}) \cdot dK \tag{10}$$

and

$$V_K^{v,B_{AC}}(t) = \oint_{K(t)} (v \times B_{AC}) \cdot dK. \tag{11}$$

The term B_{DC} can be regarded as the component of the vector field B that could result, for instance, from a mix of stationary permanent magnets and primary loops of constant current placed along the path of the wire loop given by $K(t)$. The term B_{AC}, on the other hand, is the time-varying component, and could be, for example, the net result of moving magnets and time-varying currents in primary loops placed in the neighborhood of the moving wire loop under consideration, which, in this case, plays the role of a secondary coil. If the vector field v is such that the cross product $v \times B_{DC}$ is constant along $K(t)$, then the term of the induced voltage on the wire loop due to B_{DC} is zero, because the integral of the oriented differential elements dK over curve K is also zero:

$$V_K^{v,B_{DC}}(t) = \oint_{K(t)} (v \times B_{DC}) \cdot dK = (v \times B_{DC}) \left(\oint_{K(t)} dK \right) = 0. \tag{12}$$

One particular case, in which $v \times B_{DC}$ is constant, resulting in $V_K(v, B_{DC})(t) = 0$ is when both v and B_{DC} are spatially invariant, corresponding to the situation of a rigid wire loop travelling with constant speed on a hypothetically uniform magnetic field, a configuration where no induced voltage is experimentally observed on the wire loop. It is worthwhile mentioning that even for a rigid body, the velocity vector field v is not necessarily constant along K, when the centroid $P(t)$ of $K(t)$ is moving with constant velocity v_K: if the wire loop defined by $K(t)$ is spinning, the velocity vector v on the points of $K(t)$ will be dispersed around the average velocity v_K of the wire loop, what will potentially cause a non-zero induced voltage, even while moving in a constant, uniform magnetic field.

3.2. Dynamically Induced Voltage on a Spinning Wire Loop Moving in Space

Let us consider that the wire loop given by $K(t)$ is spinning with angular frequency $\omega \neq 0$ around its centroid at position $P(t)$, while it moves (translation) in G with velocity v_K. If at some initial time t_0 the centroid is at position $P_0 = P(t_0)$, then the centroid position is given by:

$$P(t) = P_0 + \int_{t_0}^{t} v_K dt \tag{13}$$

Recognizing $K(t)$ as the image of a joint function k of the time t and of an independent parameter $q \in R$, with k continuous and differentiable in all points where the corresponding point in $K(t)$ is also differentiable, it is possible to write that:

$$K(t) = \left\{ k(q,t) \in R^3 | q \in Q = [q_1, q_2] \subset R \right\} \tag{14}$$

where, at any time t, $k(q_1,t) = k(q_2,t)$, for $K(t)$ is a closed path. Then, the point $k(q,t)$ can be expressed by its relative position r with respect to P, such that:

$$k(q,t) = P(t) + r(q,t), \tag{15}$$

Using (13), expression (15) becomes:

$$k(q,t) = P_0 + \int_{t_0}^{t} v_K dt + r(q,t). \tag{16}$$

The velocity v is given by:

$$v = \frac{dk(q,t)}{dt} = v_K + \frac{\partial r(q,t)}{\partial t} + \frac{\partial r(q,t)}{\partial q} \cdot \frac{\partial q}{\partial t} \tag{17}$$

Because the parametric variable q, which generates $K(t)$, is independent of time t, $\partial q/\partial t = 0$, then (17) is simply:

$$v = v_K + \frac{\partial r(q,t)}{\partial t} \tag{18}$$

In addition, because the loop is spinning about $P(t)$ with angular velocity vector ω, $\partial r(q,t)/\partial t$ is the radial velocity with respect to $P(t)$, which can be expressed by the cross-product of ω and $r(q,t)$:

$$\frac{\partial r(q,t)}{\partial t} = \omega \times r(q,t) \tag{19}$$

Then substituting (19) in (17), the velocity field on the point $k(q,t)$ of $K(t)$:

$$v(q,t) = v_0 + \omega \times r(q,t) \tag{20}$$

Because $K(t)$ and the relative position vectors $r(q,t)$ are spinning together and measured in the same reference frame of vector fields B and v, it is possible to abstract the specific time t and the specific point of $K(t)$ associated with parameter $q \in Q$, and simply denote the velocity vector field as:

$$v = v_K + \omega \times r \tag{21}$$

Finally, substituting (21) in (5), an expression for computing the dynamically induced voltage on a loop that spins around (its centroid) $P(t)$ that is moving with velocity v_K on an arbitrary time-varying vector field B is given by:

$$V_K^D(t) = V_K^0(t) + \oint_{K(t)} (v_K \times B) \cdot dK + \oint_{K(t)} ((\omega \times r) \times B) \cdot dK, \tag{22}$$

Now noticing that:

$$V_K^{v_K,B}(t) = \oint_{K(t)} (v_K \times B) \cdot dK \tag{23}$$

and naming:

$$V_K^{\omega,B}(t) = \oint_{K(t)} ((\omega \times r) \times B) \cdot dK \tag{24}$$

it is possible to write:

$$V_K^D(t) = V_K^0(t) + V_K^{v_K,B}(t) + V_K^{\omega,B}(t) \tag{25}$$

The dynamically induced voltage on the wire loop at instant t_1, $V_K^D(t_1)$, is then the sum of three terms:

(i) The first term, $V_K^0(t_1)$, is the induced voltage on the wire loop as if it were stationary at the position $P(t_1)$, with no angular rotation, under the effect of the time-varying magnetic field B.

(ii) The second term, $V_K^{v_K,B}(t_1)$, is the voltage that would be induced on the wire loop if it were passing with constant velocity v_K at the point $P(t_1)$, without spinning, but with the same spatial orientation it would have at time t_1 due to the spinning movement given by ω, travelling on a time-invariant magnetic field $B(t_1)$.

(iii) The third term, $V_K^{\omega,B}(t_1)$, corresponds to the voltage that would be induced on the same wire loop if it were spinning with angular velocity ω fixed at position $P(t_1)$, while being surrounded by a time-invariant magnetic field $B(t_1)$.

This physical interpretation of (25) requires some abstraction, since all voltages V_K^0, $V_K^{v_K,B}(t_1)$, and $V_K^{\omega,B}(t_1)$ are functions of the time themselves, but the expression provides a tool for the computation of the dynamically induced voltage $V_K^D(t)$.

It is still possible to consider (7) again and write that:

$$V_K^{v_K,B}(t) = V_K^{v_K,B_{DC}}(t) + V_K^{v_K,B_{AC}}(t) \tag{26}$$

and

$$V_K^{\omega,B}(t) = V_K^{\omega,B_{DC}}(t) + V_K^{\omega,B_{AC}}(t) \tag{27}$$

further breaking (25) into:

$$V_K^D(t) = V_K^0(t) + V_K^{v_K,B_{DC}}(t) + V_K^{v_K,B_{AC}}(t) + V_K^{\omega,B_{DC}}(t) + V_K^{\omega,B_{AC}}(t) \tag{28}$$

The voltage component associated with the effect of the spinning movement in the time-invariant magnetic field B_{DC}, $V_K^{\omega,B_{DC}}(t)$, for instance, is the prevalent term in rotative generators, mostly used for grid and off-grid higher power level generation, while $V_K^{v_K,B_{DC}}(t)$ is the prevalent term in linear generators used in some energy harvesting devices, as in [17, 18]. The first term of the dynamically induced voltage, $V_K^0(t)$, on the other hand, is the only relevant term in stationary inductive WPT applications, in which the relative position of the secondary coil is fixed with respect to the primary coils that produces the magnetic field B.

3.3. Case of a Non-Spinning Coil Not Subjected to an External DC Magnetic Field

In a typical DIWPT configuration for electric mobility applications, the secondary coil, as given by the curve $K(t)$, can be assumed to be a rigid body that is neither spinning nor making turns of significant magnitude along its path, and that implies an equivalent angular velocity of the secondary coil that can be neglected ($\omega \cong 0$), and thus, the term $V_K^{\omega,B}(t)$ will be approximately zero in (25).

In the same way, considering that no permanent magnets or primary coils carrying constant currents are normally, by design, placed along the path of an EV (and the secondary coil), it can be assumed that no external DC magnetic field component will act on $K(t)$, so $V_K^{v_K,B_{DC}}(t) = 0$.

Under the above conditions, the dynamically induced voltage on secondary coil on-board of an EV, $V_K^D(t)$, can be expressed as:

$$V_K^D(t) \cong V_K^0(t) + V_K^{v_K,B_{AC}}(t). \tag{29}$$

If more than one primary coil is simultaneously inducing voltage on a secondary coil, as the case of multi-transmitter WPT, the calculation of the total induced voltage should consider the superposition of the magnetic fluxes on the secondary coil, resulting from the different primary coils that are simultaneously active. So, the total dynamically induced V_K^D voltage would be the sum of as many parcels in the form of the right side of (29) as primary coils. Then, if Q primary coils are present, (29) would be replaced by:

$$V_K^D(t) \cong \sum_i^Q V_{K_i}^0(t) + \sum_i^Q V_{K_i}^{v_{K_i} B_{AC_i}}(t) \tag{30}$$

Similar to the single primary coil case, the first summation in the second term of (30) can be still recognized as the total induced voltage that would be obtained in the stationary condition, and the second summation is the total voltage deviation that is due to the relative movement of the secondary coil to each of the Q primary coils.

Normally, all primary coils are stationary, so the velocities v_{K_i} are all equal to v_K, the velocity of the secondary coil to ground reference. But the relative position of the secondary coil to each of the primary coils is different. Assuming that the primary coils are synchronously excited at the same frequency, depending on the phase excitation and relative position of the primary coils, these errors can add up constructively or destructively, increasing or reducing the dynamically induced voltage term. On the other hand, if the quasi-stationary approximation can be demonstrated to be valid in the frequency range used in automotive applications for a single primary coil, this result will also be valid if a small number N of primary coils are inducing voltage on the secondary coil, because the maximum voltage error in this approximation will be the algebraic sum of N negligible individual voltage errors. This is the case in most DIWPT automotive applications, where multiple primary coils are presumed, but only the few primary coils that are closest to the single secondary coil are used as active transmitters to that secondary coil. In this way, in the next sections, the error in the quasi-stationary approximation for the dynamically induced voltage will just be analyzed for the case of a single primary coil.

4. DYNAMICALLY INDUCED VOLTAGE UNDER SINUSOIDAL SOURCES

The dynamically induced voltage given by expression (22), or even that in the simplified case of non-spinning coils not subjected to DC magnetic fields, as in (29), may be difficult to calculate in a general case, due to the interaction of the velocity vector with the magnetic field vector in the integrands of these expressions. However, by taking into account some extra considerations, commonly found in real-world electric mobility applications of DIWPT, the dynamically induced voltage calculation can be simplified. In fact, in many circumstances it will be shown that it is even possible to neglect the terms of the induction voltage due to the relative movement of the coils, allowing the approximation of $V_K^D(t)$ by $V_K^0(t)$ only, as if the moving coil were at rest in all points of its trajectory. So, in the case of a non-spinning coil that moves at constant velocity v_K and is not subjected to a DC magnetic field component, let us further assume that the time-varying magnetic vector field B_{AC} has a periodic sinusoidal dependence on time t, with frequency f_0, being expressed as:

$$B_{AC}(p,t) = B_G(p)\cos(2\pi f_0 t + \theta_0), \quad p \in G \tag{31}$$

where the vector field B_G is not time-dependent. Without loss of generality, let us also assume that the phase $\theta_0 = 0$; then, expanding (4):

$$V_K^0(t) = -\int_S \frac{\partial B_{AC}}{\partial t} \cdot dS = 2\pi f_0 \sin(2\pi f_0 t) \int_S B_G \cdot dS, \tag{32}$$

and expanding (5):

$$\begin{aligned} V_K^{v_K, B_{AC}}(t) &= \oint_{K(t)} (v_K \times [B_G \cos(2\pi f_0 t)]) \cdot dK = \cos(2\pi f_0 t) \oint_{K(t)} (v_K \times B_G) \cdot dK \\ &= ||v_K|| \cos(2\pi f_0 t) \oint_{K(t)} (1_{v_K} \times B_G) \cdot dK \end{aligned} \tag{33}$$

where 1_{v_K} is the unity vector with the same orientation of the velocity v_K.

By applying Stoke's theorem to (33), it follows that:

$$V_K^{v_K,B_{AC}}(t) = \cos(2\pi f_0 t) \int_S \nabla \times (v_K \times B_G) \cdot dS, \tag{34}$$

where the integrand $\nabla \times (v_K \times B_G)$ can be expanded as:

$$\nabla \times (v_K \times B_G) = v_K(\nabla \cdot B_G) - B_G(\nabla \cdot v_K) + (B_G \cdot \nabla)v_K - (v_K \cdot \nabla)B_G \tag{35}$$

In (35), the first term in null because $\nabla \cdot B_G$ is null according to the Maxwell-Gauss equation for magnetism, and $\nabla \cdot v_K$ is null by hypothesis (the velocity v_K is constant in space). It can also be noticed that for vector fields x and y, $(x \cdot \nabla)y$ is the Jacobian of y, J_y, applied to x; so:

$$(v_K \cdot \nabla)B_G = J_{B_G}(v_K) = \|v_K\| J_{B_G}(1_{v_K}) \tag{36}$$

$$(B_G \cdot \nabla)v_K = J_{v_K}(B_G) \tag{37}$$

Again, because v_K is constant, $J_{v_K}(B_G) = 0$, then (35) becomes:

$$V_K^{v_K,B_{AC}}(t) = -\|v_K\| \cos(2\pi f_0 t) \int_S J_{B_G}(1_{v_K}) \cdot dS, \tag{38}$$

where J_{B_G} is the Jacobian of the magnetic field B_G acting on the coil. Expression (38) shows a more intuitive notion of $V_K^{v_K,B_{AC}}(t)$ than (33), meaning that this component of the induced voltage V_K^D is proportional to the integral on S of the variation of the vector B_G in the direction of v_K, as well as proportional to the speed $\|v_K\|$ itself.

In terms of phase delays, looking at (31) and (32), it can be noticed that the phase of V_K^0 is delayed from that of the magnetic field generated by the primary coil, B_{AC}, by an angle of $\pi/2$, while $V_K^{v_K,B_{AC}}(t)$ is either in phase or in counter-phase with B_{AC}, respectively depending whether the variation of the magnetic flux on S is negative or positive when K is incrementally displaced in the direction of v_K.

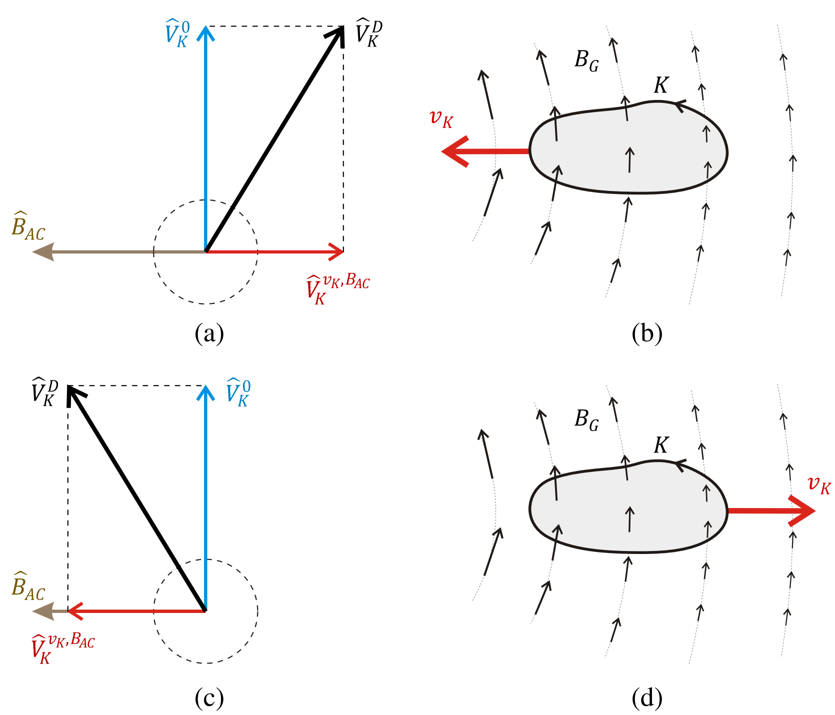

Figure 3. Phasor components of the dynamically induced voltage on a secondary coil (a) for the case when its movement is enhancing the magnetic flux on the induced coil (b). Phasor components of the dynamically induced voltage on a secondary coil (c) for the case when its movement is decreasing the magnetic flux on the induced coil (c).

Figure 3(a) shows the complex plots of the associated phasors $\hat{B}_{AC}, \hat{V}_K^0, \hat{V}_K^{v_K,B_{AC}}$ and $\hat{V}_K^D$ at the frequency f_0, for the case in which the secondary coil is in a position of the space where the tendency for the variation of the magnetic flux on S is positive in the direction of the velocity v_K (i.e., the magnetic flux that affects the coil is increasing), as illustrated in Figure 3(b). Conversely, if the secondary coil is moving in a position where the tendency for the variation of the magnetic flux on S is negative in the direction of the velocity v_K (i.e., the magnetic flux that affects the coil is decreasing), as illustrated in Figure 3(d), then $\hat{V}_K^{v_K,B_{AC}}$ will be in phase with $\hat{B}_{AC}$, as shown in Figure 3(c). In both cases, if there is a non-zero speed $\|v_K\|$, it is expected that

$$\left|\hat{V}_K^D\right| \geq \left|\hat{V}_K^0\right|. \tag{39}$$

5. APPROXIMATING DIWPT AS QUASI-STATIONARY IWPT

The magnitude ratio δ of the root-mean-square (RMS) values of $\hat{V}_K^{v_K,B_{AC}}$ (38) and $V_K^0(t)$ (32) can then be calculated as:

$$\delta = \frac{\left[V_K^{v_K,B_{AC}}\right]_{\text{RMS}}}{\left[V_K^0\right]_{\text{RMS}}} = \frac{\|v_K\|}{2\pi f_0} \frac{\left|\int_S J_{B_G}(1_{v_K}) \cdot dS\right|}{\left|\int_S B_G \cdot dS\right|}, \tag{40}$$

or, alternatively, according to (33) as:

$$\delta = \frac{\|v_K\|}{2\pi f_0} \frac{\left|\oint_{K(t)} (1_{v_K} \times B_G) \cdot dS\right|}{\left|\int_S B_G \cdot dS\right|}, \tag{41}$$

Since the ratio of the integrals, in either (40) or (41), does not depend on $\|v_K\|$ or f_0, if the magnetic net flux of B_G though the coil is not null, it follows from (41) that it will be always possible to make δ smaller than any given value by also making the ratio $\|v_K\|/f_0$ sufficiently small. That is, in a typical DIWPT system design, with the pick-up coil fixed on board of an EV, the higher the EV speed $\|v_K\|$ is, the higher the induction frequency f_0 has also to be in order to make it possible to approximate $V_K^D(t)$ by $V_K^0(t)$, treating the design as a quasi-stationary IWPT case. In the following paragraphs, a few more assumptions on the geometry of the problem are made to derive a criterion for checking the applicability of this approximation.

5.1. Simplified Criterion for Approximating DIWPT as a Quasi-Stationary WPT

In order to assess whether the ratio $\|v_K\|/f_0$ is low enough as to permit the approximation of $V_K^D(t)$ by $V_K^0(t)$, or, conversely, to establish the minimum frequency f_0 for which this approximation is valid for a given maximum speed $v_{\max}$, a relationship among $\|v_K\|$, f_0, and the geometry of the coil is initially found for an specific configuration.

So, let us first assume that the magnetic field $B_{AC}(p,t)$, acting on the secondary coil given by the curve K, is due to a single primary coil generating in the neighborhood of K, with both coils in free space, and that any point of K is at a distance $\|r\|$ from the primary coil much greater than the dimensions of the primary coil itself, as illustrated in Figure 4.

Then, the time-invariant magnetic vector field B_G of (31) can be approximated as if it were produced by an equivalent magnetic dipole, m, which can be expressed as [19]:

$$B_G(r) \cong B_m(r) = \frac{\mu_0}{4\pi}\left[\frac{3(m \cdot r)r}{\|r\|^5} - \frac{m}{\|r\|^3}\right] \tag{42}$$

which can be further developed as:

$$B_G(r) \cong B_m(r) = \frac{\mu_0\|m\|}{4\pi\|r\|^3}\left[3\cos\theta \cdot 1_r - 1_m\right] \tag{43}$$

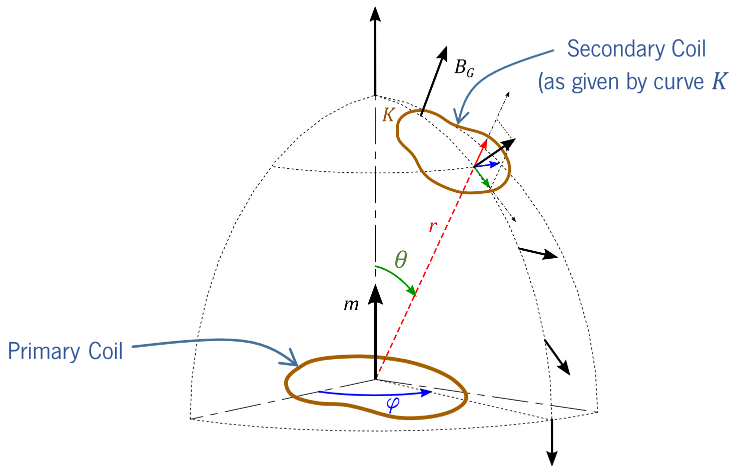

Figure 4. Special case when the primary coil can be modeled by approximation as a magnetic dipole.

where μ_0 is the uniform magnetic permeability of the free space; m is the magnetic dipole vector; 1_r is the unity vector in direction of the oriented distance r to the secondary coil, shown in red in Figure 4; and 1_m is the unity vector in the direction of the magnetic dipole m.

For our analysis, working in spherical coordinates (r, θ, φ), it is more convenient to reexpress 1_m in terms of 1_r and 1_θ, the unity vector in direction of the polar angular displacement θ, shown in green in Figure 4:

$$1_m = \cos\theta \cdot 1_r - \sin\theta \cdot 1_\theta \tag{44}$$

Then, by substituting (44) in (43) it results in that:

$$B_G(r) \cong \frac{\mu_0 \|m\|}{4\pi \|r\|^3} \left[\cos\theta \cdot 1_r - \sin\theta \cdot 1_\theta\right] \tag{45}$$

Under the above conditions,

$$\oint_{K(t)} (1_{v_K} \times B_G) \cdot dK \leq \frac{\mu_0 \|m\|}{\|r\|^3_{\min}} \ell_K \cdot \max_\theta \|(2\cos\theta \cdot 1_r + \sin\theta \cdot 1_\theta\| \tag{46}$$

where ℓ_K is the total length of the curve, and $\|r\|_{\min}$ is the minimum distance of a point in K to the position of the equivalent magnetic dipole m.

But since 1_r and 1_θ are orthogonal,

$$\|(2\cos\theta \cdot 1_r + \sin\theta \cdot 1_\theta\| = \sqrt{3\cos^2\theta + 1} \tag{47}$$

what implies in:

$$1 \leq \|(2\cos\theta \cdot 1_r + \sin\theta \cdot 1_\theta)\| \leq 2 \tag{48}$$

Hence, (46) becomes:

$$\oint_{K(t)} (1_{v_K} \times B_G) \cdot dK \leq \frac{\mu_0 \|m\|}{\|r\|^3_{\min}} 2\ell_K. \tag{49}$$

Now, for simplicity, let us further assume that K is planar and delimits an oriented area A_K, and in search for an upper bound to the magnitude ratio δ, search for a lower bound for the integral in the denominator of (41).

In a well designed DIWPT configuration, in order to optimize the magnetic time-varying flux through A_K, the power transfer will occur with the secondary coil in a nominal relative position range to the primary coil such that $1_{B_G} \cdot dA$ is either always positive or always negative, at all points on A_K, thus an equivalent angle β exists, with $\cos(\beta) \neq 0$, such that:

$$\left| \int 1_{B_G} \cdot dA \right| = \|A_K\| |\cos(\beta)| \tag{50}$$

The angle β can be seen as some mean value of the angle between the magnetic field B_G and the normal unit vector at each point of the area A_K, and in good DIWPT designs, it will be as close to zero or π as possible (i.e., the secondary coil plane will be perpendicular to the direction of the magnetic flux density B), so that the magnetic flux from primary to secondary coil is increased.

So, linking (45) into (41):

$$\left|\int_S B_G \cdot ds\right| = \frac{\mu_0\|m\|}{4\pi} \cdot \left|\int_A \frac{(2\cos\theta \cdot 1_r + \sin\theta \cdot 1_\theta) \cdot dA}{\|r\|^3}\right| \tag{51}$$

So, under the constructive assumption of $1_{B_G} \cdot dA$ always positive or always negative:

$$\left|\int_S B_G \cdot ds\right| \geq \frac{\mu_0\|m\|}{\|r\|^3_{\max}} \min_\theta \|2\cos\theta \cdot 1_r + \sin\theta \cdot 1_\theta\| \left|\int_A 1_{B_G} \cdot dA\right| \tag{52}$$

Looking at (48) again,

$$\left|\int_S B_G \cdot ds\right| \geq \frac{\mu_0\|m\|}{\|r\|^3_{\max}} \|A_K\| \cos\beta \tag{53}$$

Combining (49) and (53) into (41) results in:

$$\delta \leq \frac{\|v_K\|}{\pi f_0} \frac{\ell_K \gamma^3}{\|A_K\| \, |\cos\beta|} = \delta_L, \tag{54}$$

where γ is the ratio between $\|r\|_{\max}$ and $\|r\|_{\min}$ (the maximum and minimum distances of a point in K to the position of the equivalent magnetic dipole m, respectively):

$$\gamma = \frac{||r||_{\max}}{||r||_{\min}} \tag{55}$$

The relative error ε in approximating $V_K^D(t)$ by $V_K^0(t)$ is then given by:

$$\varepsilon = \frac{|\hat{V}_K^D| - |\hat{V}_K^0|}{|\hat{V}_K^0|} = \frac{|\hat{V}_K^D|}{|\hat{V}_K^0|} - 1 = \frac{\sqrt{\left|\hat{V}_K^0\right|^2 + \left|\hat{V}_K^{v_K,B_{AC}}\right|^2}}{\left|\hat{V}_K^0\right|} - 1 = \sqrt{1+\delta^2} - 1 \tag{56}$$

In the regions where $\hat{V}_K^{v_K,B_{AC}}$ is to be neglected relatively to $\left|\hat{V}_K^0\right|$, it is expectable that $0 < \delta < 1$, so it is possible to use the inequality:

$$\sqrt{(1+\delta^2)} \leq 1 + \frac{\delta^2}{2} \tag{57}$$

from:

$$\epsilon \leq \frac{\delta^2}{2} \leq \frac{\delta_L^2}{2} \tag{58}$$

Then, by forcing the limit value δ_L to be less than $\sqrt{2\varepsilon}$, for some desired small enough ε, it will be possible to establish a condition in which the dynamically induced voltage component $V_K^{v_K,B_{AC}}$ can be neglected relatively to V_K^0, and the induced voltage, V_K^D, approximated by its purely stationary component.

It is emphasized that this theoretical result was derived and holds, among other assumptions, provided that the primary coil geometry and distance to the secondary coil are such that the magnetic field of the primary coil can be satisfactorily modeled as that of an ideal magnetic dipole, at least in the neighborhood of the secondary coil. On the other hand, by the superposition principle, (54) would as well hold individually for each primary coil that is sourcing the magnetic field to the secondary coil (and the associated induced voltage component), if more than one primary coil is present and close enough to influence the induced voltage on the secondary coil.

5.2. Quasi-Stationary Approximation in DIWPT Design for Automotive Applications

The speed of 140 km/h and a secondary coil diameter of 0.5 m can be considered within typical ranges for respectively the maximum speed and typical coil pick-up size, for an EV.

In order to analyze the situation whereas an EV is close to the lane, it would be necessary to look back to Equations (30) and (38), and notice that:

$$V_K^D(t) \cong V_K^0(t) - \|v_K\| \cos(2\pi f_0 t) \int_S J_{B_G}(1_{v_K}) \cdot dS, \tag{59}$$

Let us assume that the secondary and primary coils are flat, parallel to the ground and located at distance h to each other. The differential element dS of the secondary coil will be perpendicular to the ground plane, which is also the plane on which the vehicle can move. Also, the ISO 4130 [20] three-dimensional reference system is equivalent to having the x and y components of dS equal to zero (z is the vertical axis), that is $dS = (0, 0, dA)$. Also, since the vehicle only moves parallel to the ground plane:

$$1_{v_K} = \frac{(v_x, v_y, 0)}{||v_K||}, \tag{60}$$

where v_x is the speed of the vehicle in the direction of the road, and v_y is its speed of lateral movement with respect to the center of the road.

So, expression (59) becomes equivalent to:

$$V_K^D(t) \cong V_K^0(t) - \cos(2\pi f_0 t) \int_S \left(\frac{\partial B_z}{\partial x} v_x + \frac{\partial B_z}{\partial y} v_y \right) dA \tag{61}$$

In these conditions, expression (32) can also be simplified to:

$$V_K^0(t) = - \int_S \frac{\partial B_{AC}}{\partial t} \cdot dS = 2\pi f_0 \sin(2\pi f_0 t) \int_S B_z dA \tag{62}$$

Considering that the vehicle is moving in the direction of the road, meaning that $v_y = 0$ and assuming that the vehicle speed v_x is constant, with the voltages expressed in terms of phasors, the relative complex error in approximating $\hat{V}_K^D$ by $\hat{V}_K^0$, δ_c will be given for $\|\hat{V}_K^0\| > 0$, by:

$$\delta_c = \frac{\hat{V}_K^D}{\hat{V}_K^0} - 1 = -j \frac{v_x}{2\pi f_0} \frac{\displaystyle\int_S \frac{\partial B_z}{\partial x} dA}{\displaystyle\int_S B_z dA} \tag{63}$$

The value of $\delta = \|\delta_c\|$ tends to increase as the secondary coil approaches a position in space where $\int_S B_x dA = 0$, that is, where the induced voltage, if the vehicle were stationary, $\hat{V}_K^0$, is zero. But this is not a matter of concern, because, by design, these positions are clearly not where DIWPT operation is normally expected.

So, when the secondary coil is positioned in such a way that the primary coil is inducing non-zero voltage on it, it is possible to write:

$$\delta_c = -j \frac{v_x}{2\pi f_0} \frac{\frac{\partial \bar{B}_z}{\partial x}(S)}{\bar{B}_z(S)} \tag{64}$$

where $\bar{B}_z(S)$ and $(\overline{\partial B_z / \delta x})(S)$ are respectively the average vertical component of the magnetic field and the average derivative of the vertical component of the magnetic field with respect to x, on the area S enclosed by the secondary coil, for any given position of the vehicle.

When the area of the secondary coil is very small, it is possible to approximate (64) by taking the limit condition $S \to 0$, so, where the amplitude of the magnetic field $B_z(x, y, z) > 0$:

$$\delta_c = -j \frac{v_x}{2\pi f_0} \frac{1}{B_z} \frac{\partial B_z}{\partial x} = -j \frac{v_x}{2\pi f_0} \frac{\partial \log B_z}{\partial x} \tag{65}$$

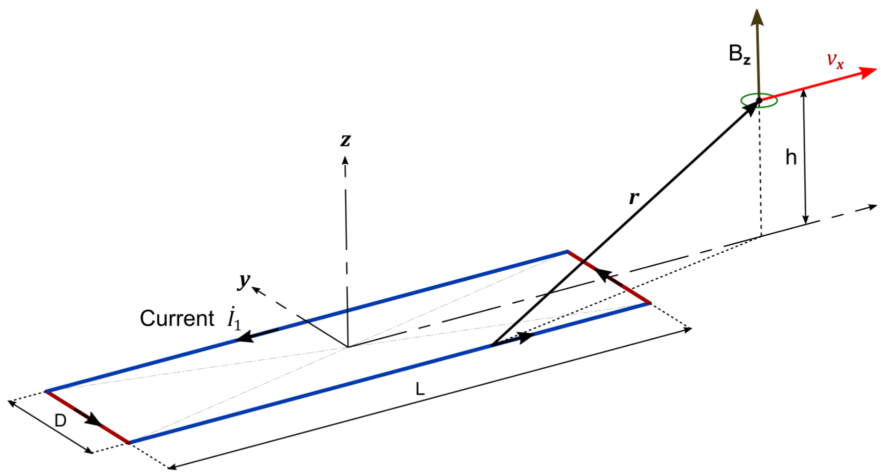

Figure 5. Rectangular primary coil in quasi-static approximation of DIWPT.

To address a popular case in automotive DIWPT applications, let us further assume that the primary coil is an oblong rectangular horizontal coil. Then, still considering that secondary coil is very small, compared to the primary coil, let us assume that it is travelling over the horizontal primary coil in center-alignment with it, at a constant height h, as shown in Figure 5.

The components of B_z respectively due to the transversal (dark red in Figure 5) and axial (blue in Figure 5) segments of the primary coil can be calculated using the Biot-Savart Law by:

$$\frac{B_{z,\text{transversal}}}{(\mu\hat{I}_1)/4\pi} = \int_{-D/2}^{D/2} \left[\frac{-(L/2-x)}{((L/2-x)^2+y^2+h^2)^{3/2}} + \frac{(L/2+x)}{((L/2+x)^2+y^2+h^2)^{3/2}} \right] dy \tag{66}$$

and

$$\frac{B_{z,\text{axial}}}{(\mu\hat{I}_1)/4\pi} = \int_{-L/2}^{L/2} \left[\frac{D}{((x\cdot u)^2+y^2+h^2)^{3/2}} \right] du \tag{67}$$

Exemplifying with primary coil dimensions $L = 2\,\text{m}$ and $D = 0.5\,\text{m}$, and $h = 0.2\,\text{m}$, and plotting the magnetic field and its gradient along the direction x, both normalized to $(\mu\hat{I}_1)/4\pi$, it is possible to plot the graphs in Figure 6(a) and Figure 6(b). The plot in Figure 6(c) is obtained by calculating the ratio of the plot values in Figure 6(b) by the corresponding value of the magnetic field in Figure 6(a), where these latter are not zero. The points of the resulting curve in the plot of Figure 6(c) will yield the relative error under the quasi-stationary approximation to the dynamically induced voltage, when being multiplied by the ratio $|v_x|/2\pi f_0$.

Visibly, the magnitude of the normalized relative error ε is only high where the magnetic field B_z is very small (close to zero), at the transitions into and out of the main design zone for wireless power transfer. However, in automotive DIWPT applications, near these transitions, the transferred power is expected to be negligible. In this example, the value of the normalized relative error, on the approach of the primary coil, is limited by the local maximum δ_m, except very close to the zero crossings of B_z, where the relative error is not important, because the power is close to zero. The relative value of the additional term observed in the dynamically induced voltage within the most relevant harvest zone of the primary coil (pink shaded) will then be limited to:

$$\delta = \frac{|v_x|}{2\pi f_0}\,\delta_m \tag{68}$$

Using the given dimensions of the coils for this example, δ_m can be numerically calculated as 2.71. So, when the vehicle travels at a speed $v_x = 140\,\text{km/h}$, if we require that $\varepsilon < 1\%$, then a sufficient condition for approximating $V_K^D(t)$ by $V_K^0(t)$, as if the secondary coil were at rest with respect to the primary coil (except very close to the zero crossing point of B_z), is that:

$$f_0 \geq \frac{2.71 v_x}{2\pi\sqrt{2}\times 1\%} \Leftrightarrow f_0 \geq 119\,\text{Hz} \tag{69}$$

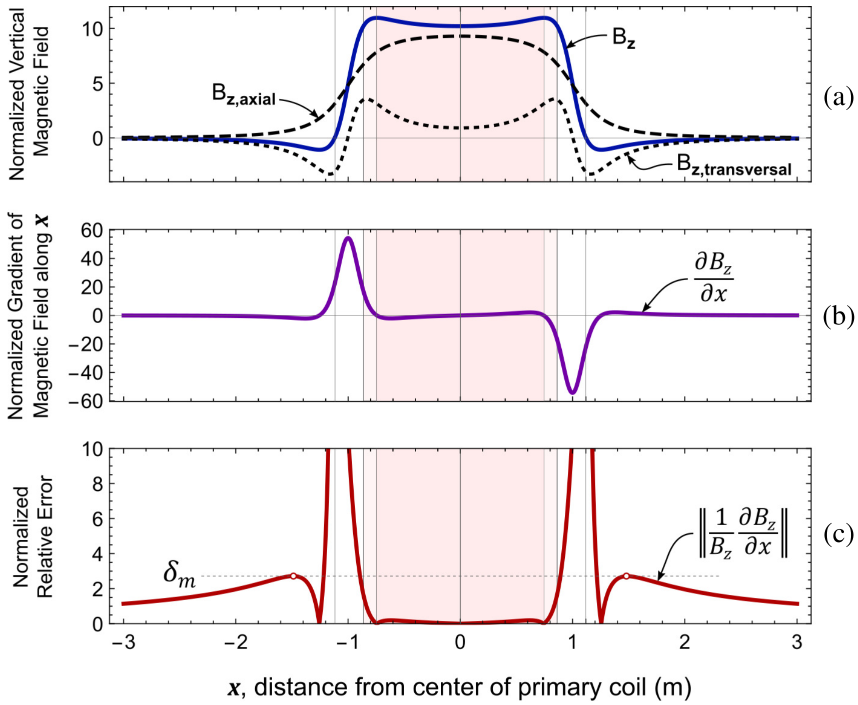

Figure 6. Error in quasi-static approximation of DIWPT using a primary rectangular coil.

Along most of the central section of the primary coil, the requirement in (67) is conservative, as it can be derived from the depressed central section of the curve of normalized error, in color red in Figure 6. On the other hand, transient elevations on the induced voltage will be expected when the secondary is transitioning into or out of the primary main power harvest zone.

In today's WPT and DIWPT applications, in order to to achieve higher levels of transferred power, the induction frequency f_0 is commonly tens of kilohertz, which is significantly higher than that required by (68). As an example, the proposed frequency band, recommended by Society of Automotive Engineers (SAE) for stationary automotive recharging, is centered at 85 kHz [21], and this band is also used in new DIWPT transfer systems for road vehicles. In these circumstances, no relevant error in the quasi-stationary approximation will be noticed. However, the actual minimum required frequency for a negligible error in the quasi-stationary approximation of the induced voltage will vary according to the specific geometry and the acceptable error epsilon defined for the design. If required, precise values for the discrepancy term of the dynamic voltage with respect to the stationary approximation can be calculated by numerically integrating (38) under the actual conditions of each DIWPT configuration.

6. CONCLUSIONS

In this paper, expressions are developed to assess the approximation of the dynamically induced voltage by the voltage that would be induced under stationary conditions, in DIWPT applications for electric vehicles. In a numeric example, it was found that at the commonly expected working frequencies and the typical maximum speed of vehicles, it can be assumed that the quasi-stationary approximation holds. The only effect unique to DIWPT, which is an expected discrepancy to this approximation, is the appearance of transient peaks in the induced voltage, when the secondary coil passes over the points of zero crossing in the magnetic field generated by the primary coil. However, since the transferred power is close to zero in the neighborhood of these points, the overall dynamic induced power is not affected. The expressions developed can also give some insight in the design of wireless power transfer systems where other higher speed applications should be considered.

ACKNOWLEDGMENT

The authors would like to acknowledge the financial and academic support received from the Centro Algoritmi of the University of Minho, the Portuguese national funding agency for science, research and technology, the FCT, and the MIT-Portugal Program. This research was also supported by the Fundação para a Ciência e a Tecnologia, Portugal under Project UIDB/04111/2020. The authors would also like to thank Prof. Pedro Alberto of the Department of Physics of the University of Coimbra, for his valuable comments and reviews.

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