Progress In Electromagnetics Research M, Vol. 114, 139–152, 2022

Electromagnetic Equivalence Principle Formulation for Optical Forces on Particles in Arbitrary Fields

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Abstract—The computation of the fields scattered by a dielectric sphere illuminated by a plane wave and the evaluation of the resultant optical forces is a classical problem that can be analytically solved using Mie theory. Whereas extending said formulation to arbitrary incident fields does not pose any conceptual difficulty, the actual computation of the scattering coefficients and force components substantially grows in complexity as soon as interactions beyond the electric dipole arise. By formulating an equivalent electromagnetic problem, we derive a set of computationally efficient formulas for the evaluation of scattering and optical forces exerted by arbitrary incident fields upon dielectric spheres in the Mie regime. As opposed to force calculations by direct integration of the Maxwell's Stress Tensor, the present formulation relies on a set of universal interaction coefficients that do not require any problem-specific integration and can therefore be all precomputed and tabulated. The proposed methods can be easily integrated with the T-Matrix method to calculate forces on non-spherical dielectric objects.

1. INTRODUCTION

Optical forces arise because of the redistribution of electromagnetic momentum that occurs when an electromagnetic wave is scattered by a polarizable obstacle. Once the scattering problem is solved, determining the forces acting on the particle amounts to calculating the flux of the Maxwell's stress tensor [1] through any surface fully enclosing particle (assuming that the background medium is homogeneous and lossless). That is the approach informing the Generalized Mie Theory of Optical Forces (GMTOF) [2] for the calculation of optical forces in the paradigmatic case of spherical particles of arbitrary size. The GMTOF formulation focuses on the interaction of spherical multipoles with one another and offers specific selection rules that reduce the computational complexity of evaluating the forces exerted on a sphere of arbitrary size. From a conceptual standpoint, the procedure to extend GMTOF to multi-particle systems is quite clear: the interactions between any two particles must be handled by expressing the fields scattered by each particle in terms of spherical multipoles concentric with the other particle. The addition theorem for vector spherical harmonics [3] (VSH) allows one to do just that, but it comes at the price of a very cumbersome and difficult to code formulation in terms of Wigner 3j and 6j symbols [4, 5].

The situation is quite different in the case of scattering and force calculations in the Rayleigh regime, or in other words in the case of small particles in which expansion terms beyond the electric dipole are negligible. In such instances the optical force on the object can be expressed in the following compact and physically transparent form [6–8]:

Received 30 August 2022, Accepted 9 November 2022, Scheduled 23 November 2022
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$$\begin{aligned}
\langle \mathbf{F} \rangle &= \frac{\alpha_R}{4} \nabla |\mathbf{E}_i|^2 + \alpha_I \frac{k_0}{\varepsilon_0} \frac{\langle \mathbf{S} \rangle}{c} + \alpha_I \frac{k_0}{\varepsilon_0} c \nabla \times \langle \mathbf{L}_S \rangle \\
\langle \mathbf{S} \rangle &= \frac{1}{2} Re \left[\mathbf{E}_i \times \mathbf{H}_i^* \right] \\
\langle \mathbf{L}_S \rangle &= \frac{\varepsilon_0}{4\,\omega\, i} \mathbf{E}_i \times \mathbf{E}_i^*
\end{aligned} \tag{1}$$

In expressions (1), α_R and α_I are the real and imaginary parts of the particle's polarizability respectively. The first term is known as the gradient force, the second term is the radiation pressure, and the less familiar last term is often referred to as spin force since it is associated to the nonuniform distribution of the spin density of the light field [6]. Aside from being physically informative and elegantly compact, what is noteworthy in the formulation (1) is the fact that it features only incident fields and intrinsic particle properties. As such, formulas (1) are eminently practical, both in terms of analysis and design. That is because formulas (1) allow for the accurate computation of optical forces even under the most complex field distributions [9–12], while entirely bypassing the onerous step of solving the scattering problem. Further, the extension to multiparticle systems in the Rayleigh regime is immediate and without any of the complications arising in the more general Mie regime. Such versatility comes at the price of a domain of applicability restricted to subwavelength particles.

For all the stated reasons we considered it of great practical utility to introduce a computational framework offering the ease of application of the Rayleigh regime approach while retaining the generality of the GMTOF. The paper is organized as follows. In Section 2, we introduce the notation and the definitions of recurring quantities. In Section 3, we outline the geometry and the properties of the scattering problem. In Section 4, we introduce an efficient method to decompose a general electromagnetic field in VSHs. The advantage of the proposed method is that no problem-specific projection integrals must be explicitly computed. The present formulation includes instead a set of universal coefficients that can be easily precalculated/tabulated. In Section 5, the scattering problem is reformulated in terms of equivalent sources. In Section 6, we exploit the equivalent sources introduced in Section 5 to obtain a general expression of the force acting on a particle in terms of only the Mie coefficients and spatial derivatives of the incident electromagnetic field. In Section 7, we draw our conclusions and highlight the advantages of the proposed methods. In Appendix A, we provide precomputed universal coefficients up to order 2. In Appendix B, we verify the consistency of the present method with well-known analytical results for dipolar particles.

2. DEFINITIONS AND NOTATION

A time harmonic dependence $e^{-i\omega t}$ is assumed throughout the paper unless otherwise specified. Vectors are indicated in boldface. Unit vectors are denoted by a circumflex accent, i.e., $\hat{r}$. ε_0 is the dielectric permittivity of vacuum, and μ_0 is the magnetic permeability of vacuum. The relative permittivity and permeability are denoted with a subscript appropriate to the region of interest, i.e., $\varepsilon_1\mu_1$, etc. Similarly, the propagation constant for region i is denoted as $k_i = \omega\sqrt{\varepsilon_0\varepsilon_i\mu_0\mu_i}$, and the intrinsic impedance is denoted as $\eta_i = \sqrt{(\mu_0\mu_i)/(\varepsilon_0\varepsilon_i)}$.

This article follows the notation of Korn and Korn [13] for spherical Bessel functions, and of Bohren and Huffman for VSH [14]. For the benefit of the reader, we repeat below the definitions of the VSHs that are recurring throughout the paper:

$$\begin{aligned}
\psi_{enm}^{(s)}(r,\theta,\phi) &= \cos m\,\phi\, P_n^m(\cos\theta) z_n^{(s)}(k\,r) \\
\psi_{onm}^{(s)}(r,\theta,\phi) &= \sin m\,\phi\, P_n^m(\cos\theta) z_n^{(s)}(k\,r) \\
\mathbf{M}_{pnm}^{(s)}(r,\theta,\phi) &= \nabla \times \left[\mathbf{r}\,\psi_{pnm}^{(s)}(r,\theta,\phi)\right] \\
\mathbf{N}_{pnm}^{(s)}(r,\theta,\phi) &= \nabla \times \mathbf{M}_{pnm}^{(s)}(r,\theta,\phi)/k
\end{aligned} \tag{2}$$

The function $z_n^{(s)}(kr)$ is a solution of the spherical Bessel equation, with a propagation constant k appropriate for the region of interest. For different values of the superscript s we have: $z_n^{(1)}(x) = j_n(x)$

(spherical Bessel function of the first kind), $z_n^{(2)}(x) = y_n(x)$ (spherical Bessel function of the second kind), and $z_n^{(3)}(x) = j_n(x)+iy_n(x) = h_n^{(1)}(x)$ (spherical Hankel function of the first kind). The functions $P_n^m(\cos\theta)$ are associated Legendre's functions of the first kind.

As a shorthand, we introduce the $(\alpha\beta\gamma)$ notation to denote partial derivatives of any vectorial function $\mathbf{V}(x, y, z)$ evaluated at the origin of the reference system:

$$\mathbf{V}^{(\alpha,\beta,\gamma)} = \frac{1}{k^{\alpha+\beta+\gamma}} \left[\frac{\partial^\alpha}{\partial x'^\alpha}\frac{\partial^\beta}{\partial y'^\beta}\frac{\partial^\gamma}{\partial z'^\gamma}\mathbf{V}\left(x', y', z'\right)\right]_{\substack{x'=0\\ y'=0\\ z'=0}} \tag{3}$$

3. FORMULATION OF THE SCATTERING PROBLEM

The geometry of interest is shown in Fig. 1, a sphere of relative permittivity ε_2 and radius R is surrounded by a background medium (henceforth "medium 1") of relative permittivity ε_1. The sphere is assumed to be centered at the origin of an appropriate spherical reference system $r\theta\phi$. The structure is illuminated by an electromagnetic field $\mathbf{E}_i\mathbf{H}_i$ consistent with Maxwell's equations in medium 1. The incident field is completely general, with the sole requirement of being non-singular in the region $r \leq R$ (in other words, all sources of the incident field are assumed to be in the region $r > R$). The internal fields are denoted by $\mathbf{E}_c\mathbf{H}_c$ and the scattered fields are denoted by $\mathbf{E}_s$, $\mathbf{H}_s$.

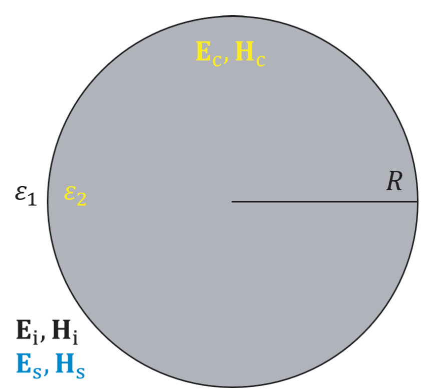


Figure 1. Layout of the scattering problem.

A generic incident field can always be expressed in a vector spherical harmonics basis as:

$$\begin{aligned}\mathbf{E}_i &= \sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[\mathbf{E}_{pnm}^{TM}+\mathbf{E}_{pnm}^{TE}\right] = \sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[E_{pnm}^{TM}\mathbf{N}_{pnm}^{(1)}+E_{pnm}^{TE}\mathbf{M}_{pnm}^{(1)}\right]\\ \mathbf{H}_i &= \sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[\mathbf{H}_{pnm}^{TM}+\mathbf{H}_{pnm}^{TE}\right] = \sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[\frac{E_{pnm}^{TM}}{i\,\eta_1}\mathbf{M}_{pnm}^{(1)}+\frac{E_{pnm}^{TE}}{i\,\eta_1}\mathbf{N}_{pmn}^{(1)}\right]\end{aligned} \tag{4}$$

For the VSHs $\mathbf{M}_{pnm}^{(1)}\mathbf{N}_{pnm}^{(1)}$ we adopt the same formulation and notation as in Bohren and Huffman [14], with the superscript (1) indicating a radial dependence in terms of spherical Bessel functions of the first kind [15] $j_n(k_1r)$. The index $p \in [eo]$ indicates even or odd azimuthal symmetry of the spherical harmonics. The constants A_{pnm}, B_{pnm}, which have units of V/m, depend on the incident field alone and owing to the orthogonality properties of the vector spherical harmonics, can be calculated by taking inner product of the vector $\mathbf{E}_i$ and the relevant basis function. In Section 4 we present an alternative and more efficient method to determine the expansion constants.

The fields in the internal region, i.e., for $r < R$, are expanded as:

$$\begin{aligned}\mathbf{E}_c &= \sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[c_n E_{pnm}^{TM}\mathbf{N}_{pnm}^{(1)} + d_n E_{pnm}^{TE}\mathbf{M}_{pnm}^{(1)}\right]\\ \mathbf{H}_c &= \frac{1}{i\,\eta_2}\sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[c_n E_{pnm}^{TM}\mathbf{M}_{pnm}^{(1)} + d_n E_{pnm}^{TE}\mathbf{N}_{pnm}^{(1)}\right]\end{aligned} \tag{5}$$

In the expansion (5) the radial dependence of the vector spherical harmonics is in terms of spherical Bessel functions of the first kind $j_n(k_2 r)$.

Finally, the scattered fields in the region $r > R$ are expressed as:

$$\begin{aligned}\mathbf{E}_s &= \sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[a_n E_{pnm}^{TM}\mathbf{N}_{pnm}^{(3)} + b_n E_{pnm}^{TE}\mathbf{M}_{pnm}^{(3)}\right]\\ \mathbf{H}_s &= \frac{1}{i\,\eta_1}\sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[a_n E_{pnm}^{TM}\mathbf{M}_{pnm}^{(3)} + b_n E_{pnm}^{TE}\mathbf{N}_{pnm}^{(3)}\right]\end{aligned} \tag{6}$$

In formulas (6) the superscript (3) indicates that the spherical harmonics have a radial dependence in terms of spherical Hankel functions of the first kind [15] $h_n^{(1)}(k_1 r)$.

In the following we will refer to the nondimensional coefficients a_n, b_n, c_n, d_n as scattering coefficients, whereas we will refer to products like $A_{pnm}a_n$ as scattering amplitudes. By imposing the continuity of the tangential fields at $r = R$, all the scattering coefficients can be easily determined. Importantly, in the linear regime, the scattering coefficients do not depend on the expansion coefficients A_{pnm}, B_{pnm} of the incident field, but only on the geometry, the material properties, and the wavelength of the problem of interest. As such, these coefficients are intrinsic properties of the system and describe its ability to scatter the various spherical harmonics at the wavelength of interest. Additionally, the scattering coefficients bear no dependence on the azimuthal parity e/o or index m. The expressions of these scattering coefficients are widely available in the literature, but for notational consistency with this work we like to refer the reader to Bohren and Huffman [14].

4. EXPANSION OF AN ARBITRARY FIELD IN VECTOR SPHERICAL HARMONICS

The completeness property of vector spherical harmonics allows for the expansion on any nonsingular field in the form (4). Exploiting the orthogonality of this vectorial basis [14], the expansion coefficients in (4) can be obtained as:

$$\begin{aligned}E_{pnm}^{TM} &= \frac{\int_0^{2\pi}\int_0^{\pi}\mathbf{E}_i\cdot\mathbf{N}_{pnm}^{(1)}\sin\theta\,d\theta\,d\phi}{\int_0^{2\pi}\int_0^{\pi}\mathbf{N}_{pnm}^{(1)}\cdot\mathbf{N}_{pnm}^{(1)}\sin\theta\,d\theta\,d\phi}\\ E_{pnm}^{TE} &= i\eta_1\frac{\int_0^{2\pi}\int_0^{\pi}\mathbf{H}_i\cdot\mathbf{N}_{pnm}^{(1)}\sin\theta\,d\theta\,d\phi}{\int_0^{2\pi}\int_0^{\pi}\mathbf{N}_{pnm}^{(1)}\cdot\mathbf{N}_{pnm}^{(1)}\sin\theta\,d\theta\,d\phi}\end{aligned} \tag{7}$$

Expressions (7) are integrals over an arbitrary spherical surface of radius a. If one chooses $a \to 0$, the electric and magnetic fields and various spherical harmonics can be replaced by their asymptotic expressions for small radii, and a Taylor expansion can be performed:

$$\begin{cases} \mathbf{E}_i(\mathbf{r})\big|_{r=a} = \sum_{s=0}^{\infty} \frac{a^s}{s!} (\hat{\mathbf{r}} \cdot \nabla')^s \, \mathbf{E}_i(\mathbf{r}')\big|_{r'=0} = \sum_{s=0}^{\infty} \sum_{\alpha+\beta+\gamma=s} (k_1\, a)^s T_{\alpha\beta\gamma}(\theta,\phi) \mathbf{E}_i^{(\alpha,\beta,\gamma)} \\ \mathbf{H}_i(\mathbf{r})\big|_{r=a} = \sum_{s=0}^{\infty} \frac{a^s}{s!} (\hat{\mathbf{r}} \cdot \nabla')^s \, \mathbf{H}_i(\mathbf{r}')\big|_{r'=0} = \sum_{s=0}^{\infty} \sum_{\alpha+\beta+\gamma=s} (k_1\, a)^s T_{\alpha\beta\gamma}(\theta,\phi) \mathbf{H}_i^{(\alpha,\beta,\gamma)} \\ T_{\alpha\beta\gamma}(\theta,\phi) = \frac{(\sin\theta\cos\phi)^{\alpha} \, (\sin\theta\sin\phi)^{\beta} \, (\cos\theta)^{\gamma}}{\alpha!\beta!\gamma!} \end{cases} \tag{8}$$

In (8) we have used the multinomial theorem [16] after expressing the fields in Cartesian coordinates and used the notation (3).

With expansions (8), in the limit we can equivalently express the coefficients (7) as:

$$\begin{cases} E_{pnm}^{TM} = \frac{(k_1 a)^{-2(n-1)}}{N_{nm}} \sum_{s=0}^{\infty} \sum_{\alpha+\beta+\gamma=s} (k_1\, a)^s \, \mathbf{E}_i^{(\alpha,\beta,\gamma)} \cdot \int_0^{2\pi} \int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi) \mathbf{N}_{pnm}^{(1)} \sin\theta \, d\theta \, d\phi \\ E_{pnm}^{TE} = i\,\eta_1 \frac{(k_1 a)^{-2(n-1)}}{N_{nm}} \sum_{s=0}^{\infty} \sum_{\alpha+\beta+\gamma=s} (k_1\, a)^s \cdot \mathbf{H}_i^{(\alpha,\beta,\gamma)} \int_0^{2\pi} \int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi) \mathbf{N}_{pnm}^{(1)} \sin\theta \, d\theta \, d\phi \end{cases} \tag{9}$$

The parameter N_{nm} in expressions (9) is given, in the limit $a \to 0$, by:

$$N_{nm} = (1+\delta_{m,0}) \frac{2\pi}{(2n+1)^2} \frac{(n+m)!}{(n-m)!} n(n+1)^2 \left[\frac{2^{n-1}(n-1)!}{(2n-1)!} \right]^2 \tag{10}$$

We introduce the following notation for the remaining vector integrals:

$$\mathbf{S}_{pnm}^{\alpha\beta\gamma} = \frac{1}{(k_1 a)^{n-1} \, N_{nm}} \int_0^{2\pi} \int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi) \mathbf{N}_{pnm}^{(1)} \sin\theta \, d\theta \, d\phi \tag{11}$$

We note that the $\mathbf{S}_{pnm}^{\alpha\beta\gamma}$ coefficients as defined by (11) are universal, in the sense that they bear no dependence on the fields being expanded, or any other problem-specific parameter. Consequently, all $\mathbf{S}_{pnm}^{\alpha\beta\gamma}$ can be precomputed and tabulated. Table A1 in Appendix A reports all $\mathbf{S}_{pnm}^{\alpha\beta\gamma}$ coefficients computed up to the quadrupolar order ($n = 2$).

Based on these considerations, the coefficients (9) reduce to the following expressions:

$$\begin{aligned} E_{pnm}^{TM} &= \sum_{\alpha'+\beta'+\gamma'=n-1} \mathbf{S}_{pnm}^{\alpha'\beta'\gamma'} \cdot \mathbf{E}_i^{(\alpha',\beta',\gamma')} \\ E_{pnm}^{TE} &= i\eta_1 \sum_{\alpha'+\beta'+\gamma'=n-1} \mathbf{S}_{pnm}^{\alpha'\beta'\gamma'} \cdot \mathbf{H}_i^{(\alpha',\beta',\gamma')} \end{aligned} \tag{12}$$

Figs. 2 and 3 show the application of the proposed expansion methods to the case of Hermite-Gauss beams. The expansion was in both cases truncated at $n = 10$.

5. EQUIVALENT ELECTROMAGNETIC SOURCES

Two different but equivalent perspectives can be adopted for the calculation of various electromagnetic quantities, such as energy, momentum, and forces. In the case of forces, one can concentrate either on the fields or on the sources of those fields. Force computations using the Maxwell's stress tensor deal exclusively with the total fields, i.e., incident plus scattered fields, whereas force computations via the Lorentz force [17] consider the interaction of the incident field with the sources of the scattered fields. From the physical point of view, such sources are polarization charges and currents induced on the scatterer. The surface equivalence principle in electromagnetics [18] offers an additional option: a

n	m	E^{TM}_{enm}	E^{TM}_{onm}	E^{TE}_{enm}	E^{TE}_{onm}
1	0	0	0	0	0
1	1	−1.42401	0	0	0. − 1.42401 *i*
2	0	0	0	0	0
2	1	0. − 0.713098 *i*	0	0	0.713098
2	2	0	0	0	0
3	0	0	0	0	0
3	1	0.426819	0	0	0. + 0.426819 *i*
3	2	0	0	0	0
3	3	0	0	0	0
4	0	0	0	0	0
4	1	0. + 0.267354 *i*	0	0	−0.267354
4	2	0	0	0	0
4	3	0	0	0	0
4	4	0	0	0	0
5	0	0	0	0	0
5	1	−0.168292	0	0	0. − 0.168292 *i*
5	2	0	0	0	0
5	3	0	0	0	0
5	4	0	0	0	0
5	5	0	0	0	0

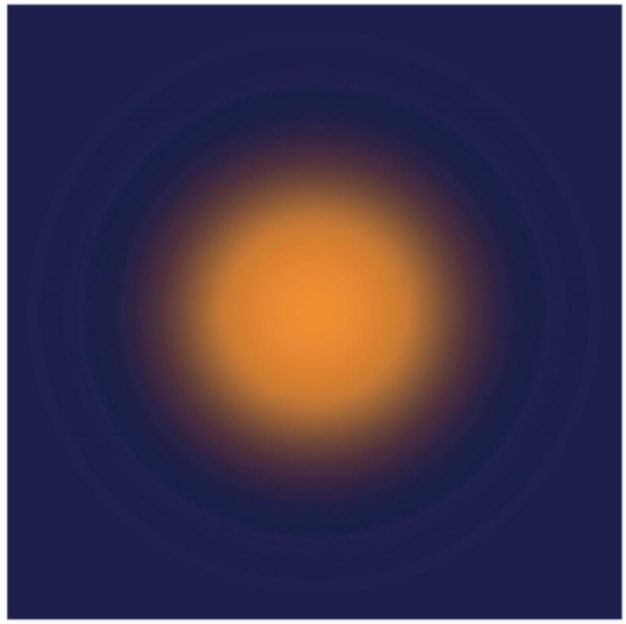


Figure 2. Expansion coefficients and intensity profile of a horizontally polarized gaussian beam with spot size $\mathbf{w_0} = \lambda$.

n	m	E^{TM}_{enm}	E^{TM}_{onm}	E^{TE}_{enm}	E^{TE}_{onm}
1	0	0. + 0.641029 *i*	0	0	0
1	1	0	0	0	0
2	0	−0.960132	0	0	0
2	1	0	0	0	0
2	2	0.168562	0	0	0. + 0.168562 *i*
3	0	0. − 1.14394 *i*	0	0	0
3	1	0	0	0	0
3	2	0. + 0.100566 *i*	0	0	−0.100566
3	3	0	0	0	0
4	0	1.18519	0	0	0
4	1	0	0	0	0
4	2	−0.0625593	0	0	0. − 0.0625593 *i*
4	3	0	0	0	0
4	4	0	0	0	0
5	0	0. + 1.10673 *i*	0	0	0
5	1	0	0	0	0
5	2	0. − 0.0389485 *i*	0	0	0.0389485
5	3	0	0	0	0
5	4	0	0	0	0
5	5	0	0	0	0

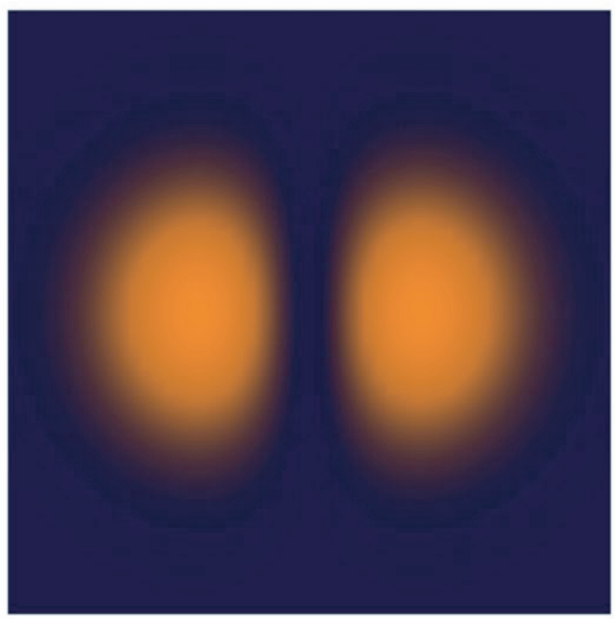


Figure 3. Expansion coefficients and intensity profile of a horizontally polarized Gauss-Hermite beam of order 1, 0 and spot size $\mathbf{w_0}=\lambda$.

scattering object and the associated physical sources of the scattered fields (i.e., dielectric polarization) can be replaced by a set of fictitious sources that radiate in the background medium and reproduce exactly the scattered fields in a certain region of interest. That is the approach that we adopt in the following. We must note that typically the reformulation of an equivalent problem does not offer any

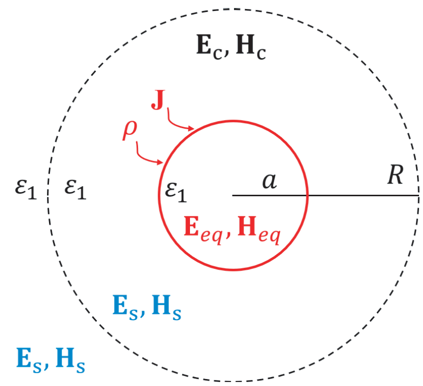


Figure 4. Layout of the equivalent problem.

computational advantage, since the full scattering problem must be solved in order to determine the equivalent sources. The present case is a lucky exception, since Mie theory offers close form scattering coefficients up to any VSH order.

Once the scattering coefficients are calculated, an equivalent problem can be formulated [18] in terms of appropriate distributions of charges and currents. With reference to Fig. 4, the idea is to find a set of equivalent sources contained in a region $r \leq a$, with $a \ll \lambda$, such that the same scattered field distribution (6) is recovered for $r \geq R$. To that end, the permittivity ε_2 is replaced by the background permittivity ε_1, and the domain of the scattered fields (6) is extended inward up to $r = a$. In the interior region $r < a$ the electric and magnetic fields can be set to any arbitrary distribution fulfilling Maxwell's equations. Any such distribution can always be expressed as a superposition of non-singular vector spherical harmonics ($\mathbf{M}^{(1)}\mathbf{N}^{(1)}$) as follows:

$$\begin{aligned}\mathbf{E}_{eq} &= \sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[u_n E_{pnm}^{TM}\mathbf{N}_{pnm}^{(1)} + v_n E_{pnm}^{TE}\mathbf{M}_{pnm}^{(1)}\right]\\ \mathbf{H}_{eq} &= \frac{1}{i\ \eta_1}\sum_{p\in[e,o]}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\left[u_n E_{pnm}^{TM}\mathbf{M}_{pnm}^{(1)} + v_n E_{pnm}^{TE}\mathbf{N}_{pnm}^{(1)}\right]\end{aligned} \tag{13}$$

The coefficients uv in (13) are chosen to ensure continuity of the tangential components of the scattered electric field at $r = a$, i.e.:

$$\left(\mathbf{E}_s|_{r=a}\right)\times\hat{\mathbf{r}} = \left(\mathbf{E}_{eq}|_{r=a}\right)\times\hat{\mathbf{r}} \tag{14}$$

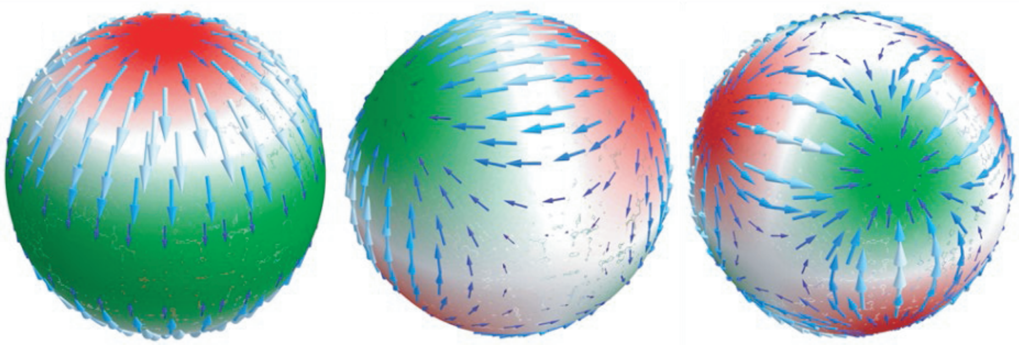
Under the constraint (14), the coefficients uv are given by:

$$\begin{aligned}u_n &= \frac{\left(\mathbf{N}_{pnm}^{(3)}\Big|_{r=a}\times\hat{\mathbf{r}}\right)\cdot\hat{\mathbf{t}}}{\left(\mathbf{N}_{pnm}^{(1)}\Big|_{r=a}\times\hat{\mathbf{r}}\right)\cdot\hat{\mathbf{t}}}\quad a_n = \frac{h_n^{(1)}(k_1a) + k_1a\,\dfrac{dh_n^{(1)}(k_1a)}{d(k_1a)}}{j_n(k_1a) + k_1a\,\dfrac{dj_n(k_1a)}{d(k_1a)}}a_n = U_n a_n\\ v_n &= \frac{\left(\mathbf{M}_{pnm}^{(3)}\Big|_{r=a}\times\hat{\mathbf{r}}\right)\cdot\hat{\mathbf{t}}}{\left(\mathbf{M}_{pnm}^{(1)}\Big|_{r=a}\times\hat{\mathbf{r}}\right)\cdot\hat{\mathbf{t}}}\quad b_n = \frac{h_n^{(1)}(k_1a)}{j_n(k_1a)}b_n = V_n b_n\end{aligned} \tag{15}$$

The unit vector $\hat{t}$ in Equation (15) represents either $\hat{\theta}$ or $\hat{\phi}$. With the expansion coefficients (15), a discontinuity remains on the radial components of the electric displacement $\mathbf{D}$ and on the tangential components of the magnetic field $\mathbf{H}$, indicating that a surface charge and a surface current distribution must be placed at $r = a$ in order to produce the correct scattered fields (6) in the external region. In particular, we have the following equivalent sources:

$$\rho = \delta(r-a)\varepsilon_0\varepsilon_1\left(\mathbf{E}_s - \mathbf{E}_{eq}\right)\cdot\hat{\mathbf{r}} \tag{16}$$

$$\mathbf{J} = \delta(r-a)\left(\mathbf{H}_s - \mathbf{H}_{eq}\right)\times\hat{\mathbf{r}} \tag{17}$$



(a) (b) (c)

Figure 5. Equivalent charges and current distributions associated with the modes (a) TM_{10}, (b) TM_{21}, and (c) TM_{32}. The red (green) areas indicate a positive (negative) charge density. The arrows indicate the equivalent surface current density.

The equivalent sources (16) and (17) obey charge conservation laws as actual charges would, i.e., $\nabla \cdot \mathbf{J} = -\partial_t \rho$. The examples in Fig. 5 show such spatial relation between equivalent charges and currents for a few modal components.

6. FORCE CALCULATIONS WITH EQUIVALENT SOURCES

In this section we compute the force exerted by the incident fields upon the equivalent sources (16) and(17). Since radius a of the auxiliary surface supporting the equivalent sources can be arbitrarily chosen, we choose a to be vanishingly small. Such a choice allows us to replace all spherical Bessel and Hankel functions by their asymptotic forms [15] for small values of the radial coordinate. The explicit expression of the equivalent surface charge density reduces to:

$$\rho \sim \delta(r-a)\ \varepsilon_0\varepsilon_1 \sum_{n=1}^{\infty}\sum_{m=0}^{n} a_n in \frac{(2n+1)!}{2^n n!}\left[E_{enm}^{TM}\cos(m\,\phi)+E_{onm}^{TM}\sin(m\,\phi)\right]\frac{P_n^m(\cos\theta)}{(k_1 a)^{n+2}} \tag{18}$$

Since the surface charge density is due to the discontinuity of the radial components of the relevant electric fields, only TM components contribute to (18). The equivalent surface current density can be written as a superposition $\mathbf{J} = \mathbf{J}^{TM} + \mathbf{J}^{TE}$ of the following TE and TM contributions as follows:

$$\begin{aligned}\mathbf{J}^{TM} =& \delta(r-a)\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{E_{enm}^{TM}}{\eta_1}a_n\frac{(2n+1)!}{2^n(n+1)!}\left[-\cos(m\phi)\frac{dP_n^m(\cos\theta)}{d\theta}\hat{\theta}+m\sin(m\phi)\frac{P_n^m(\cos\theta)}{\sin\theta}\hat{\varphi}\right]\frac{1}{(k_1a)^{n+1}}\\ &+\delta(r-a)\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{E_{onm}^{TM}}{\eta_1}a_n\frac{(2n+1)!}{2^n(n+1)!}\left[-\sin(m\phi)\frac{dP_n^m(\cos\theta)}{d\theta}\hat{\theta}-m\cos(m\phi)\frac{P_n^m(\cos\theta)}{\sin\theta}\hat{\varphi}\right]\frac{1}{(k_1a)^{n+1}}\end{aligned} \tag{19}$$

$$\begin{aligned}\mathbf{J}^{TE} =& \delta(r-a)\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{E_{enm}^{TE}}{\eta_1}b_n\frac{(2n+1)!}{2^n n!}\left[m\sin(m\phi)\frac{P_n^m(\cos\theta)}{\sin\theta}\hat{\theta}+\cos(m\,\phi)\frac{dP_n^m(\cos\theta)}{d\theta}\hat{\varphi}\right]\frac{1}{(k_1a)^{n+2}}\\ &+\delta(r-a)\sum_{n=1}^{\infty}\sum_{m=0}^{n}\frac{E_{onm}^{TE}}{\eta_1}b_n\frac{(2n+1)!}{2^n n!}\left[-m\cos(m\phi)\frac{P_n^m(\cos\theta)}{\sin\theta}\hat{\theta}+\sin(m\phi)\frac{dP_n^m(\cos\theta)}{d\theta}\hat{\varphi}\right]\frac{1}{(k_1a)^{n+2}}\end{aligned} \tag{20}$$

In the presence of a generic time-harmonic electromagnetic field $\mathbf{E_i}$, $\mathbf{H_i}$, the equivalent surface charge density (18) is subject to the following average force:

$$\langle \mathbf{F}\rangle = \frac{1}{2}Re\left[\mathbf{F}_\rho+\mathbf{F}_J\right] = \frac{1}{2}Re\left[\iiint \rho\mathbf{E}_i^* dV + \mu_0\iiint \mathbf{J}\times\mathbf{H}_i^* dV\right] \tag{21}$$

The volume integrals in (21) are extended over all space, but they reduce to surface integral owing to the delta functions in (18) and (19):

$$\mathbf{F}_\rho = \varepsilon_0\varepsilon_1 \sum_{n=1}^{\infty}\sum_{m=0}^{n} a_n i \frac{(2n+1)!}{2^n(n-1)!k_1^{n+2}} \int_0^{2\pi} \left[E_{enm}^{TM}\cos(m\,\phi) + E_{onm}^{TM}\sin(m\,\phi)\right] \int_0^{\pi} \frac{\mathbf{E}_i^*|_{r=a}}{a^n} P_n^m(\cos\theta)\sin\theta d\theta d\phi \tag{22}$$

Since the electric field in (22) is evaluated for $r = a \to 0$ and is nonsingular, we can substitute the vector Taylor expansion (8), thus obtaining the following expression for $\mathbf{F}_\rho$:

$$\mathbf{F}_\rho = \frac{\varepsilon_0\varepsilon_1}{k_1^2} \sum_{s=0}^{\infty}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\alpha+\beta+\gamma=s} a_n\,(k_1\,a)^{s-n}\left[Q_{enm}^{\alpha\beta\gamma}E_{enm}^{TM} + Q_{onm}^{\alpha\beta\gamma}E_{onm}^{TM}\right]\left[\mathbf{E}_i^{(\alpha,\beta,\gamma)}\right]^* \tag{23}$$

In Equation (23) we have defined the following numerical coefficients:

$$\begin{cases} Q_{enm}^{\alpha\beta\gamma} = \dfrac{i(2n+1)!}{2^n(n-1)!} \displaystyle\int_0^{2\pi}\int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi)\cos(m\,\phi)P_n^m(\cos\theta)\sin\theta d\theta d\phi \\ Q_{onm}^{\alpha\beta\gamma} = \dfrac{i(2n+1)!}{2^n(n-1)!} \displaystyle\int_0^{2\pi}\int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi)\sin(m\,\phi)P_n^m(\cos\theta)\sin\theta d\theta d\phi \end{cases} \tag{24}$$

We note that the $Q_{pnm}^{\alpha\beta\gamma}$ coefficients as defined by (24) are universal, in the sense that they bear no dependence on the fields being expanded, or any other problem-specific parameter. Consequently, all $Q_{pnm}^{\alpha\beta\gamma}$ can be precomputed and tabulated. Table A2 in Appendix A reports all the nonzero $Q_{pnm}^{\alpha\beta\gamma}$ coefficients up to the quadrupolar order ($n = 2$). The coefficients (24) introduce a series of restrictions. Most importantly, the coefficients (24) are identically zero for $\alpha + \beta + \gamma < n$; therefore, in the limit $a \to 0$, only the term $s = n$ must be retained in (23):

$$\mathbf{F}_\rho = \frac{\varepsilon_0\varepsilon_1}{k_1^2} \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\alpha+\beta+\gamma=n} a_n\left[Q_{enm}^{\alpha\beta\gamma}E_{enm}^{TM} + Q_{onm}^{\alpha\beta\gamma}E_{onm}^{TM}\right]\mathbf{E}_i^{*(\alpha,\beta,\gamma)} \tag{25}$$

As a consistency check, expression (25) is independent of the radius a of the arbitrary sphere selected to formulate the equivalent problem.

The second force component $\mathbf{F}_J$ is due to the interaction of the incident magnetic field with the equivalent currents (19):

$$\mathbf{F}_J = \mu_0 \iiint \mathbf{J}\times\mathbf{H}_i^* dV \tag{26}$$

The magnetic field is expanded according to (8), so that plugging (19) and (20) in (26) and rearranging the terms we obtain:

$$\begin{aligned} \mathbf{F}_J \;=\; & \frac{\mu_0}{k_1^2\eta_1} \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{\infty}\sum_{\alpha+\beta+\gamma=s} a_n\,(k_1 a)^{s-(n-1)}\left[E_{enm}^{TM}\mathbf{I}_{enm}^{TM\,\alpha\beta\gamma} + E_{onm}^{TM}\mathbf{I}_{onm}^{TM\,\alpha\beta\gamma}\right]\times\mathbf{H}_i^{*(\alpha,\beta,\gamma)} \\ & + \frac{\mu_0}{k_1^2\eta_1} \sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{s=0}^{\infty}\sum_{\alpha+\beta+\gamma=s} b_n\,(k_1 a)^{s-n}\left[E_{enm}^{TE}\mathbf{I}_{enm}^{TE\,\alpha\beta\gamma} + E_{onm}^{TE}\mathbf{I}_{onm}^{TE\,\alpha\beta\gamma}\right]\times\mathbf{H}_i^{*(\alpha,\beta,\gamma)} \end{aligned} \tag{27}$$

In (27) we have introduced the following vector coefficients:

$$\begin{cases} \mathbf{I}_{enm}^{TM\,\alpha\beta\gamma} = \dfrac{(2n+1)!}{2^n(n+1)!} \displaystyle\int_0^{2\pi}\int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi)\left[-\cos(m\,\phi)\frac{dP_n^m(\cos\theta)}{d\theta}\hat{\theta} + m\sin(m\,\phi)\frac{P_n^m(\cos\theta)}{\sin\theta}\hat{\varphi}\right]\sin\theta\,d\theta\,d\phi \\ \mathbf{I}_{onm}^{TM\,\alpha\beta\gamma} = \dfrac{(2n+1)!}{2^n(n+1)!} \displaystyle\int_0^{2\pi}\int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi)\left[-\sin(m\,\phi)\frac{dP_n^m(\cos\theta)}{d\theta}\hat{\theta} - m\cos(m\,\phi)\frac{P_n^m(\cos\theta)}{\sin\theta}\hat{\varphi}\right]\sin\theta d\theta d\phi \end{cases} \tag{28}$$

$$\begin{cases} \mathbf{I}_{enm}^{TE\,\alpha\beta\gamma} = \dfrac{(2n+1)!}{2^n n!} \displaystyle\int_0^{2\pi}\int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi) \left[m\sin(m\,\phi)\dfrac{P_n^m(\cos\theta)}{\sin\theta}\hat{\theta} + \cos(m\,\phi)\dfrac{dP_n^m(\cos\theta)}{d\theta}\hat{\varphi} \right] \sin\theta d\theta d\phi \\ \mathbf{I}_{onm}^{TE\,\alpha\beta\gamma} = \dfrac{(2n+1)!}{2^n n!} \displaystyle\int_0^{2\pi}\int_0^{\pi} T_{\alpha\beta\gamma}(\theta,\phi) \left[-m\cos(m\,\phi)\dfrac{P_n^m(\cos\theta)}{\sin\theta}\hat{\theta} + \sin(m\,\phi)\dfrac{dP_n^m(\cos\theta)}{d\theta}\hat{\varphi} \right] \sin\theta d\theta d\phi \end{cases} \tag{29}$$

Just like the coefficients $\mathbf{S}_{pnm}^{\alpha\beta\gamma}$ and $Q_{pnm}^{\alpha\beta\gamma}$, the coefficients $\mathbf{I}_{pnm}^{\alpha\beta\gamma}$ as defined by (28) and (29) are universal and can be precomputed, since they are independent of any problem-specific parameter. Tables A3 and A4 in Appendix A report all the nonzero $\mathbf{I}_{pnm}^{\alpha\beta\gamma}$ coefficients up to the quadrupolar order ($n = 2$). Noting that the coefficients (28) are identically zero for $\alpha+\beta+\gamma < n-1$, and the coefficients (29) are identically zero for $\alpha+\beta+\gamma < n$, in the limit $a \to 0$, the summations over the s index in (27) reduce to a single term:

$$\begin{aligned} \mathbf{F}_J \;=\; & \frac{\mu_0}{k_1^2\eta_1}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\alpha+\beta+\gamma=n-1} a_n \left[E_{enm}^{TM}\mathbf{I}_{enm}^{TM\,\alpha\beta\gamma} + E_{onm}^{TM}\mathbf{I}_{onm}^{TM\,\alpha\beta\gamma} \right] \times \mathbf{H}_i^{*(\alpha,\beta,\gamma)} \\ & + \frac{\mu_0}{k_1^2\eta_1}\sum_{n=1}^{\infty}\sum_{m=0}^{n}\sum_{\alpha+\beta+\gamma=n} b_n \left[E_{enm}^{TE}\mathbf{I}_{enm}^{TE\,\alpha\beta\gamma} + E_{onm}^{TE}\mathbf{I}_{onm}^{TE\,\alpha\beta\gamma} \right] \times \mathbf{H}_i^{*(\alpha,\beta,\gamma)} \end{aligned} \tag{30}$$

Combining (25) and (30), along with the coefficients (12), leads to a particularly useful result: a force expression in terms of the incident field and its derivatives only. In the proposed formulation, the treatment of multiparticle systems is straightforward. As an example, Fig. 6 shows a two-dimensional map of the force field generated by 2 Rayleigh particles, and the predicted stable optical binding sites [19]. The position of the stable binding sites is consistent with the experimentally observed formation of hexagonal lattices [20] in this type of systems.

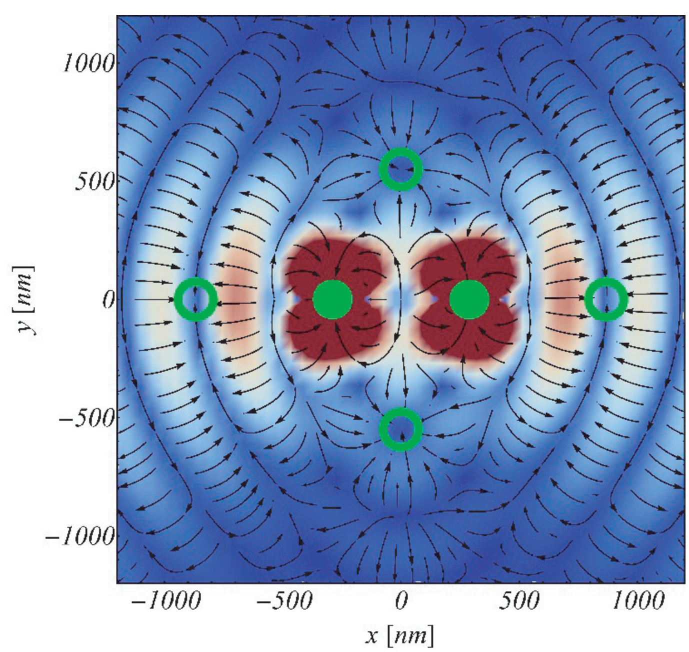


Figure 6. Optical forces generated by two Rayleigh particles (solid green disks) illuminated by an x-polarized plane wave of wavelength $\lambda = \mathbf{600}$ nm. The empty green circles indicate the stable optical binding sites for this configuration.

7. CONCLUSIONS

The analytical results presented in this paper allow one to calculate the force experienced by a Mie particle in a general electromagnetic field distribution without having to explicitly solve the scattering problem, and without any integration involving the parameters of the problem at hand. Notice in fact that all coefficients $\mathbf{S}Q\mathbf{I}$ are universal and can be precomputed. All the properties of the particle are captured by the well-known Mie coefficients $a_n b_n$. The T-Matrix method can be easily integrated in the proposed formulation to treat non-spherical objects, in which case the Mie coefficients $a_n b_n$ are multiplied by the appropriate correction factors. The present method is well-suited to treat multiparticle systems since the proposed field decomposition method is based on spatial derivatives of the fields only and hence bypasses the implementation of the translation theorem for vector spherical harmonics to evaluate the scattering of spherical multipoles from off-center particles. The proposed analytical approach provides a versatile technique to predict both numerically and analytically optomechanical interactions in complex field distributions and in multiparticle settings.

APPENDIX A. TABLE OF UNIVERSAL COEFFICIENTS $\mathbf{S}^{\alpha\beta\gamma}_{\mathbf{PNM}}$, $\mathbf{Q}^{\alpha\beta\gamma}_{\mathbf{PNM}}$, AND $\mathbf{I}^{\alpha\beta\gamma}_{\mathbf{PNM}}$

Table A1. $\mathbf{S}^{\alpha\beta\gamma}_{\mathbf{pnm}}$ coefficients up to the quadrupolar order.

n	m	α	β	γ	$\mathbf{S}^{\alpha\beta\gamma}_{enm}$	$\mathbf{S}^{\alpha\beta\gamma}_{onm}$
1	0	0	0	0	$(3/2)\hat{z}$	$\mathbf{0}$
1	1	0	0	0	$-(3/2)\hat{x}$	$-(3/2)\hat{y}$
2	0	1	0	0	$-(5/6)\hat{x}$	$\mathbf{0}$
2	0	0	1	0	$-(5/6)\hat{y}$	$\mathbf{0}$
2	0	0	0	1	$(5/3)\hat{z}$	$\mathbf{0}$
2	1	1	0	0	$-(5/6)\hat{z}$	$\mathbf{0}$
2	1	0	1	0	$\mathbf{0}$	$-(5/6)\hat{z}$
2	1	0	0	1	$-(5/6)\hat{x}$	$-(5/6)\hat{y}$
2	2	1	0	0	$(5/12)\hat{x}$	$(5/12)\hat{y}$
2	2	0	1	0	$-(5/12)\hat{y}$	$(5/12)\hat{x}$
2	2	0	0	1	$\mathbf{0}$	$\mathbf{0}$

APPENDIX B. ELECTRIC DIPOLE

As a consistency check, using formulas (23) and (30), we derive the force on a particle with electric dipolar response only ($a_1 = a_{1r} + i\, a_{1i}$). The relevant coefficients are the following:

$$\begin{cases} E^{TM}_{e10} = \dfrac{3}{2}E_{iz} \\ E^{TM}_{o10} = 0 \\ E^{TM}_{e11} = -\dfrac{3}{2}E_{ix} \\ E^{TM}_{o11} = -\dfrac{3}{2}E_{iy} \end{cases} \begin{cases} Q^{100}_{e10} = 0 \\ Q^{100}_{o10} = 0 \\ Q^{100}_{e11} = -4\, i\, \pi \\ Q^{100}_{o11} = 0 \end{cases} \begin{cases} Q^{010}_{e10} = 0 \\ Q^{010}_{o10} = 0 \\ Q^{010}_{e11} = 0 \\ Q^{010}_{o11} = -4\, i\, \pi \end{cases} \begin{cases} Q^{001}_{e10} = 4\, i\, \pi \\ Q^{001}_{o10} = 0 \\ Q^{001}_{e11} = 0 \\ Q^{001}_{o11} = 0 \end{cases} \begin{cases} \mathbf{I}^{TM\,000}_{e10} = -4\pi\, \hat{\mathbf{z}} \\ \mathbf{I}^{TM\,000}_{o10} = \mathbf{0} \\ \mathbf{I}^{TM\,000}_{e11} = 4\pi\, \hat{\mathbf{x}} \\ \mathbf{I}^{TM\,000}_{o11} = 4\pi\, \hat{\mathbf{y}} \end{cases} \quad \text{(B1)}$$

Table A2. All nonzero $\mathbf{Q}_{\mathbf{pnm}}^{\alpha\beta\gamma}$ coefficients up to the quadrupolar order.

n	m	α	β	γ	$Q_{enm}^{\alpha\beta\gamma}$	$Q_{onm}^{\alpha\beta\gamma}$
1	0	0	0	1	$4i\,\pi$	0
1	1	1	0	0	$-4i\,\pi$	0
1	1	0	1	0	0	$-4i\,\pi$
2	0	2	0	0	$-4i\,\pi$	0
2	0	0	2	0	$-4i\,\pi$	0
2	0	0	0	2	$8i\,\pi$	0
2	1	1	0	1	$-24i\,\pi$	0
2	1	0	1	1	0	$-24i\,\pi$
2	2	2	0	0	$24i\,\pi$	0
2	2	0	2	0	$-24i\,\pi$	0
2	2	0	0	2	0	0
2	2	1	1	0	0	$48i\,\pi$

Table A3. $\mathbf{I}_{\mathbf{pnm}}^{\mathbf{TM}\,\alpha\beta\gamma}$ coefficients up to the quadrupolar order.

n	m	α	β	γ	$\mathbf{I}_{enm}^{TM\,\alpha\beta\gamma}$	$\mathbf{I}_{onm}^{TM\,\alpha\beta\gamma}$
1	0	0	0	0	$-(4\pi)\hat{z}$	$\mathbf{0}$
1	1	0	0	0	$(4\pi)\hat{x}$	$(4\pi)\hat{y}$
2	0	1	0	0	$(4\pi)\hat{x}$	$\mathbf{0}$
2	0	0	1	0	$(4\pi)\hat{y}$	$\mathbf{0}$
2	0	0	0	1	$-(8\pi)\hat{z}$	$\mathbf{0}$
2	1	1	0	0	$(12\pi)\hat{z}$	$\mathbf{0}$
2	1	0	1	0	$\mathbf{0}$	$(12\pi)\hat{z}$
2	1	0	0	1	$(12\pi)\hat{x}$	$(12\pi)\hat{y}$
2	2	1	0	0	$-(24\pi)\hat{x}$	$-(24\pi)\hat{y}$
2	2	0	1	0	$(24\pi)\hat{y}$	$-(24\pi)\hat{x}$
2	2	0	0	1	$\mathbf{0}$	$\mathbf{0}$

The time averaged force is given by:

$$\begin{aligned}\langle \mathbf{F} \rangle &= \frac{1}{2} Re \left[\frac{\varepsilon_0 \varepsilon_1}{k_1^2} \sum_{m=0}^{1} \sum_{\alpha+\beta+\gamma=1} a_1 \left[E_{e1m}^{TM} Q_{e1m}^{\alpha\beta\gamma} + E_{o1m}^{TM} Q_{o1m}^{\alpha\beta\gamma} \right] \mathbf{E}_i^{*(\alpha,\beta,\gamma)} \right. \\ &\quad \left. + \frac{\mu_0}{k_1^2 \eta_1} \sum_{m=0}^{1} a_1 \left[E_{e1m}^{TM} \mathbf{I}_{e1m}^{TM\,000} + E_{o1m}^{TM} \mathbf{I}_{o1m}^{TM\,000} \right] \times \mathbf{H}_i^{*(0,0,0)} \right] \qquad \text{(B2)} \\ &= \frac{1}{2} Re \left[-\frac{3\,i\,\pi \varepsilon_0 \varepsilon_1 a_1}{k_1^3} \left(\mathbf{E}_i|_{\mathbf{r}=0} \cdot \nabla \right) \mathbf{E}_i^*|_{\mathbf{r}=0} - \frac{3\pi a_1}{k_1^2} \frac{\mathbf{E}_i|_{\mathbf{r}=0} \times \mathbf{H}_i^*|_{\mathbf{r}=0}}{c/\sqrt{\varepsilon_1}} \right] \qquad \text{(B3)}\end{aligned}$$

Formula (B2) can be further simplified using the following vector identities [6, 7, 17]:

$$\begin{cases} Re[(\mathbf{E} \cdot \nabla)\mathbf{E}^*] = \frac{1}{2}\nabla(\mathbf{E} \cdot \mathbf{E}^*) - Re\left[\mathbf{E} \times \nabla \times \mathbf{E}^*\right] = \frac{1}{2}\nabla(\mathbf{E} \cdot \mathbf{E}^*) - Im\left[\omega\,\mu_0 \mathbf{E} \times \mathbf{H}^*\right] \\ Im[(\mathbf{E} \cdot \nabla)\mathbf{E}^*] = -\frac{1}{2i}\nabla \times (\mathbf{E} \times \mathbf{E}^*) \end{cases} \qquad \text{(B4)}$$

Table A4. All nonzero $\mathbf{I}_{\mathbf{pnm}}^{\mathbf{TE}\,\alpha\beta\gamma}$ coefficients up to the quadrupolar order.

n	m	α	β	γ	$\mathbf{I}_{enm}^{TE\,\alpha\beta\gamma}$	$\mathbf{I}_{onm}^{TE\,\alpha\beta\gamma}$
1	0	1	0	0	$-(4\pi)\hat{y}$	$\mathbf{0}$
1	0	0	1	0	$(4\pi)\hat{x}$	$\mathbf{0}$
1	1	1	0	0	$\mathbf{0}$	$\mathbf{0}$
1	1	0	1	0	$(4\pi)\hat{z}$	$-(4\pi)\hat{z}$
1	1	0	0	1	$-(4\pi)\hat{y}$	$(4\pi)\hat{x}$
2	0	1	0	1	$-(12\pi)\hat{y}$	$\mathbf{0}$
2	0	0	1	1	$(12\pi)\hat{x}$	$\mathbf{0}$
2	1	2	0	0	$(12\pi)\hat{y}$	$\mathbf{0}$
2	1	0	2	0	$\mathbf{0}$	$-(12\pi)\hat{x}$
2	1	0	0	2	$-(12\pi)\hat{y}$	$(12\pi)\hat{x}$
2	1	1	1	0	$-(12\pi)\hat{x}$	$(12\pi)\hat{y}$
2	1	1	0	1	$\mathbf{0}$	$-(12\pi)\hat{z}$
2	1	0	1	1	$(12\pi)\hat{z}$	$\mathbf{0}$
2	2	2	0	0	$\mathbf{0}$	$(24\pi)\hat{z}$
2	2	0	2	0	$\mathbf{0}$	$-(24\pi)\hat{z}$
2	2	0	0	2	$\mathbf{0}$	$\mathbf{0}$
2	2	1	1	0	$-(48\pi)\hat{z}$	$\mathbf{0}$
2	2	1	0	1	$(24\pi)\hat{y}$	$-(24\pi)\hat{x}$
2	2	0	1	1	$(24\pi)\hat{x}$	$(24\pi)\hat{y}$

Using (B4), the force (B2) can be expressed as:

$$\langle \mathbf{F} \rangle = \frac{3\,\pi\varepsilon_0\varepsilon_1 a_{1i}}{k_1^3}\frac{1}{4}\nabla\left|\mathbf{E}_i\right|^2 - \frac{3\pi a_{1r}}{2k_1^2} Re\left[\frac{\mathbf{E}_i\times\mathbf{H}_i^*}{c/\sqrt{\varepsilon_1}}\right] - \frac{1}{2i}\frac{3\;\pi\varepsilon_0\varepsilon_1 a_{1r}}{2k_1^3}\nabla\times\left(\mathbf{E}_i\times\mathbf{E}_i^*\right) \tag{B5}$$

The result (B5) coincides with the well-known expression (1) once the TM_{01} Mie coefficient $a_1 = a_{1r} + i\,a_{1i}$is written in terms of the dipolar polarizability$\alpha = \alpha_r + i\,\alpha_i$:

$$\begin{cases} \alpha_r = \dfrac{3\,\pi\varepsilon_0\varepsilon_1}{k_1^3} a_{1i} \\ \alpha_I = -\dfrac{3\,\pi\varepsilon_0\varepsilon_1}{k_1^3} a_{1r} \end{cases} \tag{B6}$$

ACKNOWLEDGMENT

A. Salandrino acknowledges the support of the Air Force Office of Scientific Research (AFOSR) through the grant FA9550-20-1-0046.

REFERENCES

1. Stratton, J. A., *Electromagnetic Theory*, John Wiley & Sons, 2007.
2. Salandrino, A., S. Fardad, and D. N. Christodoulides, "Generalized Mie theory of optical forces," *JOSA B*, Vol. 29, No. 4, 855–866, 2012.
3. Felderhof, B. and R. Jones, "Addition theorems for spherical wave solutions of the vector Helmholtz equation," *Journal of Mathematical Physics*, Vol. 28, No. 4, 836–839, 1987.

4. Rotenberg, M., R. Bivins, N. Metropolis, J. K. Wooten, and L. Biedenharn, "The 3-j and 6-j symbols," *Physics Today*, Vol. 13, No. 10, 52, 1960.
5. Messiah, A., *Quantum Mechanics: Volume II*, North-Holland Publishing Company Amsterdam, 1962.
6. Albaladejo, S., M. I. Marqués, M. Laroche, and J. J. Sáenz, "Scattering forces from the curl of the spin angular momentum of a light field," *Phys. Rev. Lett.*, Vol. 102, No. 11, 113602, 2009.
7. Gordon, J. P., "Radiation forces and momenta in dielectric media," *Phys. Rev. A*, Vol. 8, No. 1, 14–21, 1973.
8. Chaumet, P. C. and M. Nieto-Vesperinas, "Time-averaged total force on a dipolar sphere in an electromagnetic field," *Opt. Lett.*, Vol. 25, No. 15, 1065–1067, 2000.
9. Fardad, S., A. Salandrino, A. Samadi, M. Heinrich, Z. Chen, and D. N. Christodoulides, "Scattering detection of a solenoidal Poynting vector field," *Opt. Lett.*, Vol. 41, No. 15, 3615–3618, 2016, [Online], Available: http://ol.osa.org/abstract.cfm?URI=ol-41-15-3615.
10. Salandrino, A. and D. N. Christodoulides, "Negative index Clarricoats-Waldron waveguides for terahertz and far infrared applications," *Opt. Express*, Vol. 18, No. 4, 3626–3631, Feb. 15, 2010.
11. Salandrino, A. and D. N. Christodoulides, "Reverse optical forces in negative index dielectric waveguide arrays," *Opt. Lett.*, Vol. 36, No. 16, 3103–3105, 2011.
12. Butler, C., S. Fardad, A. Sincore, M. Vangheluwe, M. Baudelet, and M. Richardson, *Multispectral Optical Tweezers for Molecular Diagnostics of Single Biological Cells (SPIE BiOS)*, SPIE, 2012.
13. Korn, G. A. and T. M. Korn, *Mathematical Handbook for Scientists and Engineers: Definitions, Theorems, and Formulas for Reference and Review*, Courier Corporation, 2000.
14. Bohren, C. F. and D. R. Huffman, *Absorption and Scattering of Light by Small Particles*, John Wiley & Sons, 2008.
15. Abramowitz, M. and I. A. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*, Courier Corporation, 1964.
16. Koshy, T., *Discrete Mathematics with Applications*, Elsevier, 2004.
17. Jackson, J. D., *Classical Electrodynamics*, Wiley, 1999.
18. Balanis, C. A., *Advanced Engineering Electromagnetics*, Wiley Online Library, 2012.
19. Burns, M. M., J.-M. Fournier, and J. A. Golovchenko, "Optical matter: Crystallization and binding in intense optical fields," *Science*, Vol. 249, No. 4970, 749–754, 1990.
20. Han, F. and Z. Yan, "Phase transition and self-stabilization of light-mediated metal nanoparticle assemblies," *ACS Nano*, Vol. 14, No. 6, 6616–6625, 2020.