

One-Step Absolutely Stable FDTD Methods for Electromagnetic Simulation

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Abstract—As the explicit finite-difference time-domain (FDTD) method is restricted by the well-known Courant-Friedrichs-Lewy (CFL) stability condition and is inefficient for solving numerical tasks with fine structures, various implicit methods have been proposed to tackle the problem, while many of them adopt time-splitting schemes that generally need at least two sub-steps to finish update at a full time step, and the strategies used seem to be an unnatural habit of computation compared with the most widely-used one-step methods. The procedure of splitting time step also reduces computational efficiency and makes implementation of these algorithms complex. In the present paper, two novel one-step absolutely stable FDTD methods including one-step alternating-direction-implicit (ADI) and one-step locally-one-dimensional (LOD) methods are proposed. The two proposed methods are derived from the original ADI-FDTD method and LOD-FDTD method through some linear operations applied to the original methods and are algebraically equivalent to the original methods respectively, but they both avoid the appearance of intermediate fields and are one-step method just like the conventional FDTD method. Numerical experiments are carried out for validation of the two proposed methods, and from the numerical results it can be concluded that the proposed methods can solve equation correctly and are simpler than the original methods, and their computation efficiency is close to that of the existing one-step leapfrog ADI-FDTD method.

1. INTRODUCTION

The finite-difference time-domain (FDTD) method has been considered as one of the most popular temporal methodologies for solutions of electromagnetic problems and has also been comprehensively applied to a wide range of problems [1, 2]. Being an explicit finite difference algorithm, however, the FDTD method is strictly limited by the well-known Courant-Friedrichs-Lewy (CFL) stability condition [2] which states that the time step size is positively related to spatial mesh sizes. As a result, when there are fine structures in the calculation domain, the FDTD method has to employ time steps with sizes in a strictly constrained value domain, and the iteration of a large number of time steps appears unavoidably, which is a heavy burden of computer time. In order to overcome the CFL stability condition and improve computational efficiency, various implicit difference schemes have been adopted. Many implicit FDTD methods have been proposed, and all those methods show valuable performance improvement, such as hybrid implicit-explicit (HIE) FDTD method [3, 4], Crank-Nicolson (CN) FDTD method [5, 6], alternating-direction-implicit (ADI) FDTD method [7, 8], locally-one-dimensional (LOD) FDTD method [9, 10], and Weighted Laguerre Polynomial (WLP) FDTD method [11, 12]. Among the implicit methods mentioned above, the HIE-FDTD method only applies implicit scheme to the spatial derivative in the direction along which there are fine elements, while taking general explicit scheme for the other spatial derivatives in the directions along which there are no fine structures. The HIE-FDTD method shows obvious performance improvement for problems with fine structures only in one direction.

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Nevertheless, since the HIE-FDTD method just finishes the limit from spatial cells only in one direction on the time step sizes, when there are fine structures at least in two directions, the HIE-FDTD method will also confront the same problem that the conventional FDTD method shoulders. The CN-FDTD method and WLP-FDTD method are both absolutely stable algorithms where time step sizes are not limited by spatial cell sizes in all the directions but dependent on the tolerance of numerical accuracy, yet the two methods both result in a large irreducible sparse matrix that is very expensive to solve. To improve calculation efficiency, different factor-splitting schemes are introduced into the CN-FDTD method and later are also transplanted into the WLP-FDTD method from the improved CN-FDTD methods just mentioned and realize great progress in computation efficiency, but the implementations of the new methods also become more complex and need more memory [13–16]. The ADI-FDTD method and LOD-FDTD method are also two absolutely stable FDTD methods, only need to handle low-dimensional matrices, and have been attracting the attention of researchers for a long time even till now [17–20]. However, the ADI-FDTD method that splits a time step into two sub-steps does not seem to be a natural habit of calculation as the conventional FDTD method and also makes the process of computation complex. The LOD-FDTD method also adopts time step-splitting schemes and in some cases may have more sub-steps than two for the purpose of more low numerical dispersion error, while at the same time the algorithm becomes even more complicated with the expansion of number of sub-steps. As the complexity of the splitting-step methods increases, the implementations of the two algorithms in the main computation domain, the treatment of boundary conditions, and the implementation of source and some other key elements in a algorithm all become even more difficult. On the other hand, more intermediate steps of calculation also mean more memory cost and reduction of computation efficiency. In order to improve the performance of splitting-step methods, some researchers resorted to alternative forms of the original methods without intermediate variables [21, 22], and there are also researchers developing a new form of ADI-FDTD method by eliminating redundancy in the method and simplifying it [23]. It must be pointed out that all the techniques applied in [21–23] are linear operations of the original methods.

In this paper, efforts are made to design novel absolutely stable FDTD algorithms that do not require intermediate sub-steps and do not need intermediate field variables, namely, the one-step ADI-FDTD method and the one-step LOD-FDTD method in two-dimensional situation. The two proposed one-step absolutely stable FDTD methods are derived from the original ADI-FDTD method and LOD-FDTD method through not too many steps of linear operations applied to the original methods, respectively. The two proposed methods are also algebraically equivalent to the original ADI-FDTD method and LOD-FDTD method. The proposed one-step algorithms are simpler than the original methods and should also have the same numerical property as those of the original methods in that the proposed algorithms and the original methods where the proposed methods are derived from are algebraically equivalent.

The rest of the paper is organized as follows. In Section 2, the frameworks of the two novel one-step absolutely stable FDTD methods are described in detail. In Section 3, numerical experiments are carried out to verify the two proposed methods, and discussions are also listed. Then in the final Section 4, the conclusions about the two proposed methods are presented.

2. FORMULATIONS OF THE PROPOSED METHODS

For simplicity, both the proposed ADI-FDTD method and proposed LOD-FDTD method are discussed in a 2D TE case with simple, isotropic, and lossless media.

2.1. Formulation of the Proposed One-Step ADI-FDTD Algorithm

In the assumed situation, the original ADI-FDTD method [7] is expressed as

$$E_x^{n+1/2} = E_x^n + \frac{\Delta t}{2\varepsilon} \frac{\partial H_z^n}{\partial y} \quad (1)$$

$$E_y^{n+1/2} = E_y^n - \frac{\Delta t}{2\varepsilon} \frac{\partial H_z^{n+1/2}}{\partial x} \quad (2)$$

$$H_z^{n+1/2} = H_z^n + \frac{\Delta t}{2\mu} \frac{\partial E_x^n}{\partial y} - \frac{\Delta t}{2\mu} \frac{\partial E_y^{n+1/2}}{\partial x} \quad (3)$$

$$E_x^{n+1} = E_x^{n+1/2} + \frac{\Delta t}{2\varepsilon} \frac{\partial H_z^{n+1}}{\partial y} \quad (4)$$

$$E_y^{n+1} = E_y^{n+1/2} - \frac{\Delta t}{2\varepsilon} \frac{\partial H_z^{n+1/2}}{\partial x} \quad (5)$$

$$H_z^{n+1} = H_z^{n+1/2} + \frac{\Delta t}{2\mu} \frac{\partial E_x^{n+1}}{\partial y} - \frac{\Delta t}{2\mu} \frac{\partial E_y^{n+1/2}}{\partial x} \quad (6)$$

To reach the proposed one-step ADI-FDTD method, Eq. (1) plus Eq. (4) is

$$E_x^{n+1} = E_x^n + \frac{\Delta t}{2\varepsilon} \frac{\partial (H_z^{n+1} + H_z^n)}{\partial y} \quad (7)$$

Then Eq. (3) plus Eq. (6) is

$$H_z^{n+1} = H_z^n + \frac{\Delta t}{2\mu} \frac{\partial (E_x^{n+1} + E_x^n)}{\partial y} - \frac{\Delta t}{\mu} \frac{\partial E_y^{n+1/2}}{\partial x} \quad (8)$$

Equation (5) going one time step back plus Eq. (2) is written as

$$E_y^{n+1/2} = E_y^{n-1/2} - \frac{\Delta t}{2\varepsilon} \frac{\partial (H_z^{n+1/2} + H_z^{n-1/2})}{\partial x} \quad (9)$$

Then with Eq. (6) going one time step back and expressing the earlier magnetic field with the later one, we can acquire

$$H_z^{n-1/2} = H_z^n - \frac{\Delta t}{2\mu} \frac{\partial E_x^n}{\partial y} + \frac{\Delta t}{2\mu} \frac{\partial E_y^{n-1/2}}{\partial x} \quad (10)$$

Equation (10) plus Eq. (3) is

$$H_z^{n+1/2} + H_z^{n-1/2} = 2H_z^n - \frac{\Delta t}{2\mu} \frac{\partial (E_y^{n+1/2} - E_y^{n-1/2})}{\partial x} \quad (11)$$

Then combining Eqs. (11) and (9), finally we get

$$E_y^{n+1/2} = E_y^{n-1/2} - \frac{\Delta t}{\varepsilon} \frac{\partial H_z^n}{\partial x} + \frac{\Delta t^2}{4\mu\varepsilon} \frac{\partial^2 (E_y^{n+1/2} - E_y^{n-1/2})}{\partial x^2} \quad (12)$$

Eqs. (7), (8), and (12) formulate the proposed one-step ADI-FDTD method in this paper.

The implementation of the proposed one-step ADI-FDTD method is very simple and direct. One firstly solves E_y through Eq. (12) implicitly, then implicitly solves E_x , and finally explicitly solves H_z . Of course, when E_y have been solved, one can also firstly solves H_z implicitly and then E_x explicitly.

2.2. Formulation of the Proposed One-Step LOD-FDTD Algorithm

In the situation mentioned above, the original LOD-FDTD method [10] is expressed as

$$E_x^{n+1/2} = E_x^n \quad (13)$$

$$E_y^{n+1/2} = E_y^n - \frac{\Delta t}{2\varepsilon} \frac{\partial (H_z^{n+1/2} + H_z^n)}{\partial x} \quad (14)$$

$$H_z^{n+1/2} = H_z^n - \frac{\Delta t}{2\mu} \frac{\partial (E_y^{n+1/2} + E_y^n)}{\partial x} \quad (15)$$

$$E_x^{n+1} = E_x^{n+1/2} + \frac{\Delta t}{2\varepsilon} \frac{\partial (H_z^{n+1} + H_z^{n+1/2})}{\partial y} \quad (16)$$

$$E_y^{n+1} = E_y^{n+1/2} \quad (17)$$

$$H_z^{n+1} = H_z^{n+1/2} + \frac{\Delta t}{2\mu} \frac{\partial (E_x^{n+1} + E_x^{n+1/2})}{\partial y} \quad (18)$$

Adding Eq. (13) with Eq. (16), we get

$$E_x^{n+1} = E_x^n + \frac{\Delta t}{2\varepsilon} \frac{\partial (H_z^{n+1} + H_z^{n+1/2})}{\partial y} \quad (19)$$

Replacing $E_y^{n+1/2}$ in Eq. (15) with E_y^{n+1} , naming it as modified Eq. (15) and then applying the modified Eq. (15) to Eq. (19), we acquire

$$E_x^{n+1} = E_x^n + \frac{\Delta t}{2\varepsilon} \frac{\partial (H_z^{n+1} + H_z^n)}{\partial y} - \frac{\Delta t^2}{4\mu\varepsilon} \frac{\partial^2 (E_y^{n+1} + E_y^n)}{\partial x \partial y} \quad (20)$$

Now adding Eq. (14) with Eq. (17), a new expression is

$$E_y^{n+1} = E_y^n - \frac{\Delta t}{2\varepsilon} \frac{\partial (H_z^{n+1/2} + H_z^n)}{\partial x} \quad (21)$$

Again substituting $E_y^{n+1/2}$ in Eq. (15) with E_y^{n+1} and then applying the modified Eq. (15) to Eq. (21), we get

$$E_y^{n+1} = E_y^n - \frac{\Delta t}{\varepsilon} \frac{\partial H_z^n}{\partial x} + \frac{\Delta t^2}{4\mu\varepsilon} \frac{\partial^2 (E_y^{n+1} + E_y^n)}{\partial x^2} \quad (22)$$

Adding Eq. (15) with Eq. (18), we get

$$H_z^{n+1} = H_z^{n+1/2} + \frac{\Delta t}{2\mu} \frac{\partial (E_x^{n+1} + E_x^{n+1/2})}{\partial y} - \frac{\Delta t}{2\mu} \frac{\partial (E_y^n + E_y^{n+1/2})}{\partial x} \quad (23)$$

Substituting $E_x^{n+1/2}$ in Eq. (23) with E_x^n from Eq. (13) and replacing $E_y^{n+1/2}$ in Eq. (23) with $E_y^{n+1/2}$ in Eq. (17), we get

$$H_z^{n+1} = H_z^{n+1/2} + \frac{\Delta t}{2\mu} \frac{\partial (E_x^{n+1} + E_x^n)}{\partial y} - \frac{\Delta t}{2\mu} \frac{\partial (E_y^{n+1} + E_y^n)}{\partial x} \quad (24)$$

Eqs. (20), (22), and (24) formulate the proposed one-step LOD-FDTD method.

In the proposed one-step LOD-FDTD method, E_y is solved implicitly through Eq. (22). E_x can be implicitly solved by combining Eqs. (20) and (24), and finally H_z is solved explicitly. Of course, after E_y is solved, one can also solve H_z firstly and then handle E_x at her or his will.

From the formulations of the proposed one-step ADI-FDTD method and proposed one-step LOD-FDTD method, it is evident that they both eliminate the intermediate field variables and are also simpler than the original methods. The implementations of the two proposed methods are also rather direct and noticeable. The two proposed one-step algorithms should also have the same numerical property as the original methods, such as the stability condition and numerical correctness, since the proposed algorithms are algebraically equivalent to the original methods where the proposed methods are derived from.

3. NUMERICAL VALIDATION AND DISCUSSIONS

To verify the proposed one-step ADI-FDTD method and proposed one-step LOD-FDTD method, two numerical experiments that use two different parallel plate waveguides [11] with extended scales are carried out.

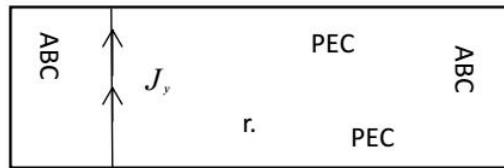


Figure 1. The schematic illustration of the two 2-D parallel plate waveguides.

In the first example, there are 120×100 cells in Fig. 1 with the sizes both set as 0.01 m along x and y directions, respectively. The electric current source is placed on $x = 10$, and the point $r(50, 5)$ is set as observation point recording E_y at each time step. The two open sides of the waveguide are truncated by the first order Mur' absorbing boundary condition. According to the CFL stability condition, the largest time step size that the conventional FDTD method can select is 23.57 ps, and in theory absolutely stable FDTD methods can adopt time step with arbitrary values. For simplicity and convenience, in this numerical example, the time step size for the conventional FDTD, the existing one-step leapfrog ADI-FDTD method [8], and the two proposed methods are all simply chosen as 20 ps.

Figure 2 plots the transient E_y at r point calculated by the conventional FDTD method, the existing one-step leapfrog ADI-FDTD method, the proposed one-step ADI-FDTD method, and the proposed one-step LOD-FDTD method with the uniform time step size. Fig. 2 shows that the numerical results calculated by the four algorithms are in good agreement and thus validates the correctness of the two proposed one-step absolutely stable FDTD methods.

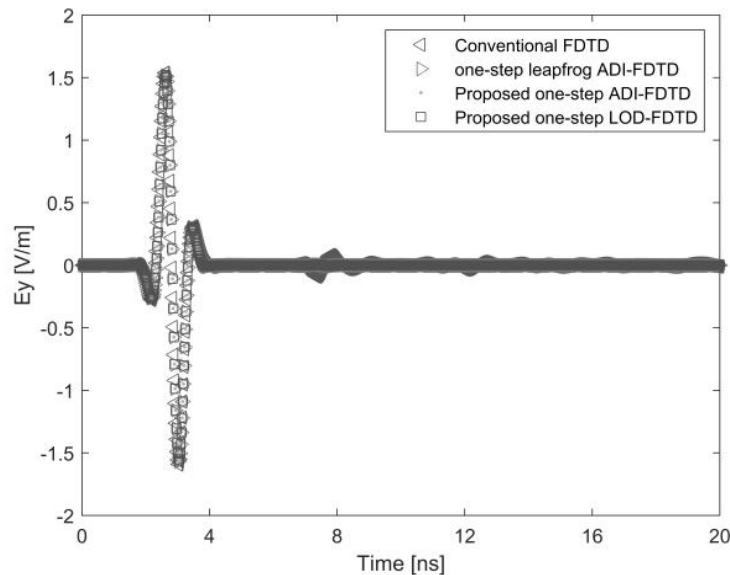


Figure 2. Transient E_y at r point solved by different algorithms.

Firstly, there are some names to be explained. Table 1 lists the running time consumed by these four methods with the uniform time step size. In Table 1 and the next Table 2, the 1step ADI-FDTD stands for the proposed one-step ADI-FDTD method, and the 1step LOD-FDTD stands for the proposed one-step LOD-FDTD method. In the next Table 2, these terms have the same meaning.

From Table 1, it can be seen that the existing one-step leapfrog ADI-FDTD method, the proposed one-step ADI-FDTD method, and the proposed one-step LOD-FDTD method take almost the same time for the numerical task, which indicates that the two proposed methods have the same computation efficiency as that of the existing one-step leapfrog ADI-FDTD method. At the same time, from Table 1, it can also be seen that the existing one-step leapfrog ADI-FDTD method and the two proposed methods

Table 1. Comparison of the computational time of different algorithms with uniform time step size.

Methods	Time step	Iterations	Time
FDTD	20 ps	1000	1.77 s
One-step leapfrog ADI-FDTD	20 ps	1000	29.17 s
Proposed 1step ADI-FDTD	20 ps	1000	31.37 s
Proposed 1step LOD-FDTD	20 ps	1000	31.54 s

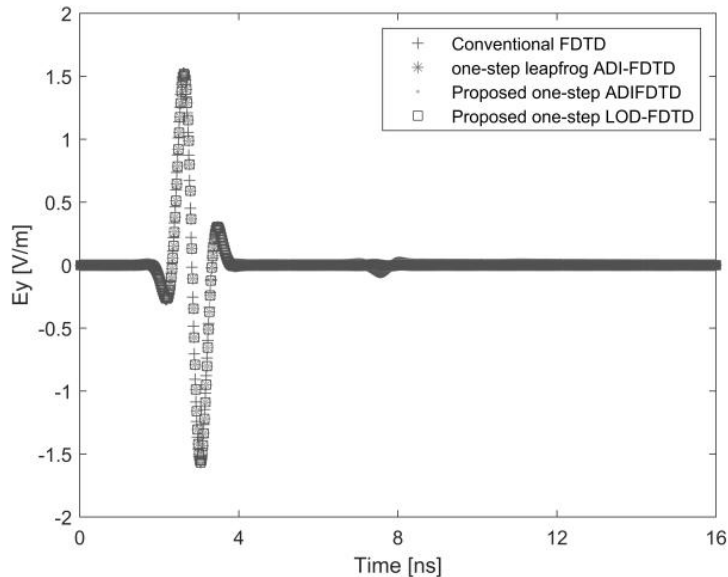
Table 2. Comparison of the computational time of different algorithms with nonuniform time step sizes.

Methods	Time step	Iterations	Time
FDTD	0.2 ps	80000	37.15 s
One-step leapfrog ADI-FDTD	20 ps	800	20.27 s
Proposed 1step ADI-FDTD	20 ps	800	23.16 s
Proposed 1step LOD-FDTD	20 ps	800	22.52 s

cost much more time than the conventional FDTD method. As those absolutely stable methods require matrix operations at every time step, at each time step the absolutely stable methods need more time to finish the update of computation. So it is unavoidable that a numerical task using those methods with the same number of iterations results in more time cost.

As a matter of fact, a run with a large time step beyond the CFL stability condition should be adopted, and then the performance of the proposed one-step methods and the original methods over the conventional FDTD method will appear, so a second numerical experiment is supplied.

In this second numerical example, the parallel plate waveguide in Fig. 1 mentioned above is still used as the test model, while the mesh size along y direction becomes 0.0001 m, and all the other conditions stay the same. The E_y at point r and CPU time costed by different methods are shown in Fig. 3 and Table 2, respectively.

**Figure 3.** Transient E_y at r point calculated by different algorithms.

As can be seen in Fig. 3, the results calculated by the conventional FDTD method, the existing one-step leapfrog ADI-FDTD method, the proposed one-step ADI-FDTD method, and the proposed one-step LOD-FDTD method all are in good agreement. In this situation, Table 2 shows that the proposed one-step ADI-FDTD method and proposed one-step LOD-FDTD method also take almost the same time as the existing one-step leapfrog ADI-FDTD method, and all these three methods cost much less time than the conventional FDTD method. As a result, it can be concluded that the proposed one-step ADI-FDTD method and proposed one-step LOD-FDTD method both actually and successfully overcome the CFL stability condition and also have approximately the same computation efficiency as the existing one-step leapfrog ADI-FDTD method.

4. CONCLUSION

In this paper, two novel one-step absolutely stable FDTD methods including the one-step ADI-FDTD method and the one-step LOD-FDTD method are proposed. Through several steps of linear operations separately applied to the original ADI-FDTD method and LOD-FDTD method, an one-step ADI-FDTD algorithm and an one-step LOD-FDTD method are proposed, and they are both algebraically equivalent to the original ADI-FDTD method and the original LOD-FDTD method, respectively. The two proposed methods both eliminate intermediate field components which are necessary in the original methods and become more concise and simpler one-step algorithms. At the same time, numerical experiments are carried out and validate that the solutions calculated by the two proposed methods are in good agreement with those of the conventional FDTD method and the existing one-step leapfrog ADI-FDTD method, and the computation efficiency of the two proposed methods is very close to that of the existing one-step leapfrog ADI-FDTD method. The two proposed methods also cost much less time than the conventional FDTD method for calculation with very thin cells in that the time step sizes can be selected in a larger domain.

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