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Self-Inductance Computation of the Thin Conical Sheet Inductor

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Abstract—In this paper, a new formula for calculating the self-inductance of a thin conical sheet inductor is given. The presented work is derived in a semi-analytical form based on the complete elliptic integrals of the first, second, and third kind plus a term to be solved numerically. The analytical formula is obtained in the special case when the thin conical sheet inductor is degenerated into a thin wall cylinder. The validation of the presented formulas is done by triple, double, single integration and by the semi-analytical formula. These self-inductance calculations of the thin conical sheet inductors can be especially useful in broadband RF applications and wireless power transfer systems where conical inductors have been used.

1. INTRODUCTION

For the first time in the literature, the calculation of the self-inductance of a thin conical sheet inductor is presented. Reviewing the relevant publications (books, monographs, published papers in the physics and in the electromagnetics), similar works could not be found except Tesla conical coil solver based on an empirical Harold Wheeler formula.

Self-inductance computations of conventional coils have been challenging from the time of Maxwell [1]. Many authors have been working of this important electrical parameter for various coil configurations (linear, circular, elliptic, etc.) [2–8]. The analytical solutions have been developed in the form of elementary functions for linear coils and for more complex configurations such as circular and elliptical coils, and the solutions can be obtained over the elliptic integrals, Bessel and Struve functions, and hypergeometric convergent series [1–8]. In modern times of powerful computers, the numerical methods have been developed to calculate such parameters (Finite Difference Method (FDM), Finite Element Method (FEM), Boundary Element Method (BEM)). Commercial software's packages are also available (Ansys, Maxwell, Fast Henry); however, there is interest to improve the self-inductance computational efficiency developing the analytical and semi-analytical methods. In this paper, a semi-analytic computation of the self-inductance for a thin conical sheet inductor is presented. This problem has not been considered so far in available publications on physics and electromagnetics (books, monographies, papers, projects, and studies). This semi-analytical computational form is based on the complete elliptic integrals of the first, second, and third kind plus a term to be solved by a numerical integration. The kernel function of this term is the continuous analytical function on the whole segment of integration. Also, this computational method is given in another form where the elliptical integral of the third kind is the combination of the elliptical integral of the first kind and the Heuman Lambda function. This form is incredibly special suitable for the computation specially in the case when the thin conical sheet is coveraged into thin wall cylindrical form [7, 8]. Therefore, this presented method is generalized and includes the case of thin wall cylindrical inductor. Wheeler derived the empirical formulae for calculation of the self-inductance in conical radio conical coils [6]. Tesla applied these empirical formulae to calculate the inductance of conical inductors in his own research. We found that

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Tesla conical coil solver based on Wheeler empirical formulas has been presented in [9], which has been used for comparative evolution. American Wire Gauge standard specifying North American wire sizing, [10], is used for the diameter of wire in [9]. The calculation of the self-inductance of a thin conical sheet is useful for conical inductors which are of ideal form for ultra-broadband applications up to 40 GHz since the conical shapes limit effects of stray capacitances and effectively substitute a series of narrow-band inductors creating high impedance over a very wide bandwidth [11–13]. The validation of the presented method is performed using six representative cases. The Mathematica files with implemented formulas are available upon request.

2. BASIC FORMULA

Let us consider a thin conical sheet inductor as shown in Figure 1 with the radii r and R, height l, and the number of turns N.

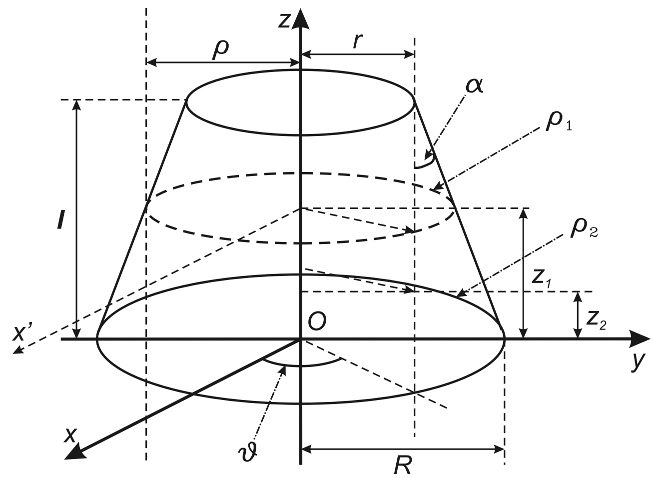

Figure 1. The thin conical sheet inductor.

The self-inductance of the thin conical sheet inductor can be calculated by,

$$L=\frac{\mu_0 N^2}{l^2}\int_0^{\pi}\int_0^{l}\int_0^{l}\frac{\rho_1\rho_2\cos(\theta)}{r_0}\,dz_1dz_2d\theta \tag{1}$$

where,

$$\frac{R-r}{l}=\operatorname{tg}(\alpha)=\eta,\quad \rho_1=R-\eta z_1,\quad \rho_2=R-\eta z_2$$
$$r_0=\sqrt{\rho_1^2+\rho_2^2-2\rho_1\rho_2\cos(\theta)+(z_1-z_2)^2}$$

Let us introduce the substitution $\theta=\pi-2\beta$, and (1) begins,

$$L=-\frac{2\mu_0 N^2}{l^2}\int_0^{\frac{\pi}{2}}\int_0^{l}\int_0^{l}\frac{(R-\eta z_1)(R-\eta z_2)\cos(2\beta)}{r_0}\,dz_1dz_2d\beta \tag{2}$$

with,

$$r_0=\sqrt{(R-\eta z_1)^2+(R-\eta z_2)^2+2(R-\eta z_1)(R-\eta z_2)\cos(2\beta)+(z_1-z_2)^2}$$

The three integrals in (2) will be solved to find semi-analytical solution. The validity of Equation (2) can be easily inspected by putting $\eta=0$ when the thin conical sheet degenerates to a thin wall solenoid [7, 8]. It will be solved as a special case.

3. CALCULATION OF SELF-INDUCTANCE OF THIN CONICAL SHEET INDUCTOR

Let us take the following transformations in (2),

$$R-\eta z_1 = t,\quad dz_1=-\frac{dt}{\eta},\quad z_1=\frac{R-t}{\eta}$$

$$z_1 = 0\to t_1=R,\quad z_1=l\to t_2=r,\quad z_1-z_2=\frac{R-\eta z_2-t}{\eta}$$

$$I_1 = \int_0^l \frac{[R-\eta z_1]\,dz_1}{r_0}=\frac{1}{\eta}\int_r^R\frac{tdt}{r_0} \tag{3}$$

with

$$r_0 = \frac{1}{\eta}\sqrt{(\eta^2+1)t^2+2t\left[\eta^2(R-\eta z_2)\cos(2\beta)-(R-\eta z_2)\right]+(\eta^2+1)(R-\eta z)^2}$$

$$r_0 = \frac{1}{\eta}\sqrt{ct^2+bt+a}=\frac{1}{\eta}R_0,\quad R_0=\sqrt{ct^2+bt+a}$$

$$c = \eta^2+1$$

$$b = 2\left(R-\eta z_2\right)\left[\eta^2\cos(2\beta)-1\right]$$

$$a = (\eta^2+1)(R-\eta z_2)^2$$

$$\Delta = 4\eta^2(R-\eta z_2)^2\left[\eta^2\sin^2(2\beta)+2\cos(2\beta)+2\right]=4\eta^2(R-\eta z_2)^2D_0$$

$$D_0 = \eta^2\sin^2(2\beta)+2\cos(2\beta)+2$$

$$I_1 = \frac{1}{\eta}\int_r^R\frac{tdt}{r_0}=\int_r^R\frac{tdt}{\sqrt{at^2+bt+a}}=\int_r^R\frac{tdt}{R_0}$$

Using [14] the integral I_1 is obtained in the analytical form over elementary functions as follows,

$$\begin{aligned}I_1 &= \int_r^R\frac{tdt}{R_0}=\left\{\frac{R_0}{c}-\frac{b}{2c}\int\frac{dt}{R_0}\right\}=\frac{1}{(\eta^2+1)}\left\{\sqrt{cR^2+bR+a}-\sqrt{cr^2+br+a}\right\}\\&\quad-\frac{(R-\eta z_2)\left[\eta^2\cos(2\beta)-1\right]}{(\eta^2+1)\sqrt{\eta^2+1}}\left\{\operatorname{asinh}\frac{\eta\left(R-\eta z_2\right)\cos(2\beta)+\eta R+z_2}{(R-\eta z_2)\sqrt{D_0}}\right.\\&\quad\left.-\operatorname{asinh}\frac{\eta\left(R-\eta z_2\right)\cos(2\beta)+\eta r+z_2-l}{(R-\eta z_2)\sqrt{D_0}}\right\}\end{aligned} \tag{4}$$

Equation (2) after the first integration is given as follows,

$$L=-\frac{2\mu_0N^2}{l^2\left(\eta^2+1\right)^{\frac{3}{2}}}\int_0^{\frac{\pi}{2}}\int_0^l I\left(R-\eta z_2\right)\cos(2\beta)\,dz_2d\beta \tag{5}$$

with,

$$\begin{aligned}I &= \sqrt{\eta^2+1}\left\{\sqrt{cR^2+bR+a}-\sqrt{cr^2+br+a}\right\}-\left(R-\eta z_2\right)\left[\eta^2\cos(2\beta)-1\right]\\&\quad\times\left\{\operatorname{asinh}\frac{\eta\left(R-\eta z_2\right)\cos(2\beta)+\eta R+z_2}{(R-\eta z)\sqrt{D_0}}-\operatorname{asinh}\frac{\eta\left(R-\eta z_2\right)\cos(2\beta)+\eta r+z_2-L}{(R-\eta z)\sqrt{D_0}}\right\}\end{aligned}$$

The second integration in (5) performs with four parts,

$$I_2 = \int_0^l I\left(R-\eta z_2\right)dz_2 = I_2^{(1)} + I_2^{(2)} + I_2^{(3)} + I_2^{(4)} \tag{6}$$

$$I_2^{(1)} = \sqrt{\eta^2+1}\int_0^l \sqrt{cR^2+bR+a}\left(R-\eta z_2\right)dz_2$$

Let us introduce in (6) the following transformations,

$$R-\eta z_2 = t,\;\; z_2=0 \to t_1 = R,\;\; z_2 = l \to t_2 = r,\;\; dz_2 = -\frac{dt}{\eta},\;\; z_2 = \frac{R-t}{\eta}$$

$$I_2^{(1)} = \sqrt{(\eta^2+1)}\int_0^l t\sqrt{(\eta^2+1)R^2 + 2Rt\left[\eta^2\cos(2\beta)-1\right] + (\eta^2+1)t^2}\,dz$$

or

$$I_2^{(1)} = \frac{\sqrt{\eta^2+1}}{\eta}\int_r^R t\sqrt{a_1+b_1t+c_1t^2}\,dt \tag{7}$$

with

$$\begin{aligned}
c_1 &= \eta^2+1\\
b_1 &= 2R\left[\eta^2\cos(2\beta)-1\right]\\
a_1 &= (\eta^2+1)R^2\\
\Delta_1 &= 4\eta^2R^2\left[\eta^2\sin^2(2\beta)+2\cos(2\beta)+2\right] = 4\eta^2R^2D_0
\end{aligned}$$

Using [14] one obtains the solution in the analytical form as follows,

$$\begin{aligned}
I_2^{(1)} &= \frac{\eta^2R^3\cos(\beta)}{3\sqrt{(\eta^2+1)^3}}\left[3\eta^2\sin^2(2\beta) + \left(\eta^2+7\right)\cos(2\beta) + \left(\eta^2+7\right)\right] - \frac{\sqrt{R^2+r^2+l^2+2Rr\cos(2\beta)}}{6\sqrt{(\eta^2+1)^3}}\\
&\times\Big[2\left(\eta^2+1\right)^2\left(R^2+r^2\right) + \left(\eta^2+1\right)Rr\left(\eta^2\cos(2\beta)-1\right) + \left(\eta^2+1\right)Rr\left(\eta^2\cos(2\beta)-1\right)\\
&-3R^2\left(\eta^2\cos(2\beta)-1\right)^2\Big] - \frac{\eta R^3\left[\eta^2\sin^2(2\beta)+2\cos(2\beta)+2\right]\left[\eta^2\cos(2\beta)-1\right]}{2\left(\eta^2+1\right)^2}\\
&\times\left[\operatorname{asinh}\frac{\eta\left(\cos(2\beta)+1\right)}{\sqrt{D_0}} - \operatorname{asinh}\frac{\eta R\cos(2\beta)+\eta r - l}{R\sqrt{D_0}}\right]
\end{aligned} \tag{8}$$

The following part is,

$$I_2^{(1)} = \sqrt{\eta^2+1}\int_0^l \sqrt{cr^2+br+a}\left(R-\eta z_2\right)dz_2$$

Using the same transformations like in (7),

$$R-\eta z_2 = t,\;\; z_2=0 \to t_1 = R,\;\; z_2 = l \to t_2 = r,\;\; dz_2 = -\frac{dt}{\eta},\;\; z_2 = \frac{R-t}{\eta}$$

one has

$$I_2^{(2)} = -\frac{\sqrt{(\eta^2+1)}}{\eta}\int_r^R t\sqrt{(\eta^2+1)r^2 + 2rt\left[\eta^2\cos(2\beta)-1\right] + (\eta^2+1)t^2}\,dt$$

or

$$I_2^{(2)} = -\frac{\sqrt{(\eta^2+1)}}{\eta}\int_r^R t\sqrt{c_2t^2+b_2t+a_2}\,dt$$

with

$$\begin{aligned}
c_2 &= \eta^2+1 \\
b_2 &= 2r\left[\eta^2\cos(2\beta)-1\right] \\
a_2 &= (\eta^2+1)r^2 \\
\Delta_2 &= 4\eta^2r^2\left[\eta^2\sin^2(2\beta)+2\cos(2\beta)+2\right]=4\eta^2r^2D_0
\end{aligned}$$

Similarly, as the integral $I_2^{(1)}$, one obtains the analytical solution as follows,

$$\begin{aligned}
I_2^{(2)} &= \frac{-1}{6\sqrt{(\eta^2+1)^3}}\sqrt{R^2+r^2+L^3+2Rr\cos(2\beta)} \\
&\left[2\left(\eta^2+1\right)^2\left(R^2+r^2\right)+\left(\eta^2+1\right)Rr\left(\eta^2\cos(2\beta)-1\right)-3r^2\left(\eta^2\cos(2\beta)-1\right)^2\right] \\
&+\frac{\eta^2r^3\cos(\beta)}{3\sqrt{(\eta^2+1)^3}}\left[3\eta^2\sin^2(2\beta)+\left(\eta^2+7\right)\cos(2\beta)+\left(\eta^2+7\right)\right] \\
&-\frac{\eta r^3\left[\eta^2\sin^2(2\beta)+2\cos(2\beta)+2\right]\left[\eta^2\cos(2\beta)-1\right]}{2\left(\eta^2+1\right)^2} \\
&\times\left[\operatorname{asinh}\frac{\eta r\cos(2\beta)+\eta R-l}{r\sqrt{D_0}}-\operatorname{asinh}\frac{\eta\left(\cos(2\beta)+1\right)}{\sqrt{D_0}}\right]
\end{aligned} \tag{9}$$

The following integration is of the third part,

$$I_2^{(3)} = -\left[\eta^2\cos(2\beta)-1\right]\int_0^l\left(R-\eta z_2\right)^2\operatorname{asinh}\frac{\eta\left(R-\eta z_2\right)\cos(2\beta)+\eta R+z_2}{(R-\eta z_2)\sqrt{D_0}}\,dz_2$$

Introducing,

$$R-\eta z_2=t,\ \ z_2=0\rightarrow t_1=R,\ \ z_2=l\rightarrow t_2=r,\ \ dz_2=-\frac{dt}{\eta},\ \ z_2=\frac{R-t}{\eta}$$

$$\begin{aligned}
I_2^{(3)} &= -\frac{\left[\eta^2\cos(2\beta)-1\right]}{\eta}\int_r^R t^2\operatorname{asinh}\frac{\eta t\cos(2\beta)+\eta R+\frac{R-t}{\eta}}{t\sqrt{D_0}}\,dt \\
&= -\frac{\left[\eta^2\cos(2\beta)-1\right]}{\eta}\int_r^R t^2\operatorname{asinh}\frac{t\left[\eta^2\cos(2\beta)-1\right]+\left(\eta^2+1\right)R}{\eta t\sqrt{D_0}}\,dt
\end{aligned} \tag{10}$$

Using the partial integration one has,

$$\begin{aligned}
v &= \frac{t^3}{3},\ \ u=\operatorname{asinh}\frac{t\left[\eta^2\cos(2\beta)-1\right]+(\eta^2+1)R}{\eta t\sqrt{D_0}} \\
du &= -\frac{\left(\eta^2+1\right)Rdt}{t\sqrt{\left(\eta^2+1\right)^2t^2+2R\left(\eta^2+1\right)\left(\eta^2\cos(2\beta)-1\right)t+\left(\eta^2+1\right)^2R^2}}
\end{aligned}$$

$$
\begin{aligned}
I_2^{(3)} &= -\frac{\left[\eta^2\cos(2\beta)-1\right]}{\eta}\left\{\frac{t^3}{3}\text{asinh}\,\frac{t\left[\eta^2\cos(2\beta)-1\right]+\left(\eta^2+1\right)R}{\eta t\sqrt{D_0}}\Bigg|_r^R\right.\\
&\left.+\frac{R\left(\eta^2+1\right)}{3}\int_r^R\frac{t^2}{\sqrt{(\eta^2+1)^2t^2+2R\left(\eta^2+1\right)\left(\eta^2\cos(2\beta)-1\right)t+(\eta^2+1)^2R^2}}\,dt\right\}\\
I_2^{(3)} &= -\frac{\left[\eta^2\cos(2\beta)-1\right]}{3\eta}\left\{R^3\text{asinh}\,\frac{\eta(\cos(2\beta)+1)}{\sqrt{D_0}}-r^3\text{asinh}\,\frac{\eta r\cos(2\beta)+\eta R+L}{r\sqrt{D_0}}\right\}\\
&+\frac{R\left[\eta^2\cos(2\beta)-1\right]\left(\eta^2+1\right)}{3\eta}\times\left\{\left[\frac{t}{2c_1}-\frac{3b_1}{4c_1^2}\right]\sqrt{a_1+bt+c_1t^2}\right.\\
&\left.+\left[\frac{3b_1^2}{8c_1^2}-\frac{a_1}{2c_1}\right]\frac{1}{\sqrt{c_1}}\,\text{asinh}\,\frac{2c_1t+b_1}{\sqrt{\Delta_1}}\right\}\Bigg|_r^R
\end{aligned}
$$

where c_1, b_1, a_1, and Δ_1 are given previously.

$$
\begin{aligned}
I_2^{(3)} &= -\frac{\left[\eta^2\cos(2\beta)-1\right]}{3\eta}\left\{R^3\text{asinh}\,\frac{\eta\left(\cos(2\beta)+1\right)}{\sqrt{D_0}}-r^3\text{asinh}\,\frac{\eta r\cos(2\beta)+\eta R+l}{r\sqrt{D_0}}\right\}\\
&-\frac{\left[\eta^2\cos(2\beta)-1\right]}{3\sqrt{\left(\eta^2+1\right)^3}}\,\eta^2R^3\cos(\beta)\left[\left(\eta^2+1\right)-3\left(\eta^2\cos(2\beta)-1\right)\right]\frac{\left[\eta^2\cos(2\beta)-1\right]}{3\sqrt{\left(\eta^2+1\right)^3}}\\
&\times\sqrt{R^2+r^2+l^2+2Rr\cos(2\beta)}\left[Rr\left(\eta^2+1\right)-3R^2\left(\eta^2\cos(2\beta)-1\right)\right]\\
&-\frac{\left[\eta^2\cos(2\beta)-1\right]}{6\eta\left(\eta^2+1\right)^2}\left[3\left(\eta^2\cos(2\beta)-1\right)^2-\left(\eta^2+1\right)^2\right]\\
&\times\left\{R^3\,\text{asinh}\,\frac{\eta(\cos(2\beta)+1)}{\sqrt{D_0}}-R^3\,\text{asinh}\,\frac{\eta R\cos(2\beta)+\eta r-l}{R\sqrt{D_0}}\right\}
\end{aligned}
\tag{11}
$$

The last one is obtained similarly as $I_2^{(3)}$,

$$
\begin{aligned}
I_2^{(4)} &= \frac{\left[\eta^2\cos(2\beta)-1\right]}{3\eta}\left\{R^3\text{asinh}\,\frac{\eta R\cos(2\beta)+\eta r-l}{R\sqrt{D_0}}-r^3\text{asinh}\,\frac{\eta(\cos(2\beta)+1)}{\sqrt{D_0}}\right\}\\
&+\frac{r\left[\eta^2\cos(2\beta)-1\right]\left(\eta^2+1\right)}{3\eta}\left\{\left[\frac{t}{2c_2}-\frac{3b_2}{4c_2^2}\right]\sqrt{a_2+b_2t+c_2t^2}\right.\\
&\left.+\left[\frac{3b_2^2}{8C_2^2}-\frac{a_2}{2c_2}\right]\frac{1}{\sqrt{c_2}}\,\text{asinh}\,\frac{2c_2t+b_2}{\sqrt{\Delta_2}}\right\}
\end{aligned}
\tag{12}
$$

where c_2, b_2, a_2, and Δ_2 are given previously.

The simplified form of (12) is given as,

$$
\begin{aligned}
I_2^{(4)} &= \frac{\left[\eta^2\cos(2\beta)-1\right]}{3\eta}\left\{R^3\,\text{asinh}\,\frac{\eta R\cos(2\beta)+\eta r-l}{R\sqrt{D_0}}-r^3\,\text{asinh}\,\frac{\eta(\cos(2\beta)+1)}{\sqrt{D_0}}\right\}\\
&+\frac{\left[\eta^2\cos(2\beta)-1\right]}{6\sqrt{\left(\eta^2+1\right)^3}}\sqrt{R^2+r^2+L^3+2Rr\cos(2\beta)}\left[Rr\left(\eta^2+1\right)-3r^2\left(\eta^2\cos(2\beta)-1\right)\right]\\
&-\frac{\left[\eta^2\cos(2\beta)-1\right]\left[3(\eta^2\cos(2\beta)-1)^2-\left(\eta^2+1\right)^2\right]}{6\eta\left(\eta^2+1\right)^2}\left\{r^3\,\text{asinh}\,\frac{\eta r\cos(2\beta)+\eta R+l}{r\sqrt{D_0}}\right.\\
&\left.-r^3\,\text{asinh}\,\frac{\eta(\cos(2\beta)+1)}{\sqrt{D_0}}\right\}
\end{aligned}
\tag{13}
$$

Putting $I_2^{(1)}$, $I_2^{(2)}$, $I_2^{(3)}$, and $I_2^{(4)}$ in (6) and using (5), the self-inductance is given by the single integration as follows,

$$L = -\frac{4\mu_0 N^2}{3l^2\left(\eta^2+1\right)^{\frac{3}{2}}} \int_0^{\frac{\pi}{2}} I_3 \cos\left(2\beta\right) d\beta \tag{14}$$

where

$$\begin{aligned} I_3 \;=\; & 2\sqrt{(\eta^2+1)}\left(R^3+r^3\right)\cos\left(\beta\right) - \sqrt{(\eta^2+1)}\left(R^2+r^2\right)\sqrt{R^2+r^2+L^3+2Rr\cos\left(2\beta\right)} \\ & +\frac{\left(\eta^2\cos\left(2\beta\right)-1\right)}{\eta}\left\{R^3 \operatorname{asinh}\frac{\eta R\cos\left(2\beta\right)+\eta r-l}{R\sqrt{D_0}} + r^3 \operatorname{asinh}\frac{\eta r\cos\left(2\beta\right)+\eta R+l}{r\sqrt{D_0}}\right. \\ & \left. -\left(R^3+r^3\right)\operatorname{asinh}\frac{\eta(\cos\left(2\beta+1\right))}{\sqrt{D_0}}\right\} \end{aligned} \tag{15}$$

Thus, one obtains the simple analytical form after two integrations. Equation (15) can be used as the single integral for any η except $\eta = 0$ because this is singularity. This case will be treated as a special case.

Before doing the last integration let us rearrange (15) as follows,

$$\begin{aligned} I_3 \;=\; & 2\sqrt{(\eta^2+1)}\left(R^3+r^3\right)\cos\left(\beta\right) - \sqrt{(\eta^2+1)}\left(R^2+r^2\right)\sqrt{R^2+r^2+L^3+2Rr\cos\left(2\beta\right)} \\ & -\frac{\left(R^3+r^3\right)}{2\eta}\left[\eta^2\cos\left(4\beta\right)-2\cos\left(2\beta\right)\right]\operatorname{asinh}\frac{\eta(\cos\left(2\beta\right)+1)}{\sqrt{D_0}} \\ & +\frac{R^3}{2\eta}\left[\eta^2\cos\left(4\beta\right)-2\cos\left(2\beta\right)\right]\operatorname{asinh}\frac{\eta R\cos\left(2\beta\right)+\eta r-l}{R\sqrt{D_0}} \\ & +\frac{R^3}{2\eta}\left[\eta^2\cos\left(4\beta\right)-2\cos\left(2\beta\right)\right]\times\operatorname{asinh}\frac{\eta R\cos\left(2\beta\right)+\eta r-l}{R\sqrt{D_0}} \\ & +\frac{\eta}{2}\left\{R^3 \operatorname{asinh}\frac{\eta R\cos\left(2\beta\right)+\eta r-l}{R\sqrt{D_0}} + r^3 \operatorname{asinh}\frac{\eta r\cos\left(2\beta\right)+\eta R+l}{r\sqrt{D_0}}\right. \\ & \left. -\left(R^3+r^3\right)\operatorname{asinh}\frac{\eta(\cos\left(2\beta\right)+1)}{\sqrt{D_0}}\right\} \end{aligned} \tag{16}$$

The three last terms in (16) do not have the analytical solution, but their kernel function is the continuous regular function without the singularities on the interval of integration $\beta \in [0; \pi/2]$. It is the function J_0 that will be solved numerically.

$$\begin{aligned} J_0 \;=\; & \frac{\eta}{2}\int_0^{\frac{\pi}{3}}\left\{R^3 \operatorname{asinh}\frac{\eta R\cos\left(2\beta\right)+\eta r-l}{R\sqrt{D_0}} + r^3 \operatorname{asinh}\frac{\eta r\cos\left(2\beta\right)+\eta R+l}{r\sqrt{D_0}}\right. \\ & \left. -\left(R^3+r^3\right)\operatorname{asinh}\frac{\eta(\cos\left(2\beta\right)+1)}{\sqrt{D_0}}\right\}d\beta \end{aligned} \tag{17}$$

Let us solve all terms in (16).

$$I_{F1} = \int_0^{\pi/2} 2\sqrt{(\eta^2+1)}\left(R^3+r^3\right)\cos\left(\beta\right)\cos\left(2\beta\right)d\beta$$

Using [14] this integral can be obtained in the analytical form as follows,

$$I_{F1} \;=\; \frac{2}{3}\sqrt{(\eta^2+1)}\left(R^3+r^3\right) \tag{18}$$

$$I_{F2} \;=\; \int_0^{\frac{\pi}{2}} \sqrt{(\eta^2+1)}\left(R^2+r^2\right)\sqrt{R^2+r^2+l^2+2Rr\cos\left(2\beta\right)}\cos\left(2\beta\right)d\beta.$$

The second integral in (16) can be transformed in the following form,

$$I_{F2} = -\sqrt{(\eta^2+1)}\left(R^2+r^2\right)\frac{2\sqrt{Rr}}{k}\int\limits_0^{\frac{\pi}{2}}\left[1-2\sin^2(\beta)\right]\Delta d\beta$$

where,

$$\begin{aligned}\Delta &= \sqrt{1-k^2\sin^2(\beta)}\\ k^2 &= \frac{4Rr}{(R+r)^2+l^2}\end{aligned}$$

The solution of this integral can be obtained in the form of the complete elliptical integrals of the first and second kind, [14] as follows,

$$I_{F2} = -\frac{2}{3k^3}\sqrt{(\eta^2+1)}\left(R^2+r^2\right)\sqrt{Rr}\left[2\left(k^2-1\right)K(k)+\left(2-k^2\right)E(k)\right] \tag{19}$$

$$I_{F3} = -\int\limits_0^{\frac{\pi}{2}}\frac{\left(R^3+r^3\right)}{2\eta}\left[\eta^2\cos(4\beta)-2\cos(2\beta)\right]\,\mathrm{asinh}\,\frac{\eta(\cos(2\beta)+1)}{\sqrt{D_0}}\,d\beta$$

Here, the partial integration is used.

$$\begin{aligned}v &= \int(\eta^2\cos(4\beta)-2\cos(2\beta))d\beta = \frac{\eta^2}{4}\sin(4\beta)-\frac{2\sin(2\beta)}{2} = \frac{\sin(2\beta)}{2}\left[\eta^2\cos(2\beta)-2\right]\\
u &= \mathrm{asinh}\,\frac{\eta(\cos(2\beta)+1)}{\sqrt{D_0}}\\
du &= \frac{2\eta\sqrt{\eta^2+1}\sin(2\beta)\cos(\beta)}{(\cos(2\beta)+1)(\eta^2\cos(2\beta)-\eta^2-2)}\,d\beta\\
I_{F3} &= -\frac{R^3+r^3}{2\eta}\left\{\sin(\beta)\cos(\beta)\left[\eta^2\cos(2\beta)-2\right]\mathrm{asinh}\,\frac{\eta(\cos(2\beta)+1)}{\sqrt{D_0}}\Bigg|_0^{\frac{\pi}{2}}\right.\\
&\quad\left.-\frac{2\eta\sqrt{\eta^2+1}}{2}\int\limits_0^{\frac{\pi}{2}}\frac{\sin(2\beta)\sin(2\beta)\cos(\beta)\left[\eta^2\cos(2\beta)-2\right]}{(\cos(2\beta)+1)(\eta^2\cos(2\beta)-\eta^2-2)}\,d\beta\right\}\\
&= \frac{\sqrt{\eta^2+1}\left(R^3+r^3\right)}{2}\int\limits_0^{\frac{\pi}{2}}\frac{(\cos^2(2\beta)-1)\cos(\beta)\left[\eta^2\cos(2\beta)-2\right]d\beta}{(\cos(2\beta)+1)(\eta^2\cos(2\beta)-\eta^2-2)}\end{aligned}$$

or

$$\begin{aligned}I_{F3} &= \frac{\sqrt{\eta^2+1}\left(R^3+r^3\right)}{2}\int\limits_0^1\frac{\left[2\eta^2t^4+(2-\eta^2)t^2\right]}{(\eta^2t^2+1)}\,d\beta\\
&= \frac{\sqrt{\eta^2+1}\left(R^3+r^3\right)}{2}\left\{\int\limits_0^1\left[2t^2-1\right]dt+\int\limits_0^1\frac{1}{(\eta^2t^2+1)}dt\right\}\end{aligned}$$

The solution of this integral is obtained in the analytical form as follows,

$$I_{F3} = \frac{\sqrt{\eta^2+1}\left(R^3+r^3\right)}{2}\left\{-\frac{1}{3}+\frac{\operatorname{arctg}(\eta)}{\eta}\right\} \tag{20}$$

$$I_{F4} = \int_0^{\frac{\pi}{2}} \frac{\left\{R^3 \operatorname{asinh}\frac{\eta R\cos(2\beta)+\eta r-l}{R\sqrt{D_0}} + r^3 \operatorname{asinh}\frac{\eta r\cos(2\beta)+\eta R+l}{r\sqrt{D_0}}\right\}}{2\eta} \times (\eta^2\cos(4\beta)-2\cos(2\beta))d\beta$$

Also, this integral will be solved by the partial integration.

$$v = \int(\eta^2\cos(4\beta)-2\cos(2\beta))d\beta = \frac{\eta^2}{4}\sin(4\beta) - \frac{2\sin(2\beta)}{2} = \frac{\sin(2\beta)}{2}\left[\eta^2\cos(2\beta)-2\right]$$

$$u = R^3 \operatorname{asinh}\frac{\eta R\cos(2\beta)+\eta r-l}{R\sqrt{D_0}} + r^3 \operatorname{asinh}\frac{\eta r\cos(2\beta)+\eta R+l}{r\sqrt{D_0}}$$

$$du = -\left\{rR\left(R^2+r^2\right)\cos(2\beta)+\left(R^4+r^4\right)+l^2\left(R^2+r^2+Rr\right)\right\}\frac{k\eta\sqrt{\eta^2+1}\sin(2\beta)}{D_0\Delta\sqrt{Rr}}\,d\beta$$

or

$$du = \frac{k\eta\sqrt{\eta^2+1}\sin(2\beta)\left\{rR\left(R^2+r^2\right)\cos(2\beta)+\left(R^4+r^4\right)+l^2\left(R^2+r^2+Rr\right)\right\}d\beta}{(\cos(2\beta)+1)(\eta^2\cos(2\beta)-\eta^2-2)\Delta\sqrt{Rr}}$$

$$I_{F4} = \frac{1}{2\eta}\left\{uv\,\Big|_0^{\frac{\pi}{2}} - \frac{k\eta\sqrt{\eta^2+1}}{2\sqrt{Rr}}\int_0^{\frac{\pi}{2}}\frac{\sin^2(2\beta)\left[\eta^2\cos(2\beta)-2\right]}{(\cos(2\beta)+1)(\eta^2\cos(2\beta)-\eta^2-2)\Delta} \times\left\{rR\left(R^2+r^2\right)\cos(2\beta)+\left(R^4+r^4\right)+l^2\left(R^2+r^2+Rr\right)\right\}d\beta\right\} \tag{21}$$

The kernel function of this integral can be obtained by using the partial decomposition as follows,

$$\frac{\left[\cos(2\beta)-1\right]\left[\eta^2\cos(2\beta)-2\right]}{(\eta^2\cos(2\beta)-\eta^2-2)\Delta} - \left\{rR\left(R^2+r^2\right)\cos(2\beta)+\left(R^4+r^4\right) + \left[\eta^2\left(R^4+r^4\right)+\eta^2l^2\left(R^2+r^2+Rr\right)-(\eta^2+2)rR\left(R^2+r^2\right)\right]\times\cos^3(2\beta) + \left[2rR\left(R^2+r^2\right)-(\eta^2+2)(\left(R^4+r^4\right)+l^2\left(R^2+r^2+Rr\right))\right]\cos(2\beta) +2\left[\left(R^4+r^4\right)+l^2\left(R^2+r^2+Rr\right)\right]\right\}/\left(\eta^2\cos(2\beta)-\eta^2-2\right) \tag{22}$$

The integral I_{F4} can be written in the following form,

$$I_{F4} = \frac{k\sqrt{\eta^2+1}}{4\sqrt{Rr}}\left\{\left(R^2+r^2\right)Rr\int_0^{\frac{\pi}{2}}\frac{\cos^2(2\beta)}{\Delta}d\beta + 2\frac{\left(R^2+r^2\right)Rr}{\eta^2}\int_0^{\pi/2}\frac{d\beta}{\Delta} + \left[R^4+r^4+l^2\left(R^2+r^2+Rr\right)\right]\int_0^{\frac{\pi}{2}}\frac{\cos(2\beta)}{\Delta}d\beta +2\left[R^4+r^4+\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right)+2\frac{\left(R^2+r^2\right)Rr}{\eta^2}\right] \times\int_0^{\pi/2}\frac{d\beta}{(\eta^2\cos(2\beta)-\eta^2-2)\Delta}\right\} \tag{23}$$

Using [14] this integral can be obtained in the form of the complete elliptical integrals of the first second and third kind, [15] as follows,

$$\begin{aligned} I_{F4} \;=\; & \frac{k\sqrt{\eta^2+1}}{4\sqrt{Rr}}\left\{\left(R^2+r^2\right)Rr\left[\frac{3k^4-8k^2+8}{3k^4}K\left(k\right)+\frac{4k^2-8}{3k^4}E\left(k\right)\right]\right. \\ & +\left[R^4+r^4+l^2\left(R^2+r^2+Rr\right)\right]\left[\frac{k^2-2}{k^2}K\left(k\right)+\frac{2}{k^2}E\left(k\right)\right] \\ & -\left[R^4+r^4+\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right)\right]\Pi\left(-\eta^2,k\right) \\ & \left.+2\frac{\left(R^2+r^2\right)Rr}{\eta^2}\left[K\left(k\right)-\Pi\left(-\eta^2,k\right)\right]\right\} \end{aligned} \tag{24}$$

Finally, the self-inductance of the thin conical sheet inductor can be obtained in the semi-analytical form as follows,

$$L=-\frac{4\mu_0 N^2}{3l^2\left(\eta^2+1\right)^{\frac{3}{2}}}\,V \tag{25}$$

where

$$\begin{aligned} V \;=\; & \frac{\sqrt{\eta^2+1}\left(R^3+r^3\right)}{2}\left\{1+\frac{\operatorname{arctg}\left(\eta\right)}{\eta}\right\}+\frac{\sqrt{\eta^2+1}}{4k^3\sqrt{Rr}}\left\{\left[(R^4+r^4+\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right))k^4\right.\right. \\ & \left.-2(R^4+r^4+4\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right))k^2+8\left(R^2+r^2\right)Rr\right]K\left(k\right) \\ & \left.+\left[2(R^4+r^4+2\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right))k^2-8\left(R^2+r^2\right)Rr\right]E\left(k\right)\right\} \\ & -\frac{k\sqrt{\eta^2+1}}{4\sqrt{Rr}}\left[R^4+r^4+\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right)\right]\Pi\left(-\eta^2,k\right) \\ & +2\frac{k\sqrt{\eta^2+1}}{4}\frac{\left(R^2+r^2\right)Rr}{\sqrt{Rr}}\frac{\left[K\left(k\right)-\Pi\left(-\eta^2,k\right)\right]}{\eta^2}+J_0 \end{aligned} \tag{26}$$

with,

$$\begin{aligned} J_0 \;=\; & \int_0^{\pi/3}\frac{\eta}{2}\left\{R^3\operatorname{asinh}\frac{\eta R\cos\left(2\beta\right)+\eta r-l}{R\sqrt{D_0}}+r^3\operatorname{asinh}\frac{\eta r\cos\left(2\beta\right)+\eta R+l}{r\sqrt{D_0}}\right. \\ & \left.-\left(R^3+r^3\right)\operatorname{asinh}\frac{\eta(\cos\left(2\beta\right)+1)}{\sqrt{D_0}}\right\}d\beta \end{aligned} \tag{27}$$

The completely elliptical integral of the third kind can be written in the following form (See Appendix A) [15] as follows,

$$\Pi\left(-\eta^2,k\right)=\frac{K(k)}{\left(\eta^2+1\right)}+\frac{\pi}{2}\frac{\eta}{\sqrt{\eta^2+1}\sqrt{\eta^2+k^2}}\left[1-\Lambda_0\left(\varepsilon,k\right)\right] \tag{28}$$

where $\Lambda_0(\varepsilon,k)$ is the Heuman Lambda function [15] given as,

$$\Lambda_0\left(\varepsilon,k\right)=\left\{K\left(k\right)E\left(\varepsilon,k'\right)-\left[K\left(k\right)-E(k)\right]F(\varepsilon,k')\right\} \tag{29}$$

with,

$$\mathrm{eps}=\arcsin\frac{1}{\sqrt{\eta^2+1}},\quad k'^2=1-k^2$$

Now,

$$V = \frac{\sqrt{\eta^2+1}\left(R^3+r^3\right)}{2}\left\{1+\frac{\operatorname{arctg}(\eta)}{\eta}\right\}+\frac{\sqrt{\eta^2+1}}{4k^3\sqrt{Rr}}\left\{\left[(R^4+r^4+\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right))k^4\right.\right.$$
$$\left.-2(R^4+r^4+4\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right))k^2+8\left(R^2+r^2\right)Rr\right]K(k)$$
$$\left.+\left[2(R^4+r^4+2\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right))k^2-8\left(R^2+r^2\right)Rr\right]E(k)\right\}$$
$$-\frac{k}{4\sqrt{\eta^2+1}\sqrt{Rr}}\left[R^4+r^4-\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right)\right]K(k)$$
$$-\frac{\pi k \operatorname{sign}(\eta)}{8\sqrt{Rr}\sqrt{\eta^2+k^2}}\frac{1-\Lambda_0(\varepsilon,k)}{\eta}\left\{\eta^2\left[R^4+r^4+l^2\left(R^2+r^2+Rr\right)\right]+\left(\eta^2+2\right)\left(R^2+r^2\right)Rr\right\}+J_0 \quad (30)$$

Equation (30) can be used to obtain the self-inductance of the thin wall solenoid. This is the special case ($\eta = 0$ or $R = r$) when the thin conical sheet degenerates into the wall solenoid.

Finding,

$$\varepsilon = \arcsin\frac{1}{\sqrt{\eta^2+1}}, \quad \frac{d\varepsilon}{d\eta} = -\frac{1}{\eta^2+1} \to -1 \text{ for } \eta \to 0 \quad (31)$$

$$\frac{\partial}{\partial\varepsilon}\Lambda_0(\varepsilon,k) = \frac{2}{\pi}\left\{K(k)\frac{\partial}{\partial\varepsilon}E\left(\varepsilon,k'\right)-\left[K(k)-E(k)\right]\frac{\partial}{\partial\varepsilon}F(\varepsilon,k')\right\}$$
$$= \frac{2}{\pi}\left\{K(k)\sqrt{1-(1-k^2)\sin^2(\varepsilon)}-\frac{\left[K(k)-E(k)\right]}{\sqrt{1-(1-k^2)\sin^2(\varepsilon)}}\right\}$$
$$= \frac{2}{\pi}\left\{K(k)\Delta_1-\frac{\left[K(k)-E(k)\right]}{\Delta}\right\} \quad (32)$$

where,

$$\Delta_1 = \sqrt{1-k'^2\sin^2(\varepsilon)} = \sqrt{1-(1-k^2)\sin^2(\varepsilon)}$$
$$\frac{\partial}{\partial\varepsilon}\Lambda_0(\varepsilon,k) \to \frac{2}{\pi k}\left[\left(k^2-1\right)K(k)+E(k)\right] \text{ for } \eta \to 0 \quad (33)$$
$$\frac{\operatorname{arctg}(\eta)}{\eta} \to 1, \ \operatorname{sign}(\eta) = 1, \text{ for } \eta \to 0_+ \quad (34)$$

From (21), (22), and (23) the limit is,

$$\lim_{\eta\to 0}\frac{1-\Lambda_0(\varepsilon,k)}{\eta} = -\lim_{\eta\to 0}\frac{d\varepsilon}{d\eta}\frac{\partial\Lambda_0(\varepsilon,k)}{\partial\varepsilon} = \frac{2}{\pi k}\left[\left(k^2-1\right)K(k)+E(k)\right] \quad (35)$$

Equation (30) gives,

$$V_0 = V(R=r) = 2R^3+\frac{R}{2k_0^3}\left\{\left[2R^2k_0^4-\left(10R^2+3l^2\right)k_0^2+8R^2\right]K(k_0)\right.$$
$$\left.+\left[\left(6R^2+3l^2\right)k_0^2-8R^2\right]E(k_0)\right\}-\frac{R}{2k_0^3}\left\{\left[2R^2k_0^4-2R^2k_0^2\right]K(k_0)-2R^2k_0^2E(k_0)\right\}$$

or

$$V_0 = 2R^3+R/(2k_0^3)\left\{\left[-\left(8R^2+3l^2\right)k_0^2+8R^2\right]K(k)+\left[\left(4R^2+3l^2\right)k_0^2-8R^2\right]E(k_0)\right\} \quad (36)$$

with

$$k_0^2 = \frac{4R^2}{4R^2+l^2}$$
$$L_0 = -\frac{4\mu_0N^2}{3l^2}V_0 \quad (37)$$

This is the formula for calculating the self-inductance of the thin wall solenoid derived from (30) in the limit case, $\eta \to 0$. Equations (36) and (37) can be simplified which leads to the well-known formula of the thin wall solenoid obtained by Lorentz [7].

$$L_0 = \frac{2\mu_0 R^2 N^2}{3l} \left\{ \frac{l}{Rk_0} K(k_0) - \frac{l^2 - 4R^2}{Rlk_0} E(k_0) - 4\frac{R}{l} \right\} \tag{38}$$

By inspection of (26) one can find that this formula is not suitable for the calculation of the self-inductance of a thin conical sheet inductor for $\eta < 10^{-4}$ because of the the term $\frac{[K(k)-\Pi(-\eta^2,k)]}{\eta^2}$ which is given in the analytical form and can considerably oscillate when η is approaching to zero. Let us give this term in the integral form,

$$\frac{\left[K(k) - \Pi\left(-\eta^2, k\right)\right]}{\eta^2} = \frac{1}{\eta^2} \left\{ \int_0^{\pi/2} \frac{d\beta}{\Delta} - \int_0^{\pi/2} \frac{d\beta}{\left[1 + \eta^2 \sin^2(\beta)\right] \Delta} \right\} = \int_0^{\pi/2} \frac{\sin^2(\beta) d\beta}{\left[1 + \eta^2 \sin^2(\beta)\right] \Delta} \tag{39}$$

On the point of view of the numerical integration near the point $\eta \to 0_+$ it is recommended to use in (26) the term $\frac{\left[K(k)-\Pi\left(-\eta^2,k\right)\right]}{\eta^2}$ instead of the term $\frac{[K(k)-\Pi(-\eta^2,k)]}{\eta^2}$ that will imply with the excellent precision near the point in the question. In this case Equation (26) can be written as follows,

$$\begin{aligned} V \;=\; & \frac{\sqrt{\eta^2+1}\left(R^3+r^3\right)}{2}\left\{1+\frac{\operatorname{arctg}(\eta)}{\eta}\right\}+\frac{\sqrt{\eta^2+1}}{4k^3\sqrt{Rr}}\left\{\Big[(R^4+r^4+\left(R^2+r^2\right)Rr\right. \\ & +l^2\left(R^2+r^2+Rr\right))k^4-2(R^4+r^4+4\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right))k^2 \\ & +8\left(R^2+r^2\right)Rr\Big]K(k)+\Big[2(R^4+r^4+2\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right))k^2 \\ & \left.-8\left(R^2+r^2\right)Rr\Big]E(k)\right\}-\frac{k\sqrt{\eta^2+1}}{4\sqrt{Rr}}\left[R^4+r^4+\left(R^2+r^2\right)Rr+l^2\left(R^2+r^2+Rr\right)\right]\Pi\left(-\eta^2,k\right) \\ & +2\frac{k\sqrt{\eta^2+1}}{4}\frac{\left(R^2+r^2\right)Rr}{\sqrt{Rr}}T+J_0 \end{aligned} \tag{40}$$

where,

$$T = \int_0^{\pi/2} \frac{\sin^2(\beta) d\beta}{\left[1 + \eta^2 \sin^2(\beta)\right] \Delta}$$

This case will be especially treated numerically and compared by the results obtained from Equations (1) and (5). Probably, the valuable approximation of the expression (40) exists that can be the point of the future research.Equation (40) is also valuable for all values of η.

Thus, the excellent method is presented for calculating the self-inductance of a thin inclined conical sheet inductor which appears for the first time in the literature in the semi-closed form.

4. NUMERICAL VALIDATIONS

To verify the validity of the new presented formula for calculating the self-inductance of the thin conical sheet we applied the following set of examples.

Example 1. The thin conical sheet with radius $R = 0.8$ m, $r = 0.2$ m, and the height $l = 0.1$ m. The number of turns is $N = 30$.

Let us begin with the basic formula (1) where the self-inductance is obtained by the triple integration.

The self-inductance is,

$$L\,(\text{Triple}) = 0.8311790201801404\,\text{mH}$$

Using Equation (5) the self-inductance is obtained by the double integration.

$$L\,(\text{Double}) = 0.8312708025112955\,\text{mH}$$

The discrepancy between these two calculations is,

$$\varepsilon = 0.011041207134633\%$$

Let us find the self-inductance obtained by the single integration (14) and (15),

$$L\,(\text{Single}) = 0.8312708025112955\,\text{mH}$$

The discrepancy between two last calculations is,

$$\varepsilon = 0\%$$

Finally, the self-inductance is calculated by the semi-analytical approach (25) and (26) or (25) and (30).

$$L\,(\text{Semi-analytical}) = 0.8312708025112955\,\text{mH}$$

The discrepancy between two last calculations is,

$$\varepsilon = 0\%$$

Thus, all results are in an excellent agreement (the same results), but there is small discrepancy for the triple integration. The Mathematica code by default is used for numerical integration.

The new semi-analytical formula is developed for thin conical sheet inductor where the wire of the turns is of negligible cross section.

For the real coils (coils with non-negligible cross section) Tesla Conical Coil software [9] is used for calculating the self-inductance of the thin conical coil whose thickness of the wire is taken into consideration. For example, the diameter of the wire $d = 0.0124\,\text{mm}$, with the diameter of the bigger conical base $B = 2R = 160\,\text{cm}$, the diameter of the smaller conical base $A = 2r = 40\,\text{cm}$, the height $h = 10\,\text{cm}$, the number of turns $N = 30$. The standard 56 AWG, [10] is used.

AWG — American Wire Gauge is the standard way to denote wire size in North America. In AWG, the larger the number is, the smaller the wire diameter and thickness are.

For 56 AWG the diameter of wire is example $d = 0.0124\,\text{mm}$,

$$L\,(\text{Tesla Conical Coil}) = 0.8\,\text{mH}$$

The discrepancy between this calculation and the exact new formula is,

$$\varepsilon = -3.75\%$$

This discrepancy can be explained by the following facts:

1) In the exact semi-analytical formula, the wire is of negligible cross section.
2) The Harold Wheeler formula is empirical.
3) In the Conical Tesla coil calculator [9] the diameter of the wire is taken into consideration.

The results of two approaches are in particularly good agreement.

Example 2. In Table 1 the calculation of the self-inductance of the thin conical sheet is given for different radii and heights. The self-inductance is normalized inductance taken as the fraction of the total self-inductance and the number of turns N^2.

In Table 1 the calculations for the thin conical sheet are given. The results can be obtained by given formulas for triple (1) and double (5) or single (14) and (15) integration as well as with the semi-analytical formulas (25) and (26) or (25) and (30). All formulas give the same results. These results can be used as the benchmark problem for testing other methods in the calculation of the self-inductance of the thin conical sheet (for example, FEM, BEM).

Example 3. The thin wall solenoid is with radius $R = r = 2\,\text{m}$, and height $l = 2\,\text{m}$. The number of turns is $N = 100$. This calculation is the special case when the thin conical sheet degenerates to the thin wall solenoid for $R = r = 2\,\text{m}$ ($\eta = 0$).

Nomenclature:

R — radii of the bigger base of the cone,

Table 1. Normalized self-inductance of the thin conical sheet.

R (m)	r (m)	l (m)	L/N^2 (μH)
5	2	0.5	7.769058773377473
7	4	2	14.04411106829513
3	8	6	9.308799530842142
8	3	2	9.308799530842142
10	1	0.1	8.555496044682762
2	12	0.02	11.782465536112989
0.05	0.01	0.2	0.04926298901408955
0.08	0.001	1	0.00779722851569193

r — radii of the smaller base of the base,
l — height of the cone,
d — diameter of the wire in [9]

Let us begin with the exact formula for calculating the self-inductance of thin wall solenoid (38) or [7, 8].

$$L\,(\text{Wall-solenoid}) = 41.49260838538719\,\text{mH}$$

Let us apply the formula for the self-inductance of the thin conical sheet (1). The case $\eta = 0$ is not singular for Equation (1). The triple integration gives,

$$L\,(\text{Triple}\,(\eta = 0)) = 41.48958270331200\,\text{mH}$$

The discrepancy between these two calculations is,

$$\varepsilon = 7.292098985649072 \cdot 10^{-3}\%$$

Applying the formula for the self-inductance of the thin conical sheet (5), the case $\eta = 0$ is also not singular for this formula. The double integration gives,

$$L\,(\text{Double}(\eta = 0)) = 41.49260838538719\,\text{mH}$$

One obtains the same result as with the exact formula (38). Obviously $\eta = 0$ is not singularity for (1) and (5), but it is the singularity for (26) and (30). In previous calculations all figures that agree are bolded. In the limit $\eta \to 0$, formula (26) or (30) becomes (36) or (38) that has been previously proven. Thus, in the case of the thin wall solenoid the best way is to directly use the formulas [7, 8] or (36) and (38).

Example 4. In this example the Conical Tesla coil calculator [9] is used where the self-inductance is calculated by the empirical Harold Wheeler formulas [6]. Let us take the thin conical sheet with $R = 2$ m, $r = 1$ m, $l = 2$ m, and $N = 100$.

From this work Equations (26) and (30) give,

$$L = 24.66985936787412\,\text{mH}$$

In Table 2 the calculation of the self-inductance of the conical coil is given for the different diameters of the wire [9], as well as with the semi-analytical formula. The Wire Gauge Conversion [10] is used to choose the diameter of the wire.

From Table 2, one can see particularly good agreement between the results obtained by the exact semi-analytical formula and those obtained by [9]. There is discrepancy between 1.013% and 5.224% which can be explain by the fact that in the presented new formula the cross section of the wire is neglected, but in [9] the thickness of the wire is taken into consideration.

Example 5. In this example the Conical Tesla coil calculator [9] is used for calculating and comparing the results of the self-inductance of thin wall solenoid with those obtained by this presented approach. Let us take the thin wall cylinder with $R = 1$ m, $r = 1$ m, $l = 1$ m, and $N = 100$. The

Table 2. Self-inductance obtained by this work (25)–(27) and Tesla conical coil solver [9].

R (m)	r (m)	l (m)	AWG [10]	d (mm)	Tesla Conical solver [9] (mH)	Equations (25)–(27) (mH)	Discrepancy (%)
2	1	2	4/0	11.684	23.39	24.66985936787412	5.188
2	1	2	1/0	8.252	23.38	24.66985936787412	5.224
2	1	2	5	4.621	23.41	24.66985936787412	5.511
2	1	2	11	2.588	23.47	24.66985936787412	4.864
2	1	2	17	1.15	23.59	24.66985936787412	4.377
2	1	2	23	0.574	23.71	24.66985936787412	3.891
2	1	2	32	0.2032	23.90	24.66985936787412	3.121
2	1	2	41	0.071	24.10	24.66985936787412	2.310
2	1	2	49	0.0297	24.26	24.66985936787412	1.661
2	1	2	56	0.0124	24.42	24.66985936787412	1.013

Table 3. Self-inductance obtained by this work and Tesla conical coil solver.

R (m)	l (m)	AWG [10]	d (m) [10]	Tesla Conical coil [9] (mH)	Equations (1), (5), and [7, 8] (mH)	Discrepancy (%)
1	1	1/0	8.252	20.90	20.74630419269360	−0.7423
1	1	5	4.621	20.81	20.74630419269360	−0.3085
1	1	11	2.588	20.80	20.74630419269360	−0.2603
1	1	17	1.15	20.84	20.74630419269360	−0.4531
1	1	23	0.574	20.91	20.74630419269360	−0.7899
1	1	32	0.2032	21.02	20.74630419269360	−1.3207
1	1	41	0.071	21.15	20.74630419269360	−1.9474
1	1	49	0.0297	21.26	20.74630419269360	−2.4776
1	1	56	0.0124	21.37	20.74630419269360	−2.9576

software [9] can also be used for the calculation of the self-inductance of the thin wall solenoid [7] and [8].

For calculating the self-inductance of thin wall solenoid, Equations (1) and (5) are used in which one puts $\eta = 0$, as well as the exact formulas [7, 8] or (36). From Table 3 one can see that the results are in a particularly good agreement where discrepancy is between 2.9576% and −0.7423%.

Example 6. Let us investigate the behavior of Equation (29) near the point $\eta = 0_+$. The radius R takes values 3, 3.01, 3.001,..., 3.000000000000001 m, $r = 3$ m, $l = 1$ m, and $N = 100$. Calculate the self-induction of the thin conical sheet until its extreme case when it becomes thin wall solenoid $R = r = 3$ m.

Table 4 presents the results obtained by (15) and (30) as well as by the exact formula (40). All figures that agree are bolded. Obviously, all results are in especially good agreement when η is near zero. In the extreme case, Equations (15), (30), and (40) give the same result,

$$L = 10.14078494647206\,\mathrm{H}$$

Table 4. Self-inductance of the thin conical sheet when η is near to zero.

R (m)	η	$L(H)$, (15)	$L(H)$, (30)	$L(H)$, (40)
3.1	$1 \cdot 10^{-1}$	**10**.35304473682088	**10**.35304473682096	**10**.14078494647206
3.01	$1 \cdot 10^{-2}$	**10.1**6366115170091	**10.1**6366115170084	**10.1**4078494647206
3.001	$1 \cdot 10^{-3}$	**10.14**308887822766	**10.14**308887822664	**10.14**078494647206
3.0001	$1 \cdot 10^{-4}$	**10.14**101550248384	**10.14**101550248384	**10.14**078494647206
3.00001	$1 \cdot 10^{-5}$	**10.140**80800348695	**10.140**80800370144	**10.140**78494647206
3.000001	$1 \cdot 10^{-6}$	**10.14078**725232260	**10.14078**725221143	**10.14078**494647206
3.0000001	$1 \cdot 10^{-7}$	**10.14078**517428976	**10.14078**517704618	**10.14078**494647206
3.00000001	$1 \cdot 10^{-8}$	**10.1407849**7490539	**10.1407849**6952954	**10.1407849**4647206
3.000000001	$1 \cdot 10^{-9}$	**10.14078494**931539	**10.14078494**877782	**10.14078494**647206
3.0000000001	$1 \cdot 10^{-10}$	**10.140784946**75639	**10.140784946**70263	**10.140784946**47206
3.00000000001	$1 \cdot 10^{-11}$	**10.140784946**50050	**10.140784946**49518	**10.140784946**47206
3.000000000001	$1 \cdot 10^{-12}$	**10.14078494647**491	**10.14078494647**432	**10.140784946472**06
3.0000000000001	$1 \cdot 10^{-13}$	**10.14078494647**371	**10.14078494647**226	**10.14078494647**206
3.00000000000001	$1 \cdot 10^{-14}$	**10.140784946472**23	**10.140784946472**16	**10.140784946472**06
3.000000000000001	$9 \cdot 10^{-16}$	**10.140784946472**06	**10.140784946472**12	**10.140784946472**06

5. CONCLUSIONS

For the first time in the literature, the new formula for calculating the self-inductance of the thin conical sheet inductor is given. The calculation is obtained in the semi-analytical form over the complete elliptical integrals of the first, second, and third kind as well as one term that does not have the analytical solution, and it must be solved numerically. The kernel function of this integral is a continuous function on the interval of integration. All procedures of the calculation are given promptly so that the potential users can easily use them choosing the appropriate formula. Six representative numerical examples are given to validate the presented method.

APPENDIX A.

In the complete elliptic integral,

$$\Pi(n,k) = \int_0^{\pi/2} \frac{d\beta}{\left[1 - n\sin^2(\beta)\right]\sqrt{1 - k^2\sin^2(\beta)}} \tag{A1}$$

with $n = -\eta^2 < 0$.

From [15] this case can be transformed as follows,

$$\begin{aligned} n &= -\eta^2 < 0, \;\; N = \frac{k^2 - n}{1 - n}, \;\; k^2 < N < 1 \\ \Pi(n,k) &= -\frac{n\left(1 - k^2\right)}{(1 - n)\left(k^2 - n\right)}\,\Pi(N,k) + \frac{k^2}{k^2 - n}\,K(k) \\ N &= \frac{k^2 - n}{1 - n} \end{aligned} \tag{A2}$$

$$(N,k) = K(k) + \frac{\pi}{2}\,\delta_2\left[1-\Lambda_0(\varepsilon,k)\right]$$

Equation (A2) gives,

$$\begin{aligned}\Pi(n,k) &= -\frac{n\left(1-k^2\right)}{(1-n)\left(k^2-n\right)}\,\Pi(N,k)+\frac{k^2}{k^2-n}\,K(k)\\ &= \frac{k^2}{k^2+\eta^2}\,K(k)+\frac{\eta^2\left(1-k^2\right)}{\left(1+\eta^2\right)\left(k^2+\eta^2\right)}K(k)+\frac{\pi}{2}\delta_2\frac{\eta^2\left(1-k^2\right)}{\left(1+\eta^2\right)\left(k^2+\eta^2\right)}\left[1-\Lambda_0(\varepsilon,k)\right]\\ &= \frac{\eta^2k^2+k^2+\eta^2-\eta^2k^2}{\left(1+\eta^2\right)\left(k^2+\eta^2\right)}K(k)+\frac{\pi}{2}\frac{\sqrt{1+\eta^2}}{|\eta|}\frac{\sqrt{k^2+\eta^2}}{1-k^2}\frac{\eta^2\left(1-k^2\right)}{\left(1+\eta^2\right)\left(k^2+\eta^2\right)}\left[1-\Lambda_0(\varepsilon,k)\right]\end{aligned}$$

where,

$$\delta_2 = \sqrt{\frac{N}{(1-N)(N-k^2)}} = \frac{\sqrt{1+\eta^2}}{|\eta|}\frac{\sqrt{k^2+\eta^2}}{1-k^2}$$

$$\Pi(n,k) = \frac{K(k)}{\left(1+\eta^2\right)}+\frac{\pi}{2}\,\mathrm{sgn}(\eta)\,\frac{\eta}{\sqrt{1+\eta^2}}\frac{1}{\sqrt{k^2+\eta^2}}\left[1-\Lambda_0(\varepsilon,k)\right] \tag{A3}$$

$$\varepsilon = \arcsin\sqrt{\frac{1-N}{1-k^2}}\ \text{ or }\ \varepsilon = \arcsin\frac{1}{\sqrt{1+\eta^2}}.$$

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