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Electromagnetic Scattering from 2-D Conducting Objects of Arbitrary Smooth Shape: Complete Mathematical Formulation of the Method of Auxiliary Sources for E-Polarized Case

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Abstract—The study investigates the mathematical background of the method of auxiliary sources (MAS) employed in electromagnetic diffraction. Here, the mathematical formulation is developed for E-polarized plane wave diffraction by perfectly conducting two-dimensional objects of arbitrary smooth shape, and the comparison with an analytical and a numerical approach is provided in the numerical part. The results reveal a quite high accuracy among all methods. The importance of the study is to develop the complete mathematical background of MAS for two-dimensional TM-polarized electromagnetic scattering problems by conducting objects. Different from the method of moments (MoM) and other integral equation approaches in electromagnetic scattering problems, here the integral equation resulting from the boundary condition on the scatterer is solved by expanding the current density as orthonormalized Hankel's function with the argument of the distance between the scatterer actual and auxiliary surfaces. The approach can be summarized by that first the sources are shifted inside the scatterer, and second, the boundary condition is employed as the total tangential electric field is zero on the surface and inside the object. Then, such expansion leads to eliminating the singularity problems by shifting the sources from the actual surface.

1. INTRODUCTION

Electromagnetic scattering and the behavior of electromagnetic waves in the vicinity of obstacles have always been of crucial importance in science and engineering [1]. Since electromagnetic waves are employed for noninvasive detection, radar applications, antenna design, and characterization of materials and surfaces with different electromagnetic or periodic properties, numerous analytical, numerical, or analytical-numerical approaches have been developed [2–6]. Therefore, apart from the full-wave simulations, new approximate and well-defined methodologies have been proposed in the literature. One of these numerical methods is the method of auxiliary sources (MAS) which is based on the solution of the integral equations resulting from the boundary condition satisfaction on the scatterer where the currents are induced [7, 8]. The singularity problem in such integral equations is handled by shifting the currents and locating them on an auxiliary surface instead of a real surface. This is one of the main advantages of the MAS compared to other methods regarding integral equation solutions. However, still, the boundary condition is satisfied on the actual surface of the scatterer. MAS has a positive dominance which is faster computation and requires less computation power by eliminating the singularity problem when the other numerical methodologies such as MoM and FEM are considered [9, 10]. Therefore, for electrically large problems, it is very convenient to employ MAS.

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In this study, the complete mathematical theory of MAS is provided for the first time. The present article investigates the E-polarized electromagnetic diffraction by arbitrary smooth-shaped perfect electric conduction surfaces. The complete and rigorous mathematical theory was developed by Georgian Mathematicians Kupradze for general cases [11, 12]. After 1967, Kupradze suggested that the theory can be realized when there would be enough computing resources in the future. Later, the method is simplified and approximated by Zaridze et al. which requires less calculation and provides satisfactory outcomes [13–16]. Now, it is the first time to employ Kupradze's method in its initial form. Here, the particular solutions to diffraction problems such as several two-dimensional cylindrical geometries are provided, and the results are compared with analytical outcomes and MoM. One of the main aims of the study is to complete and fulfill the mathematical background of the MAS for electrodynamics since it was not clear to theoretical researchers why this method works and provides accurate results because the people working on MAS have validated their method by checking and comparing the boundary and radiation conditions in the given problem. The contribution of the study is first to reveal the validity of shifting the current densities through the auxiliary surface instead of the actual scatter surface and then, second, apart from the MoM, to solve the integral equation by expressing the current density with the orthonormalized Hankel's function as the basis.

2. FORMULATION OF THE PROBLEM

In this section, the formulation of the problem and mathematical statements are provided. To solve diffraction problems, the field components need to be expressed in mathematical form first. Then, the boundary conditions on the scatterer should be satisfied. In Figure 1, the geometry of the problem is given. Here, S stands for the actual surface of the scatterer, and S_{aux} corresponds to the auxiliary surface where the current density induced on the actual surface is shifted d amount to eliminate the singularity problem.

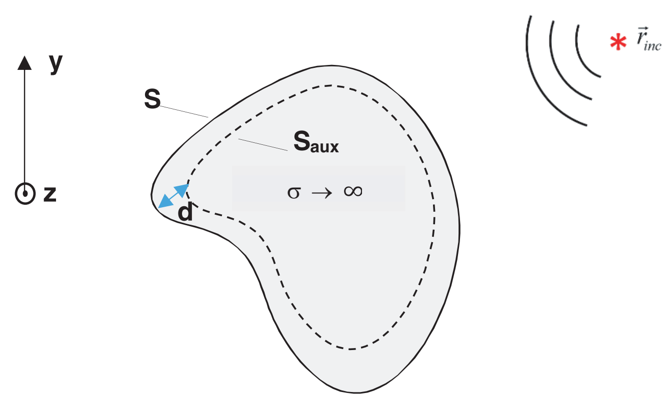

Figure 1. The geometry of the problem.

As it is known from Maxwell's equation, it is possible to express the scattered electric field in terms of Green's function convolved with the current density on the scatterer in the case of TM polarization as given in (1). It should be noted that the time dependency is $e^{-i\omega t}$ and omitted throughout the study.

$$E_{sc}(\tilde{x}) = \frac{\omega\mu}{4} \int\limits_{y} J\left(y\right) H_1^{(0)}\left(k\rho\right) dy \tag{1}$$

Here, ω is the angular frequency, μ the permeability, k the wave number, $H_1^{(0)}$ the Hankel function of the first kind and zeroth order, and $J(y)$ the electric current density on the scatter. Besides, $\rho = |\tilde{x} - y|$, $\tilde{x}$ and y are the observation and source location vectors in two-dimensional space, respectively.

The total tangential components of the electric field should vanish on the conducting scatterer surface as given below:

$$E_{sc}\left(\tilde{x}\right) + E_{inc}\left(\tilde{x}\right)\big|_{on\, S_{aux}} = 0. \tag{2}$$

Then, the following integral equation is obtained:

$$\frac{\omega\mu}{4}\int\limits_{y} J(y)\, H_1^{(0)}(k\,|\tilde{x}-y|)dy = -E_{inc}(\tilde{x}) \tag{3}$$

To solve this integral equation, there are various proposed methods in the literature. To solve (3) with Kupradze's approach, several definitions should be given here. As the first and very crucial step, a set of orthogonal functions is introduced as [11, 12]:

$$\omega_k(y) = H_1^{(0)}(k\rho_k) \tag{4}$$

Here, $\rho_k = |x_k - y|$, x_k is the variable position vector on the auxiliary surface, and it should be highlighted that the y vector denotes the scatterer surface. These functions $(\omega_k(y))$ represent the linearly independent set of functions on the S contour. However, they are not orthonormal. To express the current density as a Fourier series, we need the orthonormal set of functions. Below the orthonormalization procedure is given.

Before starting the orthonormalization procedure, the norm and the dot product between the functions are defined as follows:

$$\|X\| = \left(\int\limits_{S} |X(y)|^2\, dS\right)^{\frac{1}{2}} \tag{5}$$

$$(x_1, x_2) = \int\limits_{S} x_1(y)\,\bar{x}_2(y)\, dS \tag{6}$$

Here $X(y)$, $x_1(y)$, and $x_2(y)$ are functions in L^2 space. The bar on the function denotes the complex conjugation. After that, the orthonormalization process algorithm is given below as (7) and (8) for $\omega_k(y)$ [12].

$$\bar{\varphi}_1(y) = \frac{\bar{\omega}_1(y)}{\|\bar{\omega}_1(y)\|} \tag{7}$$

$$\bar{\varphi}_i(y) = \frac{\bar{\omega}_i(y) - \sum\limits_{k=1}^{i-1}(\bar{\omega}_i, \bar{\varphi}_k)\,\bar{\varphi}_k(y)}{\left\|\bar{\omega}_i(y) - \sum\limits_{k=1}^{i-1}(\bar{\omega}_i, \bar{\varphi}_k)\,\bar{\varphi}_k(y)\right\|} \tag{8}$$

Then, this orthonormal system can be expanded into orthogonal $\omega_k(y)$ system as (9) and (10) [11]:

$$\varphi_i(y) = \sum_{k=1}^{i} A_{ki}\omega_k(y) \tag{9}$$

$$\bar{\varphi}_i(y) = \sum_{k=1}^{i} \bar{A}_{ki}\bar{\omega}_k(y) \tag{10}$$

Here A_{ki} and $\bar{A}_{ki}$ are constant weighting coefficients. To find them, the scatterer surface is discretized, and (9) is employed for different values of y_i. Then, the system of linear algebraic equations (SLAE) is obtained. By inversion, the corresponding unknowns can be obtained as below:

$$A_{ki} = [C_{ki}]^{-1} D_i \tag{11}$$

Here, $[C_{ki}]^{-1}$ is the matrix inversion and:

$$C_{ki} = \omega_k(y_i) \tag{12}$$

$$D_i = \varphi_i(y_i) \tag{13}$$

After inversion, the constant weighting coefficients A_{ki} are obtained. Then, the current density on the scatterer surface is expressed as a Fourier series of the orthonormal basis functions $\bar{\varphi}_k(y)$ as given in (14) [11, 12]:

$$J(y) = \sum_{k=1}^{n} a_k \bar{\varphi}_k(y) \tag{14}$$

where a_k are unknown Fourier coefficients.

Here, unknown Fourier coefficients can be found by the Fourier approach by employing (6) as follows:

$$a_k = (J, \bar{\varphi}_k) = \int_S J(y)\, \varphi_k(y)\, dS \tag{15}$$

Here S stands for the scatterer surface. However, it should be noted that $J(y)$ is not known. Therefore, we need to propose another procedure such that the total electric field should vanish inside and the surface of the scatterer. Then, the following equation is obtained:

$$\frac{\omega\mu}{4} \int_y J(y)\, H_1^{(0)}(k\,|x_k - y|)dy = -E_{inc}(x_k) \tag{16}$$

Finally, (16) can be solved by the proposed approach:

First, both sides of the equation are multiplied by A_{ki} and summed as below:

$$\int_y J(y) \sum_{k=1}^{i} A_{ki}\omega_k(y)dy = -\frac{4}{\omega\mu} \sum_{k=1}^{i} A_{ki} E_{inc}(x_k) \tag{17}$$

where $\omega_k(y) = H_1^{(0)}(k\rho_k)$.

Then, summation $\sum_{k=1}^{i} A_{ki}\omega_k(y)$ at the left side of the corresponding equation is replaced by $\varphi_i(y)$, and (18) is obtained (Please look at (9)).

$$\int_y J(y)\, \varphi_i(y)dy = -\frac{4}{\omega\mu} \sum_{k=1}^{i} A_{ki} E_{inc}(x_k) \tag{18}$$

Secondly, by the definition given above, the left side of the equation becomes the Fourier coefficient and gives the ability to find these unknown Fourier coefficients.

$$a_k = \int_y J(y)\, \varphi_i(y)dy = -\frac{4}{\omega\mu} \sum_{k=1}^{i} A_{ki} E_{inc}(x_k) \tag{19}$$

As a final expression, the scattered field is defined as below:

$$E_{sc}(x) = \frac{\omega\mu}{4} \int_y H_1^{(0)}(k\rho) \sum_{k=1}^{n} a_k \bar{\varphi}_k(y)dy \tag{20}$$

The proposed approach has various advantages such as singularity elimination, approximate solution of the integral equation due to truncation and flexibility to be employed in numerous two-dimensional problems. The most important part of the theory is that the basis for the current densities is chosen as orthonormalized Hankel's function with the argument of $k\rho_k = k|x_k - y|$, and x_k is the variable position vector on the auxiliary surface. Lastly, it should be highlighted that the y vector denotes the scatterer surface.

3. NUMERICAL RESULTS

In this part, four different conducting objects are investigated. These are circular, elliptical, complex-shaped, and Cassini Oval cylinders. It should be highlighted that the study aims to fulfill the fully explained mathematical background of the MAS instead of the numerical results. However, it is crucial to compare the results with other methods such as analytical or numerical methods. In Figure 2, the normalized current density on a circular cylinder obtained by three different approaches is provided. The radius of the cylinder is a. The incidence angle (φ) is defined from x-axis. As seen from the figure, the three methods give almost the same results. The deviation is less than 0.5%.

In Figure 3, the near-electric fields for two methodologies are provided. Again the angle of incidence is 180° from the x-axis. The results are coinciding. The boundary condition for the total tangential

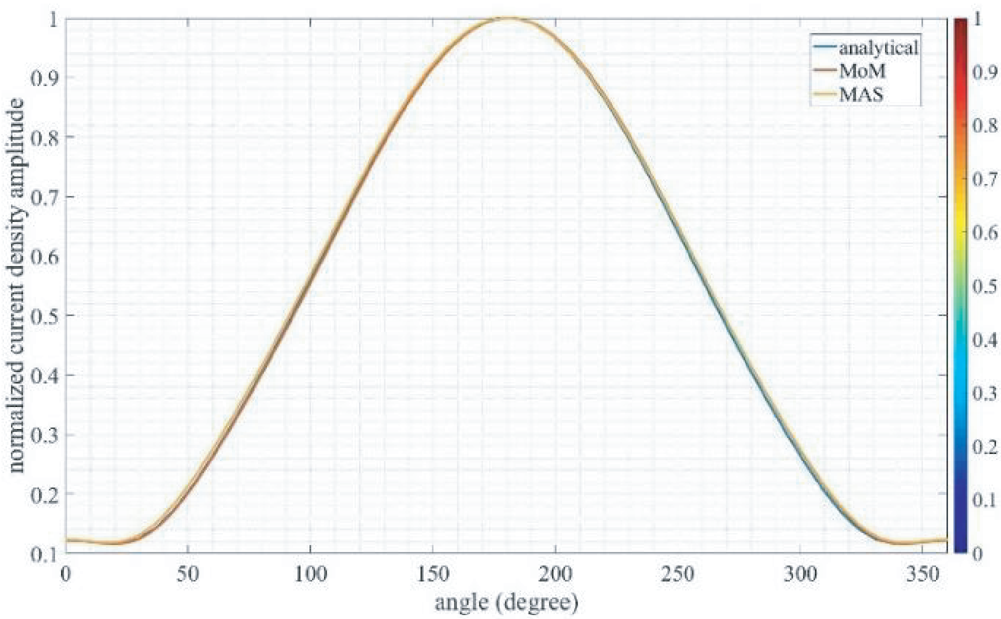

Figure 2. The normalized current distribution on the circular cylinder with respect to angle obtained by analytical, moments, and auxiliary sources methodologies [17].

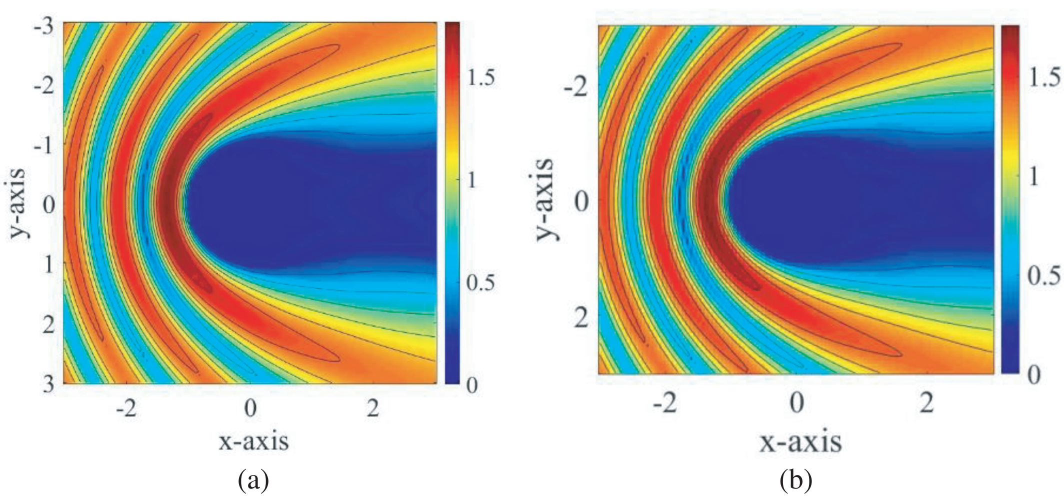

Figure 3. The electric field distribution for circular cylinder located at origin (a) MAS and (b) MoM.

electric field for both methods is satisfied, and the same shadow region is observed for the circular cylinders.

In Figure 4, normalized bi-static radar cross-section of the elliptical cylinder for E-polarized (TM-polarized case) case is provided for different wavelength values via various methods (MoM and MAS). Here, a and b correspond to lengths of the semi-major axis (x-axis) and semi-minor axis (y-axis), respectively. Only a very minor deviation is observed for high frequency which is less than 2%.

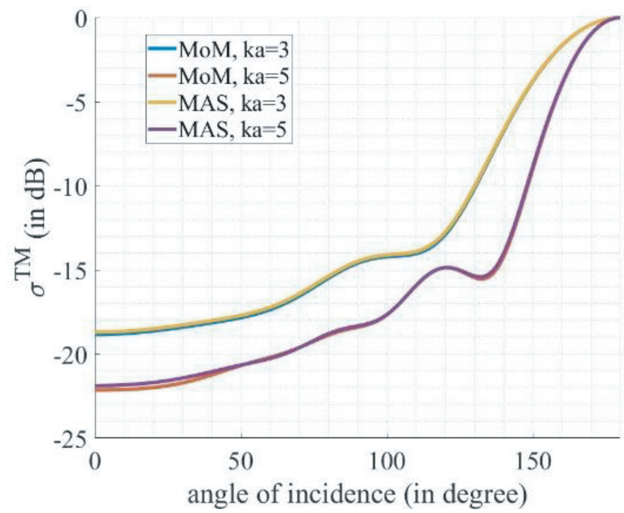

Figure 4. Bi-static radar cross-section (σ_{dB}^{TM}) for different wavelengths [18].

In Figure 5, the near electric field distribution of the elliptical cylinder is provided by MoM and MAS. They completely coincide with the given parameters.

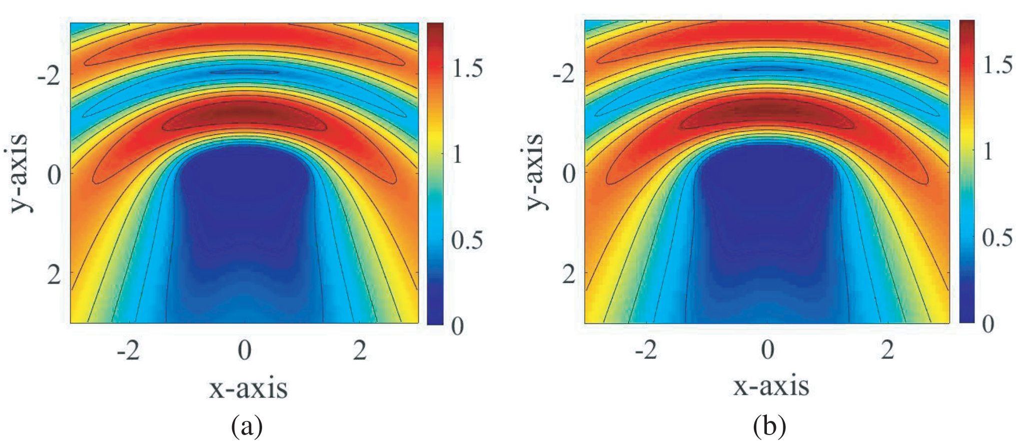

Figure 5. The electric field distribution for elliptical cylinder located at origin (a) MAS and (b) MoM.

In Figure 6, both the geometry and the diffraction by this complex geometry are provided. The dimensions of the figure are given in Table 1. For this geometry, k is chosen as 8. The boundary condition is satisfied on the surface also for higher values of wavenumber. As expected, the shadow region is observed behind the object.

In Figure 7, the Cassini Oval cylinder is investigated by MAS and MoM [19]. As it is seen, the results are coinciding. As expected, there exists a shadow region behind the object concerning the angle of incidence.

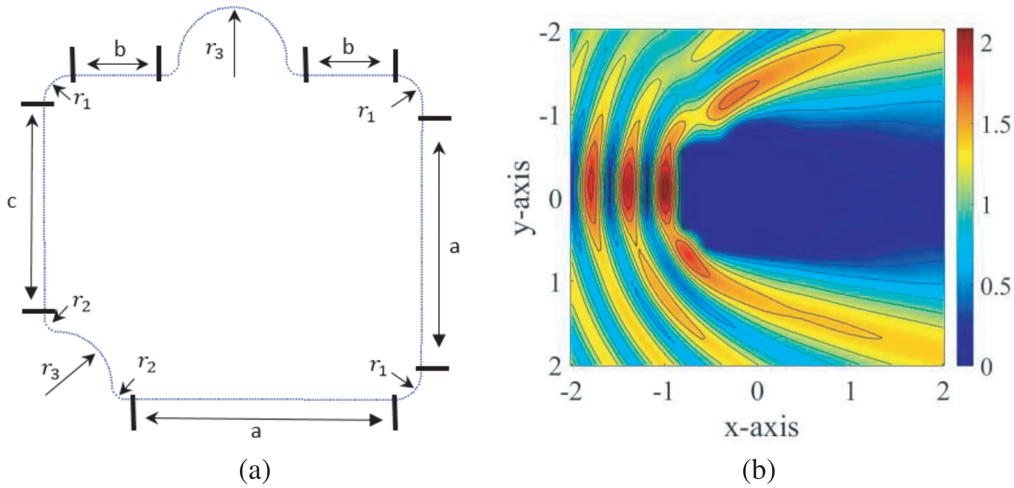

Figure 6. The geometry of the complex geometry (a) and the electric field distribution for the geometry.

Table 1. Parameters and their values for Figure 6.

Parameter	Value (m)	Parameter	Value (m)
a	1.00	r_1	0.10
b	0.35	r_2	0.05
c	0.80	r_3	0.20

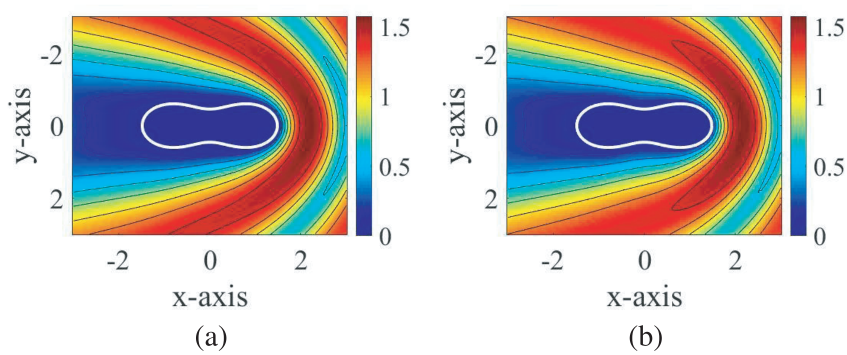

Figure 7. The Cassini shaped cylinder. (a) MAS. (b) MoM.

4. CONCLUSION

The study investigates the mathematical background of MAS employed in electromagnetic diffraction. Here, the mathematical formulation is developed for TM polarized plane wave diffraction by perfectly conducting two-dimensional objects, and the comparison with an analytical and a numerical approach is provided in the numerical part. It should be highlighted that the study aims to fulfill the fully explained mathematical background of the MAS by expressing each detail such as the expansion of the current densities, orthonormalization procedure, finding the unknown coefficients during expansion, and boundary condition satisfaction. Furthermore, several well-known perfect conducting geometries for the diffraction problems are considered to compare the outcomes. The current density, bi-static radar cross-section, and near electric field distribution of several geometries such as circular, elliptical, and complex geometries are provided in the numerical part. The main advantage of the method is to eliminate the singularity problems arising from the solution of the integral equation obtained while satisfying the boundary condition by shifting the current densities induced on the scatterer through the auxiliary surface. It is observed that the distance between the auxiliary sources and the actual surface (d) has crucial importance on the speed of the convergence and accuracy. The deviation from the analytical and moments methods is less than 3% for provided cases (current density, bi-static radar cross-section, and near electric field distribution) for all cases provided in the study. In future studies, TE polarization and 3D cases will be investigated.

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