

# Time-Harmonic Electromagnetic Fields with $\mathbf{E} \parallel \mathbf{B}$ Represented by Superposing Two Counter-Propagating Beltrami Fields

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**Abstract**—In this paper, we present a general solution for time-harmonic electromagnetic fields with its electric and magnetic fields parallel to each other ( $\mathbf{E} \parallel \mathbf{B}$  fields) in source-free vacuum and demonstrate that every time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  field is composed of the superposition of two counter-propagating Beltrami fields. We show that every  $\mathbf{E} \parallel \mathbf{B}$  field can be categorized into one of two cases depending on the time dependence of the function that describes the proportionality between the electric and magnetic fields. After presenting the mathematical definition of a Beltrami field in electromagnetism and its handedness, we perform a detailed analysis of time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  fields for each case. For the first case, we find the general solution for the  $\mathbf{E} \parallel \mathbf{B}$  fields using the angular-spectrum method and prove that every first-case  $\mathbf{E} \parallel \mathbf{B}$  field can be generated by superposing two oppositely traveling Beltrami fields with the same handedness. For the second case, we deduce the general solution for the  $\mathbf{E} \parallel \mathbf{B}$  fields by employing complex analysis and demonstrate that every time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  field is composed of two counter-propagating planar Beltrami fields with opposite handedness.

## 1. INTRODUCTION

For almost a century, Beltrami fields have attracted considerable research attention. A Beltrami field  $\mathbf{V}$  is a vector field that satisfies the equation

$$\nabla \times \mathbf{V} = \kappa \mathbf{V}, \quad (1)$$

where  $\kappa$  is a scalar function, possibly of both time and position. This equation implies that a Beltrami field is parallel to its curl everywhere. Moreover, the equation can be regarded as an eigenvalue equation involving the curl operator  $\nabla \times$ . In this interpretation,  $\mathbf{V}$  and  $\kappa$  correspond to the eigenvector and eigenvalue, respectively.

The Beltrami-field concept originated in the study of fluid dynamics [1]. For the Navier-Stokes equation, we can find solutions such that the flow velocity is parallel to the vorticity. Because the vorticity is equal to the curl of the flow velocity, these solutions are Beltrami fields. One such Beltrami field that applies to the Arnold-Beltrami-Childress (ABC) flow exhibits chaotic trajectories and has been employed in analyzing turbulence [2].

Beltrami fields have also been studied in electromagnetism. The first study dealing with electromagnetic fields that have the property expressed by Eq. (1) dates back to Silberstein [3], and the concept of such a field has since been incorporated in several branches of electromagnetism. In plasma physics, for instance, Beltrami fields have been employed in the investigation of force-free magnetic field [4]. In such a field, the current behaves as if there were no Lorentz forces in the system because the magnetic field is a Beltrami field, i.e., in a force-free field, the magnetic field is parallel to the current density, which is equivalent to the curl of the magnetic field. In electromagnetism without charged particles, the Beltrami-field concept has also been exploited for analyzing electromagnetic fields

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in chiral and bi-isotropic media, i.e., in complex materials where the electric and magnetic fields are coupled [5–7].

In this study, we focus on the relation between Beltrami fields and electromagnetic fields in which the electric and magnetic fields are parallel to each other, which we refer to as  $\mathbf{E} \parallel \mathbf{B}$  fields. Some important studies involving  $\mathbf{E} \parallel \mathbf{B}$  fields include the work of Chu and Ohkawa, who first demonstrated the possibility of  $\mathbf{E} \parallel \mathbf{B}$  fields by obtaining a solution of Maxwell's equations for transverse electromagnetic fields with  $\mathbf{E} \parallel \mathbf{B}$  [8]. However, this claim has since been confronted by a series of controversies [9–12]. Subsequently, some researchers have attempted to find general solutions for electromagnetic fields with  $\mathbf{E} \parallel \mathbf{B}$  that are subject to no specific assumption, but such solutions are yet to be derived. In the struggle to find a general solution, some solutions for  $\mathbf{E} \parallel \mathbf{B}$  fields have been obtained under specific assumptions. For instance, Zaghoul and Buckmaster [13] and Shimoda et al. [14] independently presented a general solution for transverse electromagnetic fields with  $\mathbf{E} \parallel \mathbf{B}$ . Subsequently, Uehara et al. found some nontransverse  $\mathbf{E} \parallel \mathbf{B}$  fields [15]. Recently, Nishiyama derived the general solution for a spherical  $\mathbf{E} \parallel \mathbf{B}$  field, which is a special type of nontransverse  $\mathbf{E} \parallel \mathbf{B}$  field [16]. In addition to the purely theoretical works above, the studies on practical applications using  $\mathbf{E} \parallel \mathbf{B}$  fields were presented. Evtuhov and Siegman described the method for achieving the energy uniformity in a laser cavity by using a “twisted-mode” standing wave, a special type of planar  $\mathbf{E} \parallel \mathbf{B}$  fields [17]. Subsequently, some laser techniques exploiting twisted-mode waves were proposed [18–21]. Raab et al. developed the magneto-optic trap using the electromagnetic field expressed as the superposition of three twisted-mode waves, which had the  $\mathbf{E} \parallel \mathbf{B}$  property [22].

This study deals with the analytical and mathematical aspects of  $\mathbf{E} \parallel \mathbf{B}$  fields. In the earlier theoretical works already mentioned,  $\mathbf{E} \parallel \mathbf{B}$  fields have been analyzed under various assumptions regarding the geometric properties of the electric and magnetic fields; i.e., in those papers, the electromagnetic fields were assumed to be planar or to be tangent to concentric spheres. In contrast to these earlier works, we focus here on  $\mathbf{E} \parallel \mathbf{B}$  fields that have a harmonic time dependence of the form  $e^{-i\omega t}$ , which has not been considered in the earlier works. In physics and engineering, the analysis of time-harmonic waves is important from both theoretical and practical points of view.

In this paper, our goal is to find the general solution for time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  fields in a source-free vacuum region and to demonstrate that every time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  field can be generated by the superposition of two Beltrami fields traveling in opposite directions.

The difference between this study and the earlier works is as follows. Uehara et al. presented a generic form of an  $\mathbf{E} \parallel \mathbf{B}$  field using both a scalar and a vector field, denoted by  $f$  and  $\mathbf{v}$ , respectively. They classified  $\mathbf{E} \parallel \mathbf{B}$  fields with arbitrary time dependence into three groups, corresponding to the three types of the vector field  $\mathbf{v}$ , which they refer to as “Case I”, “Case II”, and “Case III”. In contrast, this paper classifies time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  fields into two groups depending on the time dependence of the scalar function  $f$ ; thus, we need to investigate fewer cases. For the first case (which corresponds to Case II of Uehara et al.), we derive the general solutions for  $\mathbf{v}$  and  $\mathbf{E} \parallel \mathbf{B}$  field with the amplitude possibly unbounded at infinity, which were not obtained explicitly by Uehara et al. For the second case, which has not been considered directly in any of the abovementioned works, we uniquely perform the analytical derivation of the functional form of  $f$  from the condition defining this case and obtain the corresponding  $\mathbf{v}$  and  $\mathbf{E} \parallel \mathbf{B}$  field. Furthermore, unlike the earlier studies on  $\mathbf{E} \parallel \mathbf{B}$  fields, we also present a discussion that may help in experimentally generating  $\mathbf{E} \parallel \mathbf{B}$  fields; we demonstrate that, as with any other standing wave, every time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  field is composed of the superposition of two oppositely propagating waves.

This paper is organized as follows. In Section 2, we show that every time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  field can be classified into one of two groups depending on the properties of  $f$ . In addition, we deduce the partial-differential equations satisfied by  $f$  and  $\mathbf{v}$ . In Section 3, after introducing the handedness of the Beltrami fields, we derive a general mathematical expression for the first-case  $\mathbf{E} \parallel \mathbf{B}$  fields using the angular-spectrum method. Moreover, we demonstrate that a first-case  $\mathbf{E} \parallel \mathbf{B}$  field is necessarily composed of the superposition of two counter-propagating Beltrami fields with the same handedness. In Section 4, we obtain a general solution for  $\mathbf{E} \parallel \mathbf{B}$  fields belonging to the second group. Moreover, we show that the second-case  $\mathbf{E} \parallel \mathbf{B}$  fields always consist of the superposition of two counter-traveling planar Beltrami fields with opposite handedness. Finally, in Section 5, we present our general conclusions as to the findings of this paper and note some open problems.

## 2. CLASSIFICATION OF TIME-HARMONIC $\mathbf{E} \parallel \mathbf{B}$ FIELDS

Before analyzing time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  fields, we consider  $\mathbf{E} \parallel \mathbf{B}$  fields for which no requirement is placed on the temporal or spatial variations by reviewing Uehara et al.'s derivation of the four fundamental equations satisfied by the fields.

According to Uehara et al., electromagnetic fields with  $\mathbf{E} \parallel \mathbf{B}$  can be described using a time-dependent scalar field  $f$  and a time-dependent vector field  $\mathbf{v}$  [15]:

$$\mathbf{E}(\mathbf{r}; t) = E_0 \mathbf{v}(\mathbf{r}; t) \cos f(\mathbf{r}; t), \quad (2)$$

$$\mathbf{B}(\mathbf{r}; t) = B_0 \mathbf{v}(\mathbf{r}; t) \sin f(\mathbf{r}; t). \quad (3)$$

Substituting Eqs. (2) and (3) into the source-free Maxwell's equations, which are given by

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad (4)$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}, \quad (5)$$

$$\nabla \cdot \mathbf{E} = 0, \quad (6)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (7)$$

Uehara et al. obtained the following four partial-differential equations in terms of  $f$  and  $\mathbf{v}$ :

$$\frac{\partial f}{\partial t} \mathbf{v} = -c \nabla \times \mathbf{v}, \quad (8)$$

$$\frac{\partial \mathbf{v}}{\partial t} = c \nabla f \times \mathbf{v}, \quad (9)$$

$$\nabla \cdot \mathbf{v} = 0, \quad (10)$$

$$\mathbf{v} \cdot \nabla f = 0. \quad (11)$$

At this point, we assume that the  $\mathbf{E} \parallel \mathbf{B}$  field is time-harmonic, i.e., both  $\mathbf{E}$  and  $\mathbf{B}$  have the time dependence  $e^{-i\omega t}$ . Applying the time-frequency Fourier transform to Eqs. (2) and (3), we find that either  $e^{if}$  or  $\mathbf{v}$  is time-harmonic, and the other is independent of time. We can thus categorize time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  fields into two groups according to the property of  $f$ : (1) the case where  $f$  is a linear function only of  $t$  and (2) the case where  $f$  is a function only of  $\mathbf{r}$ . In this section, we derive the partial-differential equations that  $f$  and  $\mathbf{v}$  must satisfy in each case, and we solve these equations in the following sections.

First, we consider the case where  $f$  is a linear function of  $t$  with the angular frequency  $\omega$ , while  $\mathbf{v}$  is independent of time. In this case, the left-hand side of Eq. (9) is identically zero, which indicates that  $\nabla f$  and  $\mathbf{v}$  are parallel to each other unless  $\nabla f$  is zero everywhere. Equation (11) states that  $\nabla f$  is also perpendicular to  $\mathbf{v}$  unless  $\nabla f$  is zero everywhere. These two facts are consistent only if

$$\nabla f(\mathbf{r}; t) = 0. \quad (12)$$

This equation indicates that  $f$  is independent of position. Thus, the magnitude of the partial time derivative of  $f$  is identical to the angular frequency  $\omega$ , and Eqs. (8)–(11) yield the following partial-differential equation for  $\mathbf{v}$ :

$$\nabla \times \mathbf{v}(\mathbf{r}) = \pm k \mathbf{v}(\mathbf{r}), \quad (13)$$

where  $k$  is the wavenumber,  $k = \omega/c$ . Therefore,  $\mathbf{v}$  is a Beltrami field in this case. This type of  $\mathbf{E} \parallel \mathbf{B}$  field is identical to Case II considered by Uehara et al.

Next, we consider the case where  $f$  is independent of time. As the left-hand side of Eq. (8) is zero, the vector field  $\mathbf{v}$ , which is shown to be solenoidal by Eq. (10), is also an irrotational vector field, and therefore  $\mathbf{v}$  is a solution of the Laplace equation

$$\nabla^2 \mathbf{v}(\mathbf{r}; t) = 0. \quad (14)$$

In general, a solution of the Laplace equation is not constant, and the  $\mathbf{E} \parallel \mathbf{B}$  fields in this case are included in the Case III  $\mathbf{E} \parallel \mathbf{B}$  fields of Uehara et al. Applying the curl operator  $\nabla \times$  to both sides of Eq. (9), we obtain

$$|\nabla f(\mathbf{r})| = k^2, \quad (15)$$

where  $k$  is again the wavenumber,  $k = \omega/c$ . This equation is an eikonal equation for  $f$ , which is well known in geometrical optics.

### 3. $\mathbf{E} \parallel \mathbf{B}$ FIELD WITH LINEARLY TIME-DEPENDENT $f$

#### 3.1. Electromagnetic Beltrami Field

Before performing the analysis of the Case II  $\mathbf{E} \parallel \mathbf{B}$  fields, we introduce two important concepts for the analysis of the  $\mathbf{E} \parallel \mathbf{B}$  fields: the electromagnetic Beltrami field and its handedness. We consider time-harmonic electromagnetic fields whose the electric and magnetic fields satisfy

$$\nabla \times \mathbf{E} = \pm k \mathbf{E}, \quad (16)$$

$$\nabla \times \mathbf{B} = \pm k \mathbf{B}, \quad (17)$$

where the double signs in the two equations correspond to each other. These equations can be regarded as the eigenequations of the curl operator  $\nabla \times$  with the eigenvalues  $\pm k$ . Hereinafter, we refer to electromagnetic fields that satisfy these eigenequations as Beltrami fields. One of the simplest Beltrami fields is a circularly polarized wave. A left-handed circularly polarized wave satisfies the eigenvalue Eqs. (16) and (17) with the eigenvalue  $+k$ , whereas a right-handed circularly polarized wave satisfies the equations with the eigenvalue  $-k$ . We can expand the concept of handedness associated with circularly polarized waves to a general Beltrami field; we define a left-handed Beltrami field as a Beltrami field with the eigenvalue  $+k$  and a right-handed Beltrami field as a Beltrami field with the eigenvalue  $-k$ .

From Eqs. (2), (3), and (13), we find that a Case II  $\mathbf{E} \parallel \mathbf{B}$  field is a Beltrami field. The eigenvalue equation for  $\mathbf{v}$ , namely Eq. (13), determines the handedness of the Case II  $\mathbf{E} \parallel \mathbf{B}$  field; if the eigenvalue of Eq. (13) is positive, the Case II  $\mathbf{E} \parallel \mathbf{B}$  field is a left-handed Beltrami field, while if the eigenvalue is negative, it is a right-handed Beltrami field.

#### 3.2. General Solution for an $\mathbf{E} \parallel \mathbf{B}$ Field

In this subsection, we derive the general solution for a Case II  $\mathbf{E} \parallel \mathbf{B}$  field using the angular-spectrum method [23]. This method helps in understanding the physical implications of Case II  $\mathbf{E} \parallel \mathbf{B}$  and Beltrami fields.

Applying the curl operator to both sides of Eq. (8) yields the following Helmholtz equation for  $\mathbf{v}$ :

$$\nabla^2 \mathbf{v}(\mathbf{r}) + k^2 \mathbf{v}(\mathbf{r}) = 0. \quad (18)$$

The general expression for the vector field  $\mathbf{v}(\mathbf{r})$  has the form

$$\mathbf{v}(\mathbf{r}) = \int d\mathbf{p} \int d\mathbf{q} [\hat{\mathbf{v}}_1(\mathbf{p}, \mathbf{q}) + i\hat{\mathbf{v}}_2(\mathbf{p}, \mathbf{q})] e^{i\mathbf{k}(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}}, \quad (19)$$

where  $\hat{\mathbf{v}}_1$  and  $\hat{\mathbf{v}}_2$  are purely real vector-valued functions of the two purely real vector variables  $\mathbf{p} = (p_x, p_y, p_z)$  and  $\mathbf{q} = (q_x, q_y, q_z)$ , and the two vectors,  $\mathbf{p}$  and  $\mathbf{q}$ , satisfy

$$|\mathbf{p}^2 - |\mathbf{q}|^2 = 1, \quad (20)$$

$$\mathbf{p} \cdot \mathbf{q} = 0. \quad (21)$$

From the divergence-free condition, Eq. (10), and the eigenvalue equation Eq. (13), we find the following constraints on the spectra  $\hat{\mathbf{v}}_1$  and  $\hat{\mathbf{v}}_2$ . Substituting Eq. (19) into the divergence-free condition, Eq. (10), gives

$$\int d\mathbf{p} \int d\mathbf{q} k [-(\mathbf{p} \cdot \hat{\mathbf{v}}_2 + \mathbf{q} \cdot \hat{\mathbf{v}}_1) + i(\mathbf{p} \cdot \hat{\mathbf{v}}_1 - \mathbf{q} \cdot \hat{\mathbf{v}}_2)] [\hat{\mathbf{v}}_1(\mathbf{p}, \mathbf{q}) + i\hat{\mathbf{v}}_2(\mathbf{p}, \mathbf{q})] e^{i\mathbf{k}(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}} = 0. \quad (22)$$

This equation imposes the requirement that the integrand be identically zero; i.e.,

$$\mathbf{p} \cdot \hat{\mathbf{v}}_2 + \mathbf{q} \cdot \hat{\mathbf{v}}_1 = 0, \quad (23)$$

$$\mathbf{p} \cdot \hat{\mathbf{v}}_1 - \mathbf{q} \cdot \hat{\mathbf{v}}_2 = 0. \quad (24)$$

Similarly, substituting Eq. (19) into the eigenvalue Eq. (13) gives

$$\int d\mathbf{p} \int d\mathbf{q} k [-(\mathbf{q} \times \hat{\mathbf{v}}_1 + \mathbf{p} \times \hat{\mathbf{v}}_2 \pm \hat{\mathbf{v}}_1) + i(\mathbf{p} \times \hat{\mathbf{v}}_1 - \mathbf{q} \times \hat{\mathbf{v}}_2 \mp \hat{\mathbf{v}}_2)] e^{i\mathbf{k}(\mathbf{p}+\mathbf{q})\cdot\mathbf{r}} = 0, \quad (25)$$

which yields the requirement that the integrand be identically zero; that is,

$$\mathbf{q} \times \hat{\mathbf{v}}_1 + \mathbf{p} \times \hat{\mathbf{v}}_2 \pm \hat{\mathbf{v}}_1 = 0, \tag{26}$$

$$\mathbf{p} \times \hat{\mathbf{v}}_1 - \mathbf{q} \times \hat{\mathbf{v}}_2 \mp \hat{\mathbf{v}}_2 = 0. \tag{27}$$

Equations (23), (24), (26), and (27) define the geometrical relations among  $\hat{\mathbf{v}}_1$ ,  $\hat{\mathbf{v}}_2$ ,  $\mathbf{p}$ , and  $\mathbf{q}$ . Because the scalar product of  $\hat{\mathbf{v}}_1$  and the left-hand side of Eq. (26) is identical to that of  $\hat{\mathbf{v}}_2$  and left-side of Eq. (27), we find the relation

$$|\hat{\mathbf{v}}_1| = |\hat{\mathbf{v}}_2|. \tag{28}$$

As the scalar product of  $\hat{\mathbf{v}}_2$  and the left-hand side of Eq. (26) is identical to the product of  $\hat{\mathbf{v}}_1$  and left-side of Eq. (27), we find the orthogonality condition

$$\hat{\mathbf{v}}_1 \cdot \hat{\mathbf{v}}_2 = 0. \tag{29}$$

By choosing an appropriate local Cartesian coordinate with the orthonormal basis  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ , the two vectors  $\hat{\mathbf{v}}_1$  and  $\hat{\mathbf{v}}_2$  can thus be represented in the form

$$\hat{\mathbf{v}}_1 = v_0 \mathbf{e}_1, \tag{30}$$

$$\hat{\mathbf{v}}_2 = v_0 \mathbf{e}_2, \tag{31}$$

where the coefficient  $v_0$  is real. The vector variables  $\mathbf{p}$  and  $\mathbf{q}$  can be represented in the form

$$\mathbf{p} = p_1 \mathbf{e}_1 + p_2 \mathbf{e}_2 + p_3 \mathbf{e}_3, \tag{32}$$

$$\mathbf{q} = q_1 \mathbf{e}_1 + q_2 \mathbf{e}_2 + q_3 \mathbf{e}_3. \tag{33}$$

Substituting Eqs. (32) and (33) into the requirements given by Eqs. (26) and (27), we obtain

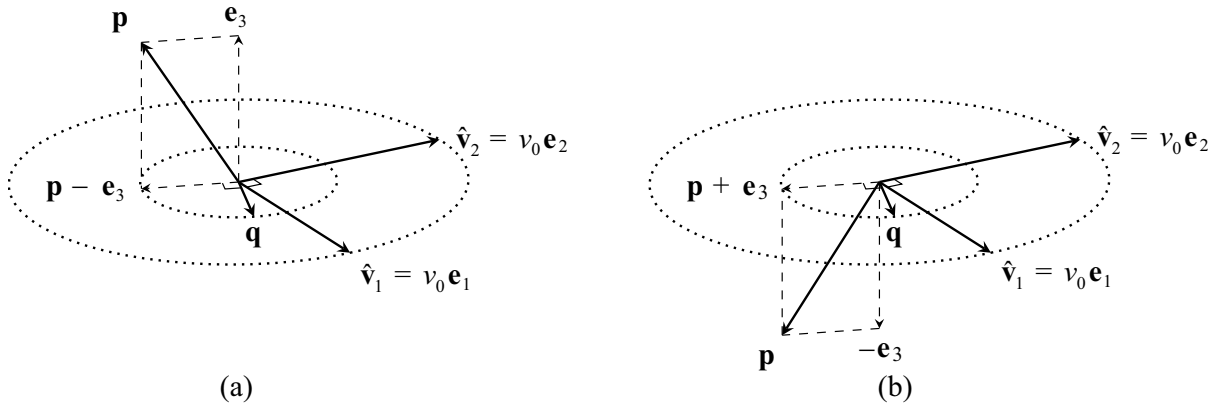
$$p_1 = q_2, \tag{34}$$

$$p_2 = -q_1, \tag{35}$$

$$p_3 = \pm 1, \tag{36}$$

$$q_3 = 0. \tag{37}$$

The vectors  $\mathbf{p}$  and  $\mathbf{q}$  defined by these four equations satisfy Eqs. (23) and (24), and the geometric requirements on  $\hat{\mathbf{v}}_1$ ,  $\hat{\mathbf{v}}_2$ ,  $\mathbf{p}$ , and  $\mathbf{q}$  are reduced to Eqs. (30)–(37). From Eqs. (34)–(37), we find the relation  $|\mathbf{p} \mp \mathbf{e}_3| = |\mathbf{q}|$ . Both  $\mathbf{p} \mp \mathbf{e}_3$  and  $\mathbf{q}$  are written as a linear combination of  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , as illustrated in Fig. 1.



**Figure 1.** Geometric relations among  $\mathbf{p}$ ,  $\mathbf{q}$ ,  $\hat{\mathbf{v}}_1$ , and  $\hat{\mathbf{v}}_2$ . (a) The case  $p_3 = 1$ . (b) The case  $p_3 = -1$ .

As the vector  $p_3 \mathbf{e}_3$  can represent any point on a sphere of radius one, denoted as  $S$ , we replace  $p_3 \mathbf{e}_3$  by the unit radial vector  $\mathbf{e}_k$  associated with the sphere  $S$ . Here, the complex vector  $\hat{\mathbf{v}}_1 + i\hat{\mathbf{v}}_2$  can be represented by a linear combination of the zenith and azimuthal unit vectors  $\mathbf{e}_\theta$  and  $\mathbf{e}_\phi$  associated with the sphere  $S$ . As the complex vector  $\mathbf{p} - \mathbf{e}_k + i\mathbf{q}$  is also represented as a linear combination of  $\mathbf{e}_\theta$  and  $\mathbf{e}_\phi$ , we can represent  $\mathbf{p} - \mathbf{e}_k + i\mathbf{q}$  in the following general form by introducing two real variables  $\xi$  and  $\eta$ :

$$\mathbf{p} - \mathbf{e}_k + i\mathbf{q} = (\eta + i\xi)(\mathbf{e}_\theta \pm i\mathbf{e}_\phi), \tag{38}$$

where the double sign corresponds to that of Eq. (13). As  $\mathbf{e}_k$ ,  $\mathbf{e}_\theta$ , and  $\mathbf{e}_\phi$  are functions only of  $\theta$  and  $\phi$ , we recognize that the vector variables  $\mathbf{p}$  and  $\mathbf{q}$  are represented by just the four variables  $\theta$ ,  $\phi$ ,  $\xi$ , and  $\eta$ . Using a complex function of  $\theta$ ,  $\phi$ ,  $\xi$ , and  $\eta$ , denoted as  $\hat{u}(\theta, \phi, \xi, \eta)$ ,  $\hat{\mathbf{v}}_1 + i\hat{\mathbf{v}}_2$  can thus be expressed in the form

$$\hat{\mathbf{v}}_1 + i\hat{\mathbf{v}}_2 = \hat{u}(\theta, \phi, \xi, \eta)(\mathbf{e}_\theta \pm i\mathbf{e}_\phi), \quad (39)$$

where the double sign also corresponds to the handedness of the Beltrami field in Eq. (13). Summarizing the results obtained so far, we find that  $\mathbf{v}(\mathbf{r})$  can be represented in the integral form

$$\mathbf{v}(\mathbf{r}) = \int_0^\pi d\theta \int_{-\pi}^\pi d\phi (\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \left[ \int_{-\infty}^\infty \int_{-\infty}^\infty \hat{u}(\theta, \phi, \xi, \eta) e^{ik(\eta+i\xi)(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}} d\xi d\eta \right] e^{ik\mathbf{e}_k \cdot \mathbf{r}}. \quad (40)$$

Here, we define the angular spectrum  $\hat{v}(\theta, \phi, \mathbf{r})$  as

$$\hat{v}(\theta, \phi, \mathbf{r}) = \int_{-\infty}^\infty \int_{-\infty}^\infty \hat{u}(\theta, \phi, \xi, \eta) e^{ik(\eta+i\xi)(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}} d\xi d\eta. \quad (41)$$

As  $\mathbf{v}(\mathbf{r})$  is a purely real vector, for every  $\mathbf{r}$ , any two angular spectra at opposite points on the sphere  $S$  must satisfy the symmetry condition

$$\hat{v}(\theta, \phi, \mathbf{r}) = \overline{\hat{v}(\pi - \theta, \pi + \phi, \mathbf{r})}. \quad (42)$$

Substituting Eq. (41) into this symmetry relation, we have

$$\int_{-\infty}^\infty \left[ \int_{-\infty}^\infty (\hat{u}(\theta, \phi, \xi, \eta) e^{ik\eta(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}} - \overline{\hat{u}(\pi - \theta, \pi + \phi, \xi, \eta)} e^{-ik\eta(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}}) d\eta \right] e^{-k\xi(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}} d\xi = 0. \quad (43)$$

For this equation to hold regardless of position  $\mathbf{r}$ , the  $\eta$ -dependence of  $\hat{u}(\theta, \phi, \xi, \eta)$  must be the Dirac delta function  $\delta(\eta)$ , which suggests that  $\hat{u}$  is actually a function of only  $\theta$ ,  $\phi$ , and  $\xi$ . Therefore, the requirement that  $\mathbf{v}(\mathbf{r})$  be a purely real vector is reduced to the equation

$$\hat{u}(\theta, \phi, \xi) = \overline{\hat{u}(\pi - \theta, \pi + \phi, \xi)}. \quad (44)$$

As a result, the angular spectrum  $\hat{v}(\theta, \phi, \mathbf{r})$  can be written in the form

$$\hat{v}(\theta, \phi, \mathbf{r}) = \int_{-\infty}^\infty d\xi \hat{u}(\theta, \phi, \xi) e^{-k\xi(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}}, \quad (45)$$

where  $\hat{u}(\theta, \phi, \xi)$  is a symmetric function satisfying Eq. (44). By employing the surface integral of  $\hat{v}(\theta, \phi, \mathbf{r})$  on the sphere  $S$ , we obtain the general solution for  $\mathbf{v}(\mathbf{r})$  as

$$\begin{aligned} \mathbf{v}(\mathbf{r}) &= \int_0^\pi d\theta \int_{-\pi}^\pi d\phi (\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \hat{v}(\theta, \phi, \mathbf{r}) e^{ik\mathbf{e}_k \cdot \mathbf{r}} \\ &= \int_0^{\frac{\pi}{2}} d\theta \int_{-\pi}^\pi d\phi \rho [\cos(k\mathbf{e}_k \cdot \mathbf{r} + \psi) \mathbf{e}_\theta \mp \sin(k\mathbf{e}_k \cdot \mathbf{r} + \psi) \mathbf{e}_\phi], \end{aligned} \quad (46)$$

where  $\rho$  and  $\psi$  are respectively the magnitude and argument of the angular spectrum  $\hat{v}(\theta, \phi, \mathbf{r})$ , i.e.,

$$\rho = |\hat{v}(\theta, \phi, \mathbf{r})|, \quad (47)$$

$$\psi = \text{Arg}(\hat{v}(\theta, \phi, \mathbf{r})). \quad (48)$$

Conversely, we can readily confirm that every vector field that satisfies Eqs. (44)–(48) is necessarily a solution for  $\mathbf{v}(\mathbf{r})$ . As a result, we obtain the general solution for a Case II  $\mathbf{E} \parallel \mathbf{B}$  field in the form

$$\mathbf{E}(\mathbf{r}; t) = E_0 \cos \omega t \int_0^{\frac{\pi}{2}} d\theta \int_{-\pi}^{\pi} d\phi \rho [\cos(k\mathbf{e}_k \cdot \mathbf{r} + \psi)\mathbf{e}_\theta \mp \sin(k\mathbf{e}_k \cdot \mathbf{r} + \psi)\mathbf{e}_\phi], \quad (49)$$

$$\mathbf{B}(\mathbf{r}; t) = \mp B_0 \sin \omega t \int_0^{\frac{\pi}{2}} d\theta \int_{-\pi}^{\pi} d\phi \rho [\cos(k\mathbf{e}_k \cdot \mathbf{r} + \psi)\mathbf{e}_\theta \mp \sin(k\mathbf{e}_k \cdot \mathbf{r} + \psi)\mathbf{e}_\phi]. \quad (50)$$

Looking now at Eq. (46), we see that the superposition principle is valid for these solutions; the superposition of multiple Case II  $\mathbf{E} \parallel \mathbf{B}$  fields yields a Case II  $\mathbf{E} \parallel \mathbf{B}$  field as long as all the original  $\mathbf{E} \parallel \mathbf{B}$  fields have the same handedness.

Next, we investigate the functional form of the angular spectrum  $\hat{v}(\theta, \phi, \mathbf{r})$  given by Eq. (45). If the angular spectrum exists for any  $\mathbf{r}$ , due to the nature of the integral transform, the angular spectrum  $\hat{v}(\theta, \phi, \mathbf{r})$  is an entire function of the complex variable  $(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}$ , where  $\mathbf{e}_\theta \pm i\mathbf{e}_\phi$  is a complex constant vector for fixed  $(\theta, \phi)$ . Conversely, if the angular spectrum is reduced to an entire function of  $(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}$  for every fixed  $(\theta, \phi)$ , the integral of Eq. (46) yields a solution for  $\mathbf{v}(\mathbf{r})$ . This is easily confirmed as follows. Suppose that for a given  $(\theta, \phi)$ , an entire function of  $(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}$  is written as  $\Phi_{\theta\phi}((\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r})$ . Then, the curl of the vector field given by Eq. (46) is computed as

$$\begin{aligned} & \nabla \times \mathbf{v}(\mathbf{r}) \\ &= \nabla \times \left[ \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi (\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \Phi_{\theta\phi}((\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}) e^{ik\mathbf{e}_k \cdot \mathbf{r}} \right] \\ &= \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi [(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \times (\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \Phi'_{\theta\phi}((\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}) + ik\mathbf{e}_k \times (\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \Phi_{\theta\phi}((\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r})] e^{ik\mathbf{e}_k \cdot \mathbf{r}} \\ &= \pm k\mathbf{v}(\mathbf{r}). \end{aligned} \quad (51)$$

We now present a specific example of a Beltrami field  $\mathbf{v}(\mathbf{r})$ . Let  $\hat{u}$  be given by

$$\hat{u}(\theta, \phi, \xi) = \delta'(\xi), \quad (52)$$

where  $\delta'(\xi)$  denotes the first derivative of the Dirac delta function. From Eq. (45), we have the angular spectrum

$$\hat{v}(\theta, \phi, \mathbf{r}) = -k(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}, \quad (53)$$

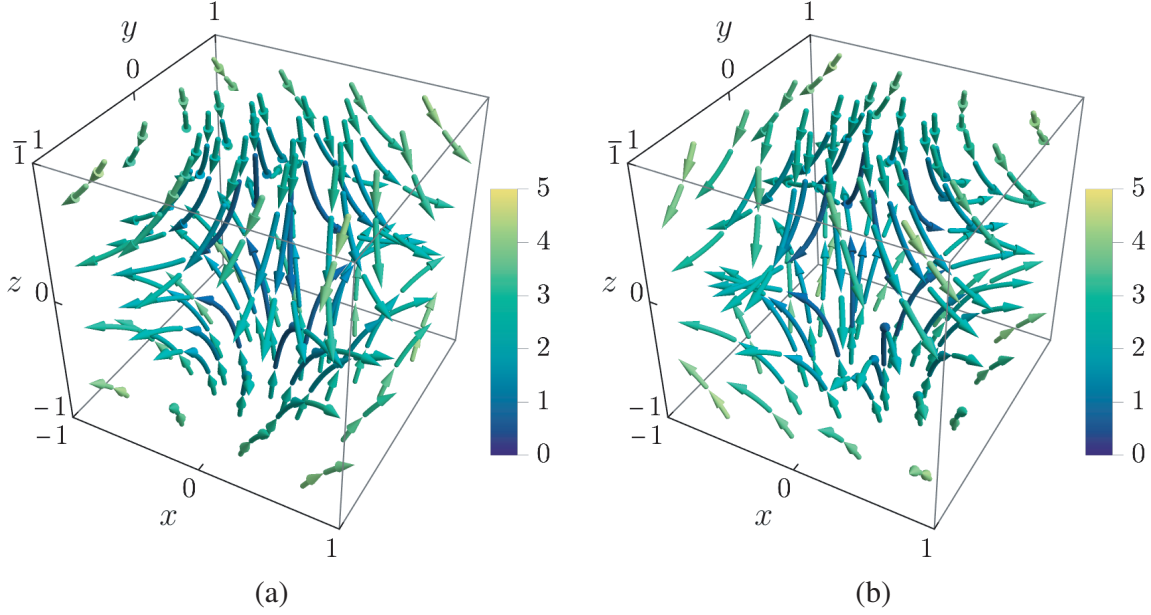
which satisfies the symmetry relation of Eq. (42). The Beltrami field  $\mathbf{v}(\mathbf{r})$  is represented in the integral form

$$\mathbf{v}(\mathbf{r}) = \int_0^{\pi} d\theta \int_{-\pi}^{\pi} d\phi (\mathbf{e}_\theta \pm i\mathbf{e}_\phi) [-k(\mathbf{e}_\theta \pm i\mathbf{e}_\phi) \cdot \mathbf{r}] e^{ik\mathbf{e}_k \cdot \mathbf{r}}. \quad (54)$$

As the integral on the right-hand side of Eq. (54) cannot be evaluated analytically, we calculate numerically the field distributions of the left- and right-handed Beltrami fields, and the numerical results are shown in Fig. 2.

Finally, we note the physically relevant solutions. If the angular spectrum  $\hat{v}(\theta, \phi, \mathbf{r})$  depends on position  $\mathbf{r}$ , the vector field  $\mathbf{v}(\mathbf{r})$  given by Eq. (46) diverges to infinity as  $|\mathbf{r}| \rightarrow \infty$ , and the corresponding electromagnetic field also diverges at infinity. Thus, the resulting electromagnetic field is not physically acceptable. Physically realizable electromagnetic fields can be obtained only if the  $\xi$  dependence of  $\hat{u}(\theta, \phi, \xi)$  is a delta function  $\delta(\xi)$ , and the angular spectrum  $\hat{v}$  is provided as a function of only  $\theta$  and  $\phi$ . Therefore, the physically relevant solution for  $\mathbf{v}(\mathbf{r})$  can be represented in the form

$$\mathbf{v}(\mathbf{r}) = \int_0^{\frac{\pi}{2}} d\theta \int_{-\pi}^{\pi} d\phi |\hat{v}(\theta, \phi)| [\cos(k\mathbf{e}_k \cdot \mathbf{r} + \text{Arg}(\hat{v}(\theta, \phi)))\mathbf{e}_\theta \mp \sin(k\mathbf{e}_k \cdot \mathbf{r} + \text{Arg}(\hat{v}(\theta, \phi)))\mathbf{e}_\phi]. \quad (55)$$



**Figure 2.** Field distributions of the left- and right-handed Beltrami fields given by Eq. (54) with  $k = 1$ . Arrow color indicates the magnitude of the vector at each point. (a) Left-handed Beltrami field. (b) Right-handed Beltrami field.

### 3.3. Decomposition of an $\mathbf{E} \parallel \mathbf{B}$ Field into Two Oppositely Propagating Beltrami Fields

As no energy flow occurs anywhere in an  $\mathbf{E} \parallel \mathbf{B}$  field, it is a special type of standing wave. Usually, we generate a standing wave by generating two oppositely propagating electromagnetic fields and superposing them. In this subsection, we demonstrate that as with other standing waves, a Case II  $\mathbf{E} \parallel \mathbf{B}$  field can be generated by superposing two propagating waves.

The general solution for a Case II  $\mathbf{E} \parallel \mathbf{B}$  field represented by Eqs. (49) and (50) can be rewritten as

$$\begin{aligned} \mathbf{E} = & \frac{E_0}{2} \int_0^{\frac{\pi}{2}} d\theta \int_{-\pi}^{\pi} d\phi \rho [\cos(k\mathbf{e}_k \cdot \mathbf{r} - \omega t + \psi)\mathbf{e}_\theta \mp \sin(k\mathbf{e}_k \cdot \mathbf{r} - \omega t + \psi)\mathbf{e}_\phi] \\ & + \frac{E_0}{2} \int_0^{\frac{\pi}{2}} d\theta \int_{-\pi}^{\pi} d\phi \rho [\cos(k\mathbf{e}_k \cdot \mathbf{r} + \omega t + \psi)\mathbf{e}_\theta \mp \sin(k\mathbf{e}_k \cdot \mathbf{r} + \omega t + \psi)\mathbf{e}_\phi], \end{aligned} \quad (56)$$

$$\begin{aligned} \mathbf{B} = & \frac{B_0}{2} \int_0^{\frac{\pi}{2}} d\theta \int_{-\pi}^{\pi} d\phi \rho [\pm \sin(k\mathbf{e}_k \cdot \mathbf{r} - \omega t + \psi)\mathbf{e}_\theta + \cos(k\mathbf{e}_k \cdot \mathbf{r} - \omega t + \psi)\mathbf{e}_\phi] \\ & + \frac{B_0}{2} \int_0^{\frac{\pi}{2}} d\theta \int_{-\pi}^{\pi} d\phi \rho [\mp \sin(k\mathbf{e}_k \cdot \mathbf{r} + \omega t + \psi)\mathbf{e}_\theta - \cos(k\mathbf{e}_k \cdot \mathbf{r} + \omega t + \psi)\mathbf{e}_\phi]. \end{aligned} \quad (57)$$

On the right-hand sides of Eqs. (56) and (57), the first terms are denoted by  $\mathbf{E}^f$  and  $\mathbf{B}^f$ , respectively, and the second terms by  $\mathbf{E}^b$  and  $\mathbf{B}^b$ , respectively. Neither of the two electromagnetic fields,  $(\mathbf{E}^f, \mathbf{B}^f)$  and  $(\mathbf{E}^b, \mathbf{B}^b)$ , exhibits the  $\mathbf{E} \parallel \mathbf{B}$  property, and they are time-reversed (or phase-conjugated) waves to each other; thus, they propagate in opposite directions to each other [24]. In addition, the two electromagnetic fields satisfy the eigenvalue Eqs. (16) and (17), and the eigenvalues for  $(\mathbf{E}^f, \mathbf{B}^f)$  and  $(\mathbf{E}^b, \mathbf{B}^b)$  correspond,



which means that the two electromagnetic fields  $(\mathbf{E}^f, \mathbf{B}^f)$  and  $(\mathbf{E}^b, \mathbf{B}^b)$  are Beltrami fields with the same handedness. These facts show that every Case II  $\mathbf{E} \parallel \mathbf{B}$  field can be decomposed into two Beltrami fields with the same handedness that propagate in opposite directions to each other. Conversely, for any propagating Beltrami field, the time-reversed field is also a Beltrami field, and the superposition of the two waves becomes a Case II  $\mathbf{E} \parallel \mathbf{B}$  field. This decomposition of the  $\mathbf{E} \parallel \mathbf{B}$  field implies that when we attempt to obtain a Case II  $\mathbf{E} \parallel \mathbf{B}$  field experimentally (i.e., one that is actually constrained to be a physically relevant solution of Eq. (55)), we should generate two appropriate propagating electromagnetic fields that satisfy Eqs. (56) and (57) and superpose them.

Finally, we can understand the physical implications of the two Beltrami fields  $(\mathbf{E}^f, \mathbf{B}^f)$  and  $(\mathbf{E}^b, \mathbf{B}^b)$  in terms of their angular-spectrum representations. The integrands of  $\mathbf{E}^f$  and  $\mathbf{B}^f$ , respectively, can be rewritten as

$$\begin{aligned} & \frac{E_0}{2} \rho [\cos(k\mathbf{e}_k \cdot \mathbf{r} - \omega t + \psi)\mathbf{e}_\theta \mp \sin(k\mathbf{e}_k \cdot \mathbf{r} - \omega t + \psi)\mathbf{e}_\phi] \\ = & \frac{E_0}{2} \rho \cos \psi [\cos(k\mathbf{e}_k \cdot \mathbf{r} - \omega t)\mathbf{e}_\theta \mp \sin(k\mathbf{e}_k \cdot \mathbf{r} - \omega t)\mathbf{e}_\phi] \\ & + \frac{E_0}{2} \rho \sin \psi [-\sin(k\mathbf{e}_k \cdot \mathbf{r} - \omega t)\mathbf{e}_\theta \mp \cos(k\mathbf{e}_k \cdot \mathbf{r} - \omega t)\mathbf{e}_\phi], \end{aligned} \quad (58)$$

$$\begin{aligned} & \frac{B_0}{2} \rho [\pm \sin(k\mathbf{e}_k \cdot \mathbf{r} - \omega t + \psi)\mathbf{e}_\theta + \cos(k\mathbf{e}_k \cdot \mathbf{r} - \omega t + \psi)\mathbf{e}_\phi] \\ = & \frac{B_0}{2} \rho \cos \psi [\pm \sin(k\mathbf{e}_k \cdot \mathbf{r} - \omega t)\mathbf{e}_\theta + \cos(k\mathbf{e}_k \cdot \mathbf{r} - \omega t)\mathbf{e}_\phi] \\ & + \frac{B_0}{2} \rho \sin \psi [\pm \cos(k\mathbf{e}_k \cdot \mathbf{r} - \omega t)\mathbf{e}_\theta - \sin(k\mathbf{e}_k \cdot \mathbf{r} - \omega t)\mathbf{e}_\phi], \end{aligned} \quad (59)$$

where we note that neither  $\rho \cos \psi$  nor  $\rho \sin \psi$  has any variation in the  $\mathbf{e}_k$ -direction. The electromagnetic field described by the first terms on right-hand sides of Eqs. (58) and (59) represents a plane wave with the amplitude variation  $\rho \cos \psi$ , which propagates in the  $\mathbf{e}_k$ -direction. Moreover, the electromagnetic field everywhere rotates along the  $\mathbf{e}_k$ -direction with the angular frequency  $\omega$ . We recognize the electromagnetic field as an inhomogeneous, circularly polarized wave propagating in the  $\mathbf{e}_k$ -direction. We can also confirm that the other electromagnetic field described by the second terms is also an inhomogeneous, circularly polarized wave traveling in the  $\mathbf{e}_k$ -direction. Thus, the Beltrami field  $(\mathbf{E}^f, \mathbf{B}^f)$  can be interpreted as the superposition of inhomogeneous, circularly polarized waves propagating in any possible direction corresponding to the northern hemisphere, where  $\theta \in (0, \pi/2]$  and  $\phi \in [-\pi, \pi]$ . Similarly, we can confirm that the other Beltrami field  $(\mathbf{E}^b, \mathbf{B}^b)$  is composed of the superposition of inhomogeneous, circularly polarized waves propagating in any possible direction corresponding to the southern hemisphere, where  $\theta \in [\pi/2, \pi)$  and  $\phi \in [-\pi, \pi]$ .

## 4. $\mathbf{E} \parallel \mathbf{B}$ FIELD WITH TIME-INDEPENDENT $f$

### 4.1. General Solution for the $\mathbf{E} \parallel \mathbf{B}$ Field

In this subsection, we examine  $\mathbf{E} \parallel \mathbf{B}$  fields with time-independent  $f$ . As opposed to the previous case, this case does not correspond exactly to any of the cases presented by Uehara et al. In Eq. (14), we observed that  $f$  satisfies an eikonal equation. We know that the solution of an eikonal equation corresponds to the trajectory of a light ray, which travels in a straight line in vacuum. Thus, each of the field lines of  $\nabla f$  is straight. We find that any two field lines of  $\nabla f$  do not intersect because if the two did intersect at a point  $\mathbf{r}_0$ , the magnitude of  $\nabla f$  would be zero at that point, which contradicts the fact that the eikonal Eq. (14) holds everywhere. Here, we assume without loss of generality that  $\nabla f$  lies along the  $z$ -direction, which yields  $\nabla f = k\mathbf{e}_z$ .

Equation (11) states that the  $z$ -component of  $\mathbf{v}$  is zero. Due to the requirement that  $\mathbf{v}$  be irrotational (i.e.,  $\nabla \times \mathbf{v} = 0$ ), both the  $x$ - and  $y$ -components of  $\mathbf{v}$  are functions only of  $x$ ,  $y$ , and  $t$ . As the time derivative of  $\mathbf{v}$  is perpendicular to  $\nabla f = k\mathbf{e}_z$ ,  $\mathbf{v}$  rotates along the  $z$ -direction with the angular frequency  $\omega = ck$ . Thus, by introducing a two-dimensional Laplacian field  $\mathbf{u}(x, y)$ ,  $\mathbf{v}$  can be

represented in the form

$$\mathbf{v}(x, y; t) = \mathbf{u}(x, y) \cos \omega t + (\mathbf{e}_z \times \mathbf{u}(x, y)) \sin \omega t, \quad (60)$$

which implies that the  $\mathbf{E} \parallel \mathbf{B}$  field considered in the present section is a special case of the planar Case III field analyzed by Nishiyama [16]. We can find such a two-dimensional Laplacian field  $\mathbf{u}(x, y)$  by employing the technique described by Nishiyama. Here, we consider a complex function  $g(x + iy)$  whose real and imaginary parts are equivalent to the  $x$ - and  $y$ -components of  $\mathbf{u}(x, y)$ , respectively; i.e.,  $g(x + iy) = u_x + iu_y$ . The requirement that  $\mathbf{u}$  be irrotational yields

$$\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} = 0, \quad (61)$$

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} = 0. \quad (62)$$

These equations correspond to the Cauchy-Riemann equations for the complex function  $g(x + iy)$ . Therefore,  $g(x + iy)$  is an entire function, and the  $x$ - and  $y$ -components of the Laplacian field  $\mathbf{u}$  correspond respectively to the real and imaginary parts of an arbitrary entire function. Thus, the general solution for an  $\mathbf{E} \parallel \mathbf{B}$  field with  $\partial f / \partial t = 0$  can be represented in the form

$$\mathbf{E}(\mathbf{r}; t) = E_0 \rho [\cos(\omega t + \psi) \mathbf{e}_x + \sin(\omega t + \psi) \mathbf{e}_y] \cos kz, \quad (63)$$

$$\mathbf{B}(\mathbf{r}; t) = B_0 \rho [\cos(\omega t + \psi) \mathbf{e}_x + \sin(\omega t + \psi) \mathbf{e}_y] \sin kz, \quad (64)$$

where  $\rho$  and  $\psi$  are the magnitude and argument of an entire function of  $x + iy$ , respectively. We now note that the superposition principle is applicable to these solutions due to the nature of entire functions; the superposition of multiple  $\mathbf{E} \parallel \mathbf{B}$  fields with time-independent  $f$  produces an  $\mathbf{E} \parallel \mathbf{B}$  field with time-independent  $f$ , provided that  $\nabla f$  of each original  $\mathbf{E} \parallel \mathbf{B}$  field points in the same direction.

A specific example of  $\mathbf{v}(\mathbf{r}; t)$  is now considered. Suppose that the entire complex function is defined by

$$g(x + iy) = e^{-(x+iy)^2}. \quad (65)$$

The corresponding  $\mathbf{v}(\mathbf{r}; t)$  is calculated from Eq. (60) as

$$\mathbf{v}(\mathbf{r}; t) = e^{-x^2+y^2} [\cos(2xy - \omega t) \mathbf{e}_x - \sin(2xy - \omega t) \mathbf{e}_y]. \quad (66)$$

A series of snapshots of the field distribution of  $\mathbf{v}(\mathbf{r}; t)$  is shown in Fig. 3.

As with the previous  $\mathbf{E} \parallel \mathbf{B}$  field case, we note the physically reasonable solutions. As an electromagnetic field in a physical system is required to be a bounded function, both  $u_x$  and  $u_y$  must be bounded functions. Applying Liouville's theorem to  $g(x + iy)$ , we find that  $g(x + iy)$  must be a constant function. Therefore, a physically relevant  $\mathbf{E} \parallel \mathbf{B}$  field with  $\partial f / \partial t = 0$  is constrained to have the form

$$\mathbf{E}(z; t) = E_0 (\cos \omega t \mathbf{e}_x + \sin \omega t \mathbf{e}_y) \cos kz, \quad (67)$$

$$\mathbf{B}(z; t) = B_0 (\cos \omega t \mathbf{e}_x + \sin \omega t \mathbf{e}_y) \sin kz, \quad (68)$$

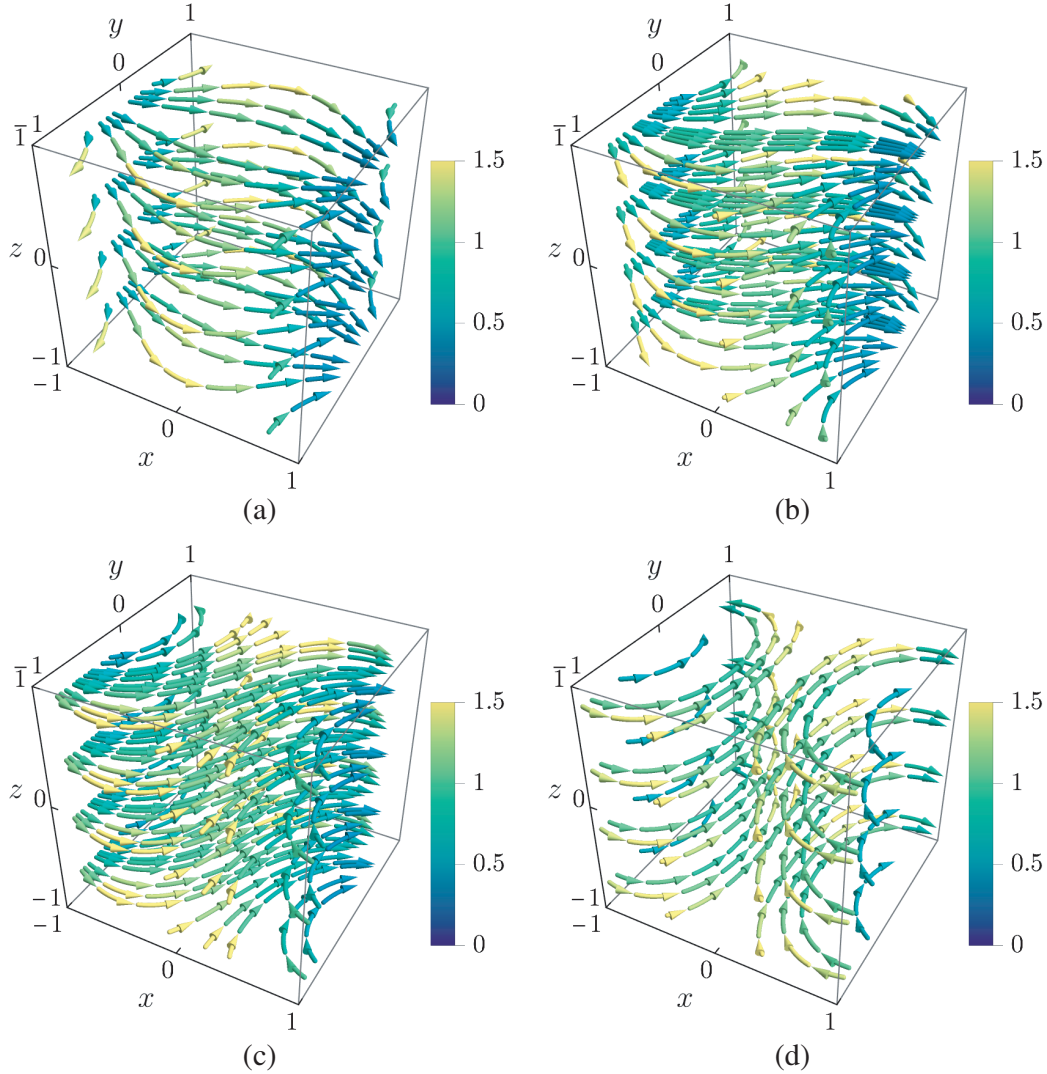
where  $E_0$  and  $B_0$  are constants. This physically acceptable result corresponds to the solution for the Case I  $\mathbf{E} \parallel \mathbf{B}$  field presented by Uehara et al.

#### 4.2. Decomposition of $\mathbf{E} \parallel \mathbf{B}$ Field into Two Counter-Propagating Planar Beltrami Fields

The general solution for the  $\mathbf{E} \parallel \mathbf{B}$  field given by Eqs. (63) and (64) can be rewritten as

$$\begin{aligned} \mathbf{E}(\mathbf{r}; t) &= \frac{E_0}{2} \rho [\cos(kz - \omega t - \psi) \mathbf{e}_x - \sin(kz - \omega t - \psi) \mathbf{e}_y] \\ &\quad + \frac{E_0}{2} \rho [\cos(-kz - \omega t - \psi) \mathbf{e}_x - \sin(-kz - \omega t - \psi) \mathbf{e}_y], \end{aligned} \quad (69)$$

$$\begin{aligned} \mathbf{B}(\mathbf{r}; t) &= \frac{B_0}{2} \rho [\sin(kz - \omega t - \psi) \mathbf{e}_x + \cos(kz - \omega t - \psi) \mathbf{e}_y] \\ &\quad + \frac{B_0}{2} \rho [-\sin(-kz - \omega t - \psi) \mathbf{e}_x - \cos(-kz - \omega t - \psi) \mathbf{e}_y]. \end{aligned} \quad (70)$$



**Figure 3.** Snapshots of the time-harmonic vector field described by Eq. (66) at different times. Arrow color indicates the magnitude of the vector at each point. (a)  $t = 0$ . (b)  $t = \pi/6\omega$ . (c)  $t = \pi/3\omega$ . (d)  $t = \pi/2\omega$ .

The first terms on the right-hand sides of Eqs. (69) and (70) are denoted as  $\mathbf{E}^f$  and  $\mathbf{B}^f$ , respectively, and the second terms on the right-hand sides of Eqs. (69) and (70) are denoted as  $\mathbf{E}^b$  and  $\mathbf{B}^b$ , respectively. The electromagnetic field  $(\mathbf{E}^f, \mathbf{B}^f)$  satisfies the eigenvalue equations

$$\nabla \times \mathbf{E}^f = k\mathbf{E}^f, \tag{71}$$

$$\nabla \times \mathbf{B}^f = k\mathbf{B}^f, \tag{72}$$

whereas the electromagnetic field  $(\mathbf{E}^b, \mathbf{B}^b)$  satisfies

$$\nabla \times \mathbf{E}^b = -k\mathbf{E}^b, \tag{73}$$

$$\nabla \times \mathbf{B}^b = -k\mathbf{B}^b, \tag{74}$$

This confirms that the two electromagnetic fields  $(\mathbf{E}^f, \mathbf{B}^f)$  and  $(\mathbf{E}^b, \mathbf{B}^b)$  are Beltrami fields with left- and right-handedness, respectively. Applying the expansions in Eqs. (58) and (59) to the planar Beltrami field  $(\mathbf{E}^f, \mathbf{B}^f)$ , we recognize the electromagnetic field  $(\mathbf{E}^f, \mathbf{B}^f)$  as the superposition of two inhomogeneous, left-handed circularly polarized waves propagating in the  $+z$ -direction. Similarly, we find that the other

Beltrami field ( $\mathbf{E}^b, \mathbf{B}^b$ ) consists of two right-handed circularly polarized waves with inhomogeneous amplitudes, traveling in the  $-z$ -direction. We can summarize these results concerning the decomposition of an  $\mathbf{E} \parallel \mathbf{B}$  field with time-independent as follows: Every  $\mathbf{E} \parallel \mathbf{B}$  field with  $\partial f / \partial t = 0$  can be decomposed into counter-propagating left- and right-handed planar Beltrami fields, each of which is composed of the superposition of two inhomogeneous, circularly polarized waves.

## 5. CONCLUSIONS

In this study, we derived general solutions of the time-harmonic Maxwell's equations such that the electric and magnetic fields are parallel to each other. We also demonstrated that every time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  field is composed of the superposition of two oppositely propagating Beltrami fields.

We started with the classification of time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  fields. After reviewing Uehara et al.'s expressions for  $\mathbf{E} \parallel \mathbf{B}$  fields, i.e.,  $\mathbf{E} = E_0 \mathbf{v} \cos f$  and  $\mathbf{B} = B_0 \mathbf{v} \sin f$ , we classified time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  fields into one of two types, corresponding to the cases where  $f$  is a linear function of time (first case) or  $f$  is independent of time (second case). Before analyzing the first-case  $\mathbf{E} \parallel \mathbf{B}$  field, which is a particular type of Beltrami fields, we defined the handedness of a Beltrami field by expanding the concept of the handedness of circularly polarized waves. Subsequently, we obtained the general solution for the first-case  $\mathbf{E} \parallel \mathbf{B}$  field in integral form and thereby demonstrated that every first-case  $\mathbf{E} \parallel \mathbf{B}$  field is composed of two oppositely propagating Beltrami fields with the same handedness. Next, we derived the general solution for the second-case  $\mathbf{E} \parallel \mathbf{B}$  field and proved that every second-case  $\mathbf{E} \parallel \mathbf{B}$  field always consists of two counter-propagating, planar Beltrami fields with opposite handedness.

The findings of this work can guide further studies on two pending issues, namely (a) finding general solutions for time-harmonic  $\mathbf{E} \parallel \mathbf{B}$  fields close to charge or current distributions and (b) deriving general solutions for  $\mathbf{E} \parallel \mathbf{B}$  fields with an arbitrary time dependence. In addition to the results of the present work, further mathematical approaches are necessary for analyzing these issues. Solving the first problem requires the development of a Green's function method that can be used to find the fundamental solution of the nonhomogeneous Helmholtz equation that describes time-harmonic electromagnetic fields in the vicinity of charged particles and current sources. Solving the second problem requires exploiting time-frequency Fourier-transform techniques. Using the Fourier transform of  $\mathbf{E} \parallel \mathbf{B}$  fields having an arbitrary time dependence, the problem can be reduced to the calculus of vector fields with a single-frequency time dependence, which we studied herein.

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