

# Diffraction Radiation Generated by a Density-Modulated Electron Beam Flying over the Periodic Boundary of the Medium Section.

## IV. Structures of Finite Thickness

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**Abstract**—The paper is focused on reliable modeling and analysis of the effects connected with the resonant transformation of the field of a plane and density-modulated electron beam flying over the periodic rough boundary of a natural or artificial medium. In the paper, such a medium occupies a part of the half-space, limited in thickness. Therefore, the numerous effects appeared due to transverse (in the thickness of the periodic structure) resonances, and the coupling of eigen regimes of two different periodic interfaces also contributes to the anomalies appearing due to excitation of the surface eigen waves of the periodic boundary interface that had been discussed in previous papers of this series.

### 1. INTRODUCTION

As we wrote in the first of three papers [1–3] of this series, Vavilov-Cherenkov radiation (VChR) [4] and Smith-Purcell radiation (SPR) [5] are among the most significant physical phenomena discovered in the 20th century. Classic works [6–9] are devoted to the theory of these phenomena and questions of their practical use. The surge of interest to them in modern science is mainly due to a growing list of relevant physical and applied problems, whose solution is facilitated or could be facilitated by new knowledge about the effects appearing in the processes of diffraction radiation (VChR or SPR). We are interested in this topic because in several computational experiments when working with sufficiently reliable mathematical models and corresponding computational schemes, the results that proved the possibility of anomalously high levels of coherent diffraction radiation during the flight of a density-modulated electron beam over a periodic boundary separating the usual artificial environment and environment with a specific frequency dispersion of permittivity and permeability have been obtained. It is not enough to simply record such results, and they also need to be convincingly explained in order to make them interesting, to disclose their potential for prospective consumers working in the field of diffraction electronics, diagnostics of charged particle beams, and monitoring of natural or artificial materials and environments of practical interest. The works [1–3] are just devoted to this problem, and there are the main components that are solved: (i) a mathematical model has been constructed and implemented in software that allows obtaining reliable numerical data on all the features of the processes of coherent diffraction radiation during the flight of a plane density-modulated electron beam over a periodic boundary of a half-space occupied by a dispersion medium [1]; (ii) the conditions for the excitation of the so-called eigen modes of the corresponding periodic structure were studied, and those of them which could provoke the effects with an anomalously high level of VChR and SPR were determined [2]; (iii) problem-oriented computational experiments confirming the conclusions based on

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*Received 27 April 2021, Accepted 10 June 2021, Scheduled 23 June 2021*

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an analytical consideration of model problems were carried out; they once again demonstrated the possibility of implementing regimes of spatial-frequency transformations of the electron beam field having undoubted practical interest [3].

All the results of works [1–3] concerned the system ‘flat density-modulated electron beam — the periodic boundary of the half-space occupied by the dispersion medium’. Obviously, with necessity they had to be ‘extended’ to more realistic systems in which the dispersion half-space would give way to periodic structures with the same material parameters, but limited in thickness, or the geometric variety of multilayer structures arising due to their wide application in various areas of science and technology. This is the provenance and background of the problem considered in this paper. We start it with a brief presentation of the model problem, which is the key for all situations of interest here and was considered in detail in [1].

In [1, 10], we had presented the algorithm for the accurate solution to the boundary value problem

$$\left\{ \begin{array}{l} [\partial_y^2 + \partial_z^2 + \varepsilon(g, k) \mu(g, k) k^2] U(g, k) = 0; \quad g = \{y, z\} \in \Omega_{int} \\ \mathbf{E}_{tg}(q, k), \quad \mathbf{H}_{tg}(q, k) \quad \text{are continuous when crossing } \Sigma^{\varepsilon, \mu} \\ \text{and virtual boundaries } y = 0, \quad y = -h; \quad q = \{x, y, z\} \\ U \{\partial_z U\}(y, l, k) = \exp(2\pi i \zeta) U \{\partial_z U\}(y, 0, k) \quad \text{for } -h \leq y \leq 0 \end{array} \right., \quad (1a)$$

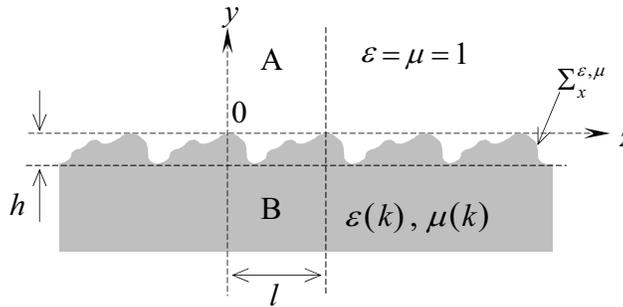
$$\begin{aligned} U(g, k) &= V_0(g, k) + U^+(g, k) = V_0(g, k) + \sum_{n=-\infty}^{\infty} U_n^+(g, k) \\ &= \exp(-i\Gamma_0^+ y) \varphi_0(z) + \sum_{n=-\infty}^{\infty} R_n(k) \exp(i\Gamma_n^+ y) \varphi_n(z); \quad g \in \bar{A}, \end{aligned} \quad (1b)$$

$$U(g, k) = U^-(g, k) = \sum_{n=-\infty}^{\infty} U_n^-(g, k) = \sum_{n=-\infty}^{\infty} T_n(k) \exp(-i\Gamma_n^-(y+h)) \varphi_n(z); \quad g \in \bar{B} \quad (1c)$$

describing the diffraction of a plane  $H$ -polarized ( $E_x^i = H_y^i = H_z^i = 0, \partial_x = 0$ ) wave  $V_0(g, k) = H_x^i(g, k) = \exp(-i\Gamma_0^+ y) \varphi_0(z)$ , coming from an upper half-space to the periodic rough boundary  $\Sigma^{\varepsilon, \mu} = \Sigma_x^{\varepsilon, \mu} \times (-\infty < x < \infty)$ ,  $\Sigma_x^{\varepsilon, \mu} = \{g : y = f(z), -h \leq f(z) \leq 0, f(z) = f(z+l)\}$ , separating vacuum ( $\varepsilon = \mu = 1.0$ ) and a dispersive (in a common case) medium with constitutive parameters  $\varepsilon(k), \mu(k) : \text{Im}\varepsilon(k) = \text{Im}\mu(k) = 0$  (see Fig. 1). Here,  $U(g, k) = H_x(g, k)$ ,

$$E_y(g, k) = -\frac{\eta_0}{ik\varepsilon(g, k)} \partial_z H_x(g, k), \quad \text{and} \quad E_z(g, k) = \frac{\eta_0}{ik\varepsilon(g, k)} \partial_y H_x(g, k) \quad (2)$$

— are the nonzero components of the total field  $\{\mathbf{E}(g, k), \mathbf{H}(g, k)\}$ ,  $g = \{y, z\}$ ,  $\partial_x = 0$ , formed by the system ‘boundary  $\Sigma^{\varepsilon, \mu}$  — electron flow’;  $\Omega_{int} = \{g \in \mathbb{R} : -h < y < 0\}$ ,  $A = \{g \in \mathbb{R} : y > 0\}$ ,  $B = \{g \in \mathbb{R} : y < -h\}$ ,  $R = \{g = \{y, z\} \in \mathbb{R}^2 : 0 < z < l\}$ , and  $\bar{G}$  is the closure of the domain  $G$ .  $\Gamma_n^+ = \sqrt{k^2 - \Phi_n^2}$ ,  $\text{Re}\Gamma_n^+ \geq 0, \text{Im}\Gamma_n^+ \geq 0$  and  $\Gamma_n^- = \sqrt{k^2 \varepsilon(k) \mu(k) - \Phi_n^2}$ ,  $\varepsilon^{-1}(k) \text{Re}\Gamma_n^- \geq 0, \text{Im}\Gamma_n^- \geq 0$  are



**Figure 1.** Periodic boundary between two media: upper half-space is filled with vacuum and lower half-space is filled with dispersive material.

vertical propagation constants of spatial harmonics  $U_n^+(g, k)$  and  $U_n^-(g, k)$ , outgoing upward (in half-space occupied by vacuum) and downward (in half-space occupied by a dispersive medium) from the boundary  $\Sigma_x^{\varepsilon, \mu}$ ; functions  $\varphi_n(z) = l^{-1/2} \exp(i\Phi_n z)$ ,  $n = 0, \pm 1, \pm 2, \dots$  form a complete (in space  $L_2(0, l)$ ) orthonormal system in the cross-section of the Floquet channel R;  $\Phi_n = (n + \zeta)2\pi/l$ ,  $\zeta : \text{Im}\zeta = 0$  is a numeric parameter, characterizing the phase shift of the field of spatial harmonics  $U_n^+(g, k)$  and  $U_n^-(g, k)$  along one period of the structure;  $k = 2\pi/\lambda$  is a frequency parameter;  $\lambda$  is a wavelength in free space. The choice of the branches of the square roots  $\Gamma_n^\pm$  is due to the so-called partial radiation conditions [1, 11–14], according to which the fields  $U^\pm(g, k)$  should not contain harmonics arriving (transferring energy) from  $y = \pm\infty$  to the boundary  $\Sigma_x^{\varepsilon, \mu}$ . There are no current sources in problem (1) (the right-hand side in the Helmholtz equation is equal to zero). The periodic boundary is excited by a homogeneous (in the case  $\text{Im}\Gamma_0^+ = 0$ ) or inhomogeneous (in the case  $\text{Im}\Gamma_0^+ > 0$ ) plane wave  $V_0(g, k)$ .

How is the problem of analyzing the effects of diffraction radiation arising during a density-modulated electron beam flight over a periodic boundary  $\Sigma_x^{\varepsilon, \mu}$  and the model problem of the electrodynamic theory of grating (1) related? Suppose that over the boundary  $\Sigma_x^{\varepsilon, \mu}$  the density-modulated electron beam is flying, and its instantaneous charge density can be written as  $\rho\delta(y - c) \exp[i(k/\beta)z]$ ,  $c \geq 0$  (see Fig. 1). Here,  $\rho$  and  $k$  are the modulation amplitude and modulation frequency of the beam, and  $\beta < 1$  is its relative velocity. The electromagnetic field of such an electron beam is  $H$ -polarized field ( $E_x^{\text{beam}} = H_y^{\text{beam}} = H_z^{\text{beam}} = 0$ ,  $\partial_x = 0$ ) with [9, 15]

$$H_x^{\text{beam}}(g, k) = 2\pi\rho\beta \exp \left\{ i \left[ \sqrt{k^2 - (k/\beta)^2} |y - c| + (k/\beta) z \right] \right\} [|y - c|/(y - c)]; \quad y \neq c.$$

From this representation it follows that in the presence of a plane (in case  $h = 0$ ) or periodic ( $h > 0$ ) boundary  $\Sigma_x^{\varepsilon, \mu}$  between vacuum and dispersive medium, the density-modulated electron beam with  $-2\pi\rho\beta\sqrt{l} \exp[-kc\sqrt{(1/\beta)^2 - 1}] = 1$  and  $k/\beta = \Phi_0 = \zeta 2\pi/l$  (for such value of  $\Phi_0$ ,  $\Gamma_0^+$  is imaginary) is generated in domains  $y > 0$  and  $y < -h$   $H$ -polarized field; its  $H_x$ -components  $U^+(g, k)$  and  $U^-(g, k)$  are defined from the solution  $U(g, k)$  to boundary value problem (1). Indeed, under the conditions specified above  $H_x^{\text{beam}}(g, k) = V_0(g, k) = H_x^i(g, k)$  for  $y < c$ , the electromagnetic field of electron beam and the field of an inhomogeneous plane wave exciting the boundary  $\Sigma_x^{\varepsilon, \mu}$  coincide. Consequently, the secondary fields arising as a result of such excitation also coincide.

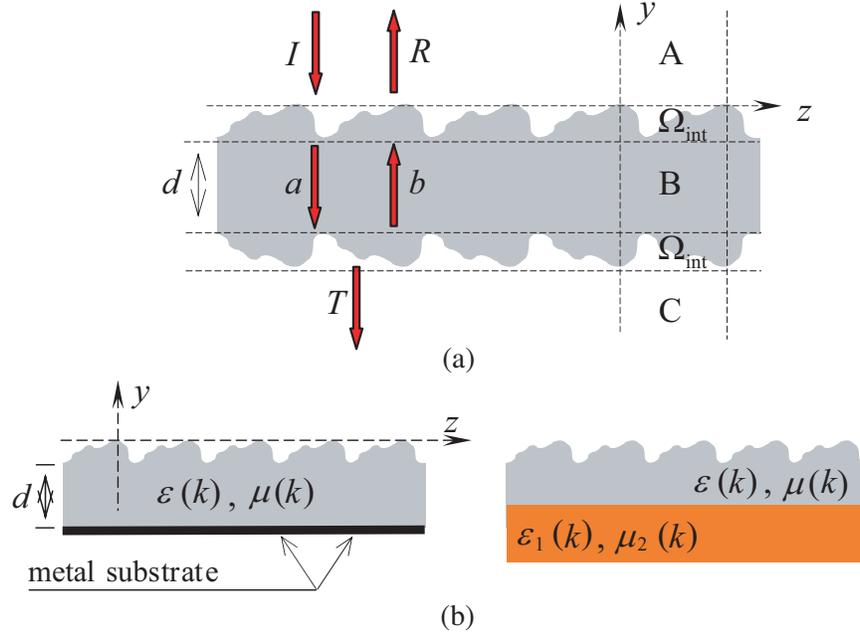
This allowed us, while solving numerically the problem (1) for artificial plasma-like medium in lower half-space  $y < f(z)$ , to investigate anomalous and resonant phenomena, accompanying the excitation of Vavilov-Cherenkov or Smith-Purcell radiation, and unambiguously relate these phenomena to the excitation of ‘unusual true eigen waves’ of the periodic boundary  $\Sigma_x^{\varepsilon, \mu}$  [2, 3]. In this paper, we continue our analysis, extending it to the structures with similar materials and geometric parameters, but limited in thickness (see, for example, Fig. 2). The relevance of such an analysis is due, first of all, to the fact that such structures are widely used in optics, microwave, and antenna technology.

Same as in [1–3], the time dependence  $t$  for harmonic processes considered in this work is defined by the factor  $\exp(-i\omega t)$  omitted everywhere, and  $\omega$  is a circular frequency. The dimensions of the SI system of all mentioned physical quantities are also omitted.

## 2. BASIC ELECTROMAGNETIC MODELS

For solving problem (1), we used the method of analytical regularization [1, 10, 11, 13, 14, 16, 17], which reduces the boundary value problems of the diffraction theory to the Fredholm operator equations of the second kind, allowing us to calculate all interesting characteristics of electromagnetic wave resonant scattering with any required accuracy. In the case of the problem (1), the role of such an operator equation is played by the infinite system of linear algebraic equations with respect to sets of complex amplitudes  $\{R_n\}_{n=-\infty}^{\infty}$ ,  $\{T_n\}_{n=-\infty}^{\infty}$  from the space of infinite sequences  $\tilde{l}_2 = \{a = \{a_n\}_{n=-\infty}^{\infty} : \sum_n |a_n|^2 (1 + |n|) < \infty\}$  [13, 16].

Obviously, quite formal substitutions  $V_0(g, k) \rightarrow V_p^A(g, k) = \exp(-i\Gamma_p^+ y) \varphi_p(z)$ ,  $p = 0, \pm 1, \pm 2, \dots$ ,  $R_n \rightarrow R_{np}^{AA}$ ,  $T_n \rightarrow T_{np}^{BA}$  or  $V_0(g, k) \rightarrow V_p^B(g, k) = \exp(i\Gamma_p^-(y + h)) \varphi_p(z)$ ,  $g \in B$ ,  $R_n \rightarrow T_{np}^{AB}$ ,  $T_n \rightarrow R_{np}^{BB}$ , lead us to consideration of the situations of a scattering of homogeneous and inhomogeneous plane waves  $V_p^A(g, k)$  or  $V_p^B(g, k)$  arriving onto the boundary  $\Sigma_x^{\varepsilon, \mu}$  from a region A or from a region B. So there are



**Figure 2.** Layered structures with periodic interfaces: (a) a slab of dispersive material; (b) a coat on a metal substrate and a two-layer structure made of dispersive material.

no fundamental changes in the problem (1). Thus, within the framework of the previously developed algorithm, we obtain the possibility to rigorously calculate the generalized scattering matrices:

$$\begin{aligned}
 R^{AA} &= \{R_{np}^{AA}\}_{n,p=-\infty}^{\infty}, & T^{BA} &= \{T_{np}^{BA}\}_{n,p=-\infty}^{\infty} \\
 \text{and } R^{BB} &= \{R_{np}^{BB}\}_{n,p=-\infty}^{\infty}, & T^{AB} &= \{T_{np}^{AB}\}_{n,p=-\infty}^{\infty},
 \end{aligned} \tag{3}$$

These matrices provide us with the operators, limited on a pair of spaces  $\tilde{l}_2 \rightarrow \tilde{l}_2$  operators [13, 16, 18], which determine all the electrodynamic characteristics of the considered periodic structure.

The extensive library of operators (3) (for many different values of constitutive parameters  $\varepsilon(k)$ ,  $\mu(k)$  and configurations of boundaries  $\Sigma_x^{\varepsilon,\mu}$ ) may be sufficient to rigorously solve problems of type (1) within the framework of the method of generalized scattering matrices [11, 14, 16, 19, 20] for a considerable number of layered structures with periodic rough and plane interfaces (see, for example, Fig. 2). The construction of the corresponding resulting operator equations of the second kind on a pair of spaces  $\tilde{l}_2 \rightarrow \tilde{l}_2$  follows the same standard schema [14] in all cases. Below, we briefly discuss the implementation of this schema only for one of the simplest structures whose geometry is shown in Fig. 2(a).

The boundary value problem describing the scattering of a plane  $H$ -polarized wave  $V_0(g, k) = H_x^i(g, k) = \exp(-i\Gamma_0^+ y)\varphi_0(z)$  arriving at the structure in Fig. 2(a) from a half-space  $y > 0$  can be written in the following form:

$$\left\{ \begin{array}{l}
 [\partial_y^2 + \partial_z^2 + \varepsilon(g, k)\mu(g, k)k^2] U(g, k) = 0; \quad g = \{y, z\} \in \Omega_{int} \\
 \mathbf{E}_{tg}(q, k), \quad \mathbf{H}_{tg}(q, k) \text{ are continuous when crossing } \Sigma^{\varepsilon,\mu} \\
 \text{and virtual boundaries } y = 0, \quad y = -h, \quad y = -(h+d), \quad y = -(2h+d), \\
 U\{\partial_z U\}(y, l, k) = \exp(2\pi i\zeta) U\{\partial_z U\}(y, 0, k) \text{ for } -h \leq y \leq 0 \text{ and} \\
 -2h-d \leq y \leq -h-d
 \end{array} \right. , \tag{4a}$$

$$\begin{aligned}
 U(g, k) &= V_0(g, k) + U^+(g, k) = V_0(g, k) + \sum_{n=-\infty}^{\infty} U_n^+(g, k) \\
 &= \exp(-i\Gamma_0^+ y) \varphi_0(z) + \sum_{n=-\infty}^{\infty} R_n(k) \exp(i\Gamma_n^+ y) \varphi_n(z); \quad g \in \bar{A}, \quad (4b)
 \end{aligned}$$

$$U(g, k) = \sum_{n=-\infty}^{\infty} [a_n(k) \exp(-i\Gamma_n^-(y+h)) + b_n(k) \exp(i\Gamma_n^-(y+h+d))] \varphi_n(z); \quad g \in \bar{B}, \quad (4c)$$

$$U(g, k) = U^-(g, k) = \sum_{n=-\infty}^{\infty} U_n^-(g, k) = \sum_{n=-\infty}^{\infty} T_n(k) \exp(-i\Gamma_n^-(y+h)) \varphi_n(z); \quad g \in \bar{C}. \quad (4d)$$

The method of generalized scattering matrices connects the sets of amplitude coefficients  $I = \{\delta_0^n\}_{n=-\infty}^{\infty}$  ( $\delta_0^n$  — is Kronecker symbol),  $R = \{R_n\}_{n=-\infty}^{\infty}$ ,  $T = \{T_n\}_{n=-\infty}^{\infty}$ ,  $a = \{a_n\}_{n=-\infty}^{\infty}$  and  $b = \{b_n\}_{n=-\infty}^{\infty}$  via the system of operator equations

$$\begin{cases} R = R^{AA}I + T^{AB}E^B b \\ a = T^{BA}I + R^{BB}E^B b \\ b = R^{BB}E^B a \\ T = T^{AB}E^B a \end{cases}, \quad (5)$$

In Eq. (5), all stages of the formation of the response of a structure to excitation by a wave  $V_0(g, k)$ , corresponding to a set of amplitude coefficients  $I$ , are clearly presented. So, for example, the first of the equations of system in Eq. (5) can be read as follows: the field of the wave  $U^+(g, k)$  in reflection domain (it corresponds to a set of amplitudes  $R$ ) is the sum of the fields, the first of which is formed as a result of wave  $V_0(g, k)$  reflection by the boundary  $\Sigma_x^{\varepsilon, \mu}$  (operator  $R^{AA}$ ), and the second is determined by the field of a wave leaving an imaginary boundary  $y = -(h+d)$  (a set of amplitudes  $b$  corresponds to this wave), influenced by a regular region  $B$  (operator  $E^B$ ) and transforming boundary  $\Sigma_x^{\varepsilon, \mu}$  (operator  $T^{AB}$ ).

The system in Eq. (5) is reduced to the Fredholm operator equation

$$a = T^{BA}I + R^{BB}E^B R^{BB}E^B a \quad (6)$$

(the operator  $E^B = \{\delta_n^p \exp(i\Gamma_n^- d)\}_{n,p=-\infty}^{\infty}$ ,  $\text{Im } \Gamma_n^- \geq 0$  is compact [13, 14, 16, 21] on a pair of spaces  $\tilde{l}_2 \rightarrow \tilde{l}_2$ ) and further simple recalculating formulas, defining the sets of complex amplitudes  $b$ ,  $R$ , and  $T$  in the space of infinite sequences  $\tilde{l}_2$ . Truncation of Equation (6) reduces the solution of problem (4) to inversion of the finite (of the order  $2N+1$ ) well-conditioned system of linear algebraic equations. The error of the solutions obtained in the implementation of such a computational scheme, estimated by the norm of the space  $\tilde{l}_2$  [16], is of the order of magnitude  $\exp(-Nd/l)$ .

### 3. A PERIODIC COAT OF DISPERSIVE MATERIAL BACKED WITH A METAL SUBSTRATE. EIGEN WAVES

Information about the eigen waves of the periodic structures is important for the correct analysis of the physical nature of the diffraction radiation processes [13]. Below, the focus is on the differences in the spectra of eigen waves of a semi-infinite structure (see [1–3] and Fig. 1) and structures of finite thickness, and the examples of such a geometry are presented in Fig. 2. These differences are mainly due to the reflection of waves from the boundaries separating layers with different constitutive parameters. Their characteristic features are most easily noted by considering one specific structure. Let it be a periodic dielectric coat backed with the metal substrate (left fragment of Fig. 2(b)). The boundary value problem describing the scattering of a plane  $H$ -polarized wave  $V_0(g, k) = H_x^i(g, k) = \exp(-i\Gamma_0^+ y) \varphi_0(z)$  arriving

at such a structure from a half-space  $y > 0$  can be written in the following form:

$$\left\{ \begin{array}{l} [\partial_y^2 + \partial_z^2 + \varepsilon(g, k) \mu(g, k) k^2] U(g, k) = 0; \quad g = \{y, z\} \in \Omega_{int} \\ \mathbf{E}_{tg}(g, k)|_{y=-h-d} = 0 \text{ and } \mathbf{E}_{tg}(g, k), \mathbf{H}_{tg}(g, k) \text{ are continuous when crossing} \\ \Sigma^{\varepsilon, \mu} \text{ and virtual boundaries } y = 0, \quad y = -h \\ U\{\partial_z U\}(y, l, k) = \exp(2\pi i \zeta) U\{\partial_z U\}(y, 0, k) \quad \text{for } -h \leq y \leq 0 \end{array} \right. , \quad (7a)$$

$$\begin{aligned} U(g, k) &= V_0(g, k) + U^+(g, k) = V_0(g, k) + \sum_{n=-\infty}^{\infty} U_n^+(g, k) \\ &= \exp(-i\Gamma_0^+ y) \varphi_0(z) + \sum_{n=-\infty}^{\infty} R_n(k) \exp(i\Gamma_n^+ y) \varphi_n(z); \quad g \in \bar{A}, \end{aligned} \quad (7b)$$

$$U(g, k) = \sum_{n=-\infty}^{\infty} [a_n(k) \exp(-i\Gamma_n^-(y+h)) + b_n(k) \exp(i\Gamma_n^-(y+h+d))] \varphi_n(z); \quad g \in \bar{B}. \quad (7c)$$

Here,  $\Omega_{int} = \{g \in \mathbb{R} : -h-d < y < 0\}$ ,  $A = \{g \in \mathbb{R} : y > 0\}$ ,  $B = \{g \in \mathbb{R} : -h-d < y < -h\}$ .

Clearly, in the case of a plane (non-transforming) boundary  $y = f(z) \equiv 0$  ( $h = 0$ ), homogeneous problem (7) (problem (7) with  $V_0(g, k) \equiv 0$ ) is reduced to an infinite set of independent homogeneous systems of linear algebraic equations

$$\left\{ \begin{array}{l} R_n = a_n + b_n \exp(i\Gamma_n^- d) \\ R_n \Gamma_n^+ = [-a_n + b_n \exp(i\Gamma_n^- d)] \Gamma_n^- \varepsilon^{-1}(k) \quad ; \quad n = 0, \pm 1, \pm 2, \dots \\ -a_n \exp(i\Gamma_n^- d) + b_n = 0 \end{array} \right. \quad (8)$$

with respect to unknown complex amplitudes  $R_n$  and  $a_n, b_n$ . The equations of systems (8) are obtained by matching in the plane  $y = 0$  the tangential components of the field  $\{\mathbf{E}(g, k), \mathbf{H}(g, k)\}$  — the  $H_x$ -component is equal to  $U(g, k)$ , and the  $E_z$ -component is connected with  $U(g, k)$  via the relation in Eq. (2) — and satisfying the condition  $\mathbf{E}_{tg}(g, k) = 0$  on the metal substrate. Dispersion equations follow from Eq. (8)

$$[1 + \exp(i2\Gamma_n^- d)] \Gamma_n^+ = -[1 - \exp(i2\Gamma_n^- d)] \Gamma_n^- \varepsilon^{-1}(k); \quad n = 0, \pm 1, \pm 2, \dots \quad (9)$$

Non-trivial solutions  $\Phi_n(\bar{\zeta}_n) = (n + \bar{\zeta}_n)2\pi/l$  to Equation (9) for each fixed value of  $k > 0$  determine an infinite set of practically identical eigen waves of a ‘periodic’ structure:

$$\begin{aligned} U(g, \bar{\zeta}_n) &= \begin{cases} U_n^+(g, \bar{\zeta}_n) = l^{-1/2} R_n \exp[i(\Gamma_n^+(\bar{\zeta}_n)y + \Phi_n(\bar{\zeta}_n)z)]; & y > 0 \\ l^{-1/2} \exp[i\Phi_n(\bar{\zeta}_n)z] [a_n \exp[-i\Gamma_n^-(\bar{\zeta}_n)y] + b_n [i\Gamma_n^-(\bar{\zeta}_n)(y+d)]]; & -d < y < 0, \end{cases} \\ \Gamma_n^+(\bar{\zeta}_n) &= \sqrt{k^2 - \Phi_n^2(\bar{\zeta}_n)}, \quad \Gamma_n^-(\bar{\zeta}_n) = \sqrt{k^2 \varepsilon(k) \mu(k) - \Phi_n^2(\bar{\zeta}_n)}, \quad n = 0, \pm 1, \pm 2, \dots \end{aligned} \quad (10)$$

Here  $\bar{\zeta}_n = -n + \Phi_n(\bar{\zeta}_n)l/2\pi$  are the propagation constants of eigen waves in Eq. (10). The nature of these waves determines not only the specific value of a complex, in the general case, magnitude of  $\bar{\zeta}$ , but also its position on the two-sheeted surface  $F_n$  of a two-valued function  $\Gamma_n^+(\zeta)$  (see details in [1, 2, 13, 22]). On the real axis  $\text{Re}\zeta$  in the first (physical) sheet of this surface, the inequalities  $\text{Re}\Gamma_n^+ \geq 0$  and  $\text{Im}\Gamma_n^+ \geq 0$  hold.

Since the electrodynamic characteristics of structures are significantly influenced only by eigen waves, with propagation constants  $\bar{\zeta}$  located on the first (physical) sheets of Riemannian surface, which is the natural region of variation of the corresponding spectral parameter [3], below in analyses of the dispersion Equation (9), we will dwell precisely on the conditions for the existence of such waves. Let the range of variation of the frequency parameter  $k$  be such that  $\text{Im}\Gamma_n^-(\bar{\zeta}_n) > 0$ . Then, for sufficiently large values of  $d$ , Equation (9) almost exactly coincides with dispersion Equation (6) from [2]. This means that surface ‘true eigen waves’ of the boundary  $y = 0$  separating the vacuum and the half-space filled

with a dispersive medium will necessarily reveal themselves in the case of the structure of finite thickness considered here. In particular, when studying the electrodynamic characteristics of the object depicted on the left fragment of Fig. 2(b), with of plasma-like media of layer, characterized with dispersion law

$$\varepsilon(k) = 1 - k_\varepsilon^2/k^2 \quad \text{and} \quad \mu(k) = 1 - k_\mu^2/k^2, \quad (11)$$

we will certainly see the influence of eigen waves, ‘continuing’ for a periodic boundary the so-called ‘true eigen waves’

$$U(g, \bar{\zeta}_n^\pm); \quad \Phi_n(\bar{\zeta}_n^\pm) = \pm \frac{k}{k_\varepsilon} \sqrt{\frac{(k_\varepsilon^2 - k^2)(k_\varepsilon^2 - k_\mu^2)}{k_\varepsilon^2 - 2k^2}} \quad (12)$$

of plane boundary  $y = 0$ , having area of existence that is limited by intervals  $k_\varepsilon k_\mu (k_\mu^2 + k_\varepsilon^2)^{-1/2} < k < \sqrt{0.5} k_\varepsilon = k^{\text{sing}}$  for  $k_\varepsilon > k_\mu$  and  $\sqrt{0.5} k_\varepsilon < k < k_\varepsilon k_\mu (k_\mu^2 + k_\varepsilon^2)^{-1/2}$  for  $k_\varepsilon < k_\mu$  [2].

Let now the range of variation of the frequency parameter  $k$  is such that  $\text{Im} \Gamma_n^-(\bar{\zeta}_n) = 0$ . Equation (9) can be written as follows:

$$\cos(\Gamma_n^- d) \Gamma_n^+ = i \sin(\Gamma_n^- d) \Gamma_n^- \varepsilon^{-1}(k); \quad n = 0, \pm 1, \pm 2, \dots \quad (13)$$

Assuming the value of  $\text{Re} \Gamma_n^- \varepsilon^{-1}(k)$  to be nonnegative (this condition is required only in the case of a semi-infinite dispersive medium, in the case of structures limited in thickness, the signs of the real parts of the propagation constants of plane waves in layers can be chosen arbitrarily), we obtain from Eq. (13) the equation

$$\Gamma_n^+ = itg(\Gamma_n^- d) \Gamma_n^- \varepsilon^{-1}(k), \quad tg(\Gamma_n^- d) \geq 0; \quad n = 0, \pm 1, \pm 2, \dots, \quad (14)$$

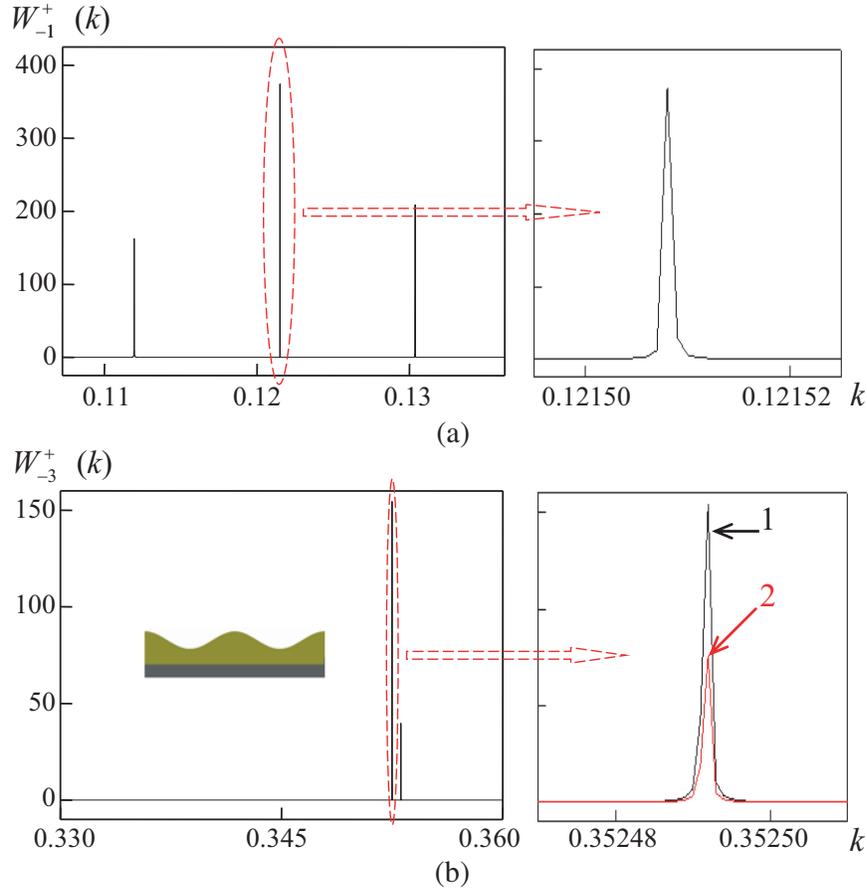
providing easy calculation of points  $\Phi_n(\bar{\zeta}_n)$  and  $\bar{\zeta}_n$  of eigen waves spectra in Eq. (10) from the first sheet of the surface  $F_n$  (here  $\text{Re} \Gamma_n^+ \geq 0$  and  $\text{Im} \Gamma_n^+ \geq 0$ ). The waves  $U(g, \bar{\zeta}_n)$  corresponding to these points are also ‘true eigen waves’, but their field, decreasing exponentially with distance from the structure in the domain A, freely oscillates with constant and possibly very large amplitudes  $a_n$  and  $b_n$  in the region B. It looks like ‘trapped’ in this area.

The spectrum of eigen waves determined by Equation (14) is much richer than the spectrum of ‘true eigen waves’ of a semi-infinite structure. It is known [1, 13, 23] that the eigen waves of structures with plane boundaries ‘smoothly transform’ into eigen waves of the corresponding structures with periodic boundaries with a smooth change in the height of their profiling. Therefore, taking into account the above conclusion, one should expect that the physics of the processes of scattering of plane waves say, in the case of the structures shown in Fig. 2, will be much richer in different resonance effects than in the case of a dispersive half-space. However even in this latter case, quite a few of them were noted [3].

#### 4. A PERIODIC COAT OF DISPERSIVE MATERIAL BACKED WITH A METAL SUBSTRATE. SMITH-PURCELL RADIATION

Let a plane, density-modulated electron beam flies over the structure shown in the left fragment of Fig. 2(b). Moreover, let the constitutive parameters of the medium filling the slab  $-d - h \leq y \leq f(z)$  be given by relations in Eq. (11). The physics of the process of interaction of the beam with the periodic metal-backed coat is described by the boundary value problem (7), where in this case [1, 3]  $\Phi_0 = \zeta 2\pi/l = k/\beta$ ,  $k$ , and  $0 < \beta < 1$  — are the modulation frequency and the relative beam rate. The modulation frequency sets the wavenumber  $k = 2\pi/\lambda$ , and  $\lambda$  is the length of the waves that make up the field of diffraction radiation (Smith-Purcell radiation) into free space  $y > 0$ . The intensity of this radiation is determined by the quantities  $W_n^+(k) = |R_n|^2 \text{Re} \Gamma_n^+ |\Gamma_0^+|^{-1}$ ,  $W_n^+(k)$  — is the energy transformed into the spatial harmonics of the periodic structure propagating without damping ( $\text{Im} \Gamma_n^+ = 0$ ). Only negative numbers  $n$  can correspond to such harmonics in the above given formula for  $\Phi_0$ .

Figure 3 shows the electrodynamic characteristics of the structure with parameters  $k_\varepsilon = 0.5$ ,  $k_\mu = 0.4$ ,  $h = 0.1$ ,  $d = 4\pi$ . Here, as elsewhere below  $l = 2\pi$  and  $f(z) = 0.5h(\cos z - 1)$ . Electron flow relative speed is  $\beta = 0.12$ , and the analyzed frequency range is  $0.1 \leq k \leq 0.36$ . Within this range  $U_n^+(g, k)$ ,  $n = -1, -2, -3$  harmonics propagate without attenuation in the bands  $0.10725 \leq k \leq 0.13625$ ,



**Figure 3.** Dependences (a)  $W_{-1}^+(k)$  and (b)  $W_{-3}^+(k)$ , characterizing the intensity of diffraction radiation into free space  $y > f(z)$ : 1 — metal-backed coat; 2 — semi-infinite structure.

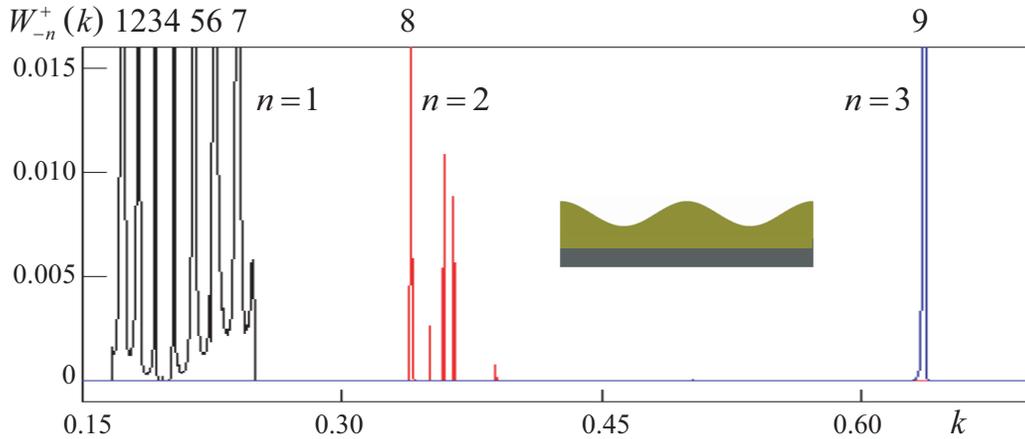
$0.2145 \leq k \leq 0.2725$  and  $k \geq 0.3215$ , correspondingly. In calculations the discretization step of parameter  $k$  is chosen equal to  $10^{-6}$ .

The value  $W_{-2}^+(k)$  within the whole frequency band where the harmonic  $U_{-2}^+(g, k)$  is propagating does not exceed  $10^{-5}$ . The quantities  $W_{-1}^+(k)$  and  $W_{-3}^+(k)$  in the corresponding ranges do not rise higher than  $10^{-2}$ , with the exception of a finite number of points and their small vicinity, where the intensity of diffraction radiation can be anomalously high. And in the case of  $W_{-3}^+(k)$ , these points almost exactly coincide with the spectral points associated with the ‘true eigen waves’ of a semi-infinite structure having the same constitutive parameters. These points are localized in the frequency range from  $k_\varepsilon k_\mu (k_\varepsilon^2 + k_\mu^2)^{-1/2} \approx 0.312$  to  $k^{\text{sing}} = \sqrt{0.5} k_\varepsilon \approx 0.354$  [2], and here, within this range, a structure limited in thickness and a semi-infinite structure with practically the same efficiency transform the field of an electron beam into the field of the  $U_{-3}^+(g, k)$  harmonic propagating in the region  $y > f(z)$  infinitely far from the structure. As an example confirming this statement, we have presented on the right fragment of Fig. 3(b), drawn by the red curve, the dependence  $W_{-3}^+(k)$  for the corresponding semi-infinite structure.

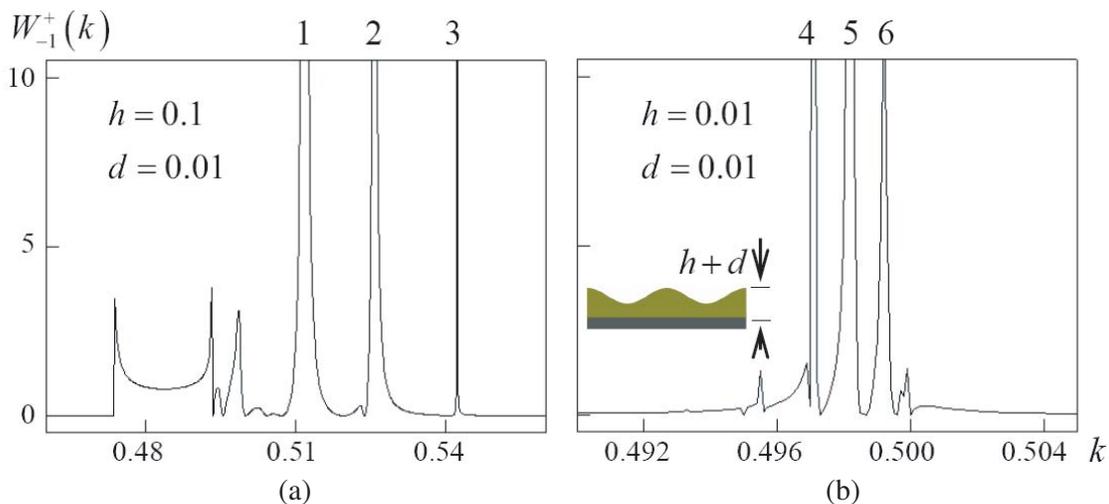
The frequencies corresponding to such regimes are close to those for which, in the case of semi-infinite structures, the existence of the so-called ‘something like leaky waves’ was noted [2]. In general, the considered example confirms the previously formulated thesis that the physics of plane wave scattering processes in the case of the structures shown in Fig. 2 will be much richer in different resonance effects than in the case of a dispersive half-space. Attention should be paid to the  $Q$ -factor of the modes, upon the excitation of which, the efficiency of diffraction radiation reaches its maximum values. It is very high, and the corresponding high and narrow resonance bursts against the background

of the surrounding insignificant values of  $W_n^+(k)$  allow us to hope that such modes can be used in measuring schemes focused on a very accurate determination of the parameters of the electron beam or material parameters of artificial plasma-like media.

For  $k_\varepsilon = k_\mu = 0.9$  and  $\beta = 0.2$  in the range  $0.15 \leq k \leq 0.7$ , considered in Fig. 4, the frequency intervals, where harmonics  $U_{-n}^+(g, k)$ ,  $n = 1, 2, 3$  propagate without attenuation, do not overlap each other and are given by such inequalities:  $0.16675 \leq k \leq 0.24975$  for  $n = 1$ ,  $0.33350 \leq k \leq 0.49975$  for  $n = 2$  and  $k \geq 0.50025$  for  $n = 3$ . The dependences  $W_{-n}^+(k)$  are clipped off here at the level of 0.016, and the peaks of these dependences with a sampling step of the frequency parameter  $k$  equal to 0.00025 have the following values: 1 —  $W_{-1}^+ \approx 2.67$ , 2 —  $W_{-1}^+ \approx 36.6$ , 3 —  $W_{-1}^+ \approx 0.17$ , 4 —  $W_{-1}^+ \approx 0.63$ , 5 —  $W_{-1}^+ \approx 140.9$ , 6 —  $W_{-1}^+ \approx 58.1$ , 7 —  $W_{-1}^+ \approx 23.6$ , 8 —  $W_{-2}^+ \approx 1.82$ , 9 —  $W_{-3}^+ \approx 185.6$ . Some of these values can be called anomalously high, and as follows from [2], their appearance is not associated with resonance frequencies inherent in both infinite and limited in thickness periodic structures. All sharp changes in the characteristics of diffraction radiation here are due to the ‘synchronism’ [3] with the ‘true eigen waves’ which are supported only by structures limited in thickness.



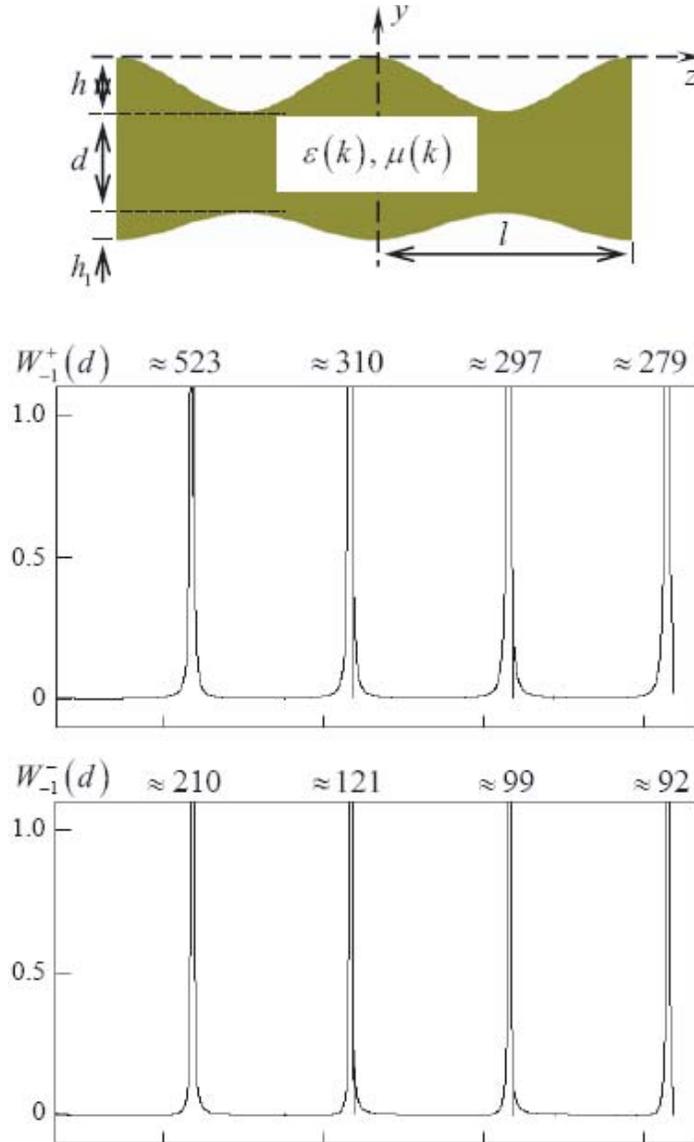
**Figure 4.** Dependences  $W_{-n}^+(k)$ ,  $n = 1, 2, 3$ , characterizing the intensity of diffraction radiation into free space  $y > f(z)$  when  $k_\varepsilon = k_\mu = 0.9$ ,  $h = 0.1$ ,  $d = 4\pi$ ,  $\beta = 0.2$ .



**Figure 5.** Dependences  $W_{-1}^+(k)$ , characterizing the intensity of diffraction radiation into free space  $y > f(z)$ ,  $k_\varepsilon = 0.5$ ,  $k_\mu = 1.0$ ,  $\beta = 0.9$  and  $d = 0.01$ : (a)  $h = 0.1$ , (b)  $h = 0.01$ .

In the frequency range  $0.46 \leq k \leq 0.56$  and for  $k_\varepsilon = 0.5$ ,  $k_\mu = 1.0$ ,  $\beta = 0.9$ , only one harmonic —  $U_{-1}^+(g, k)$  — can propagate in the reflection zone of a periodic structure without attenuation — its lower threshold point is  $k_{-1}^+ \approx 0.4737$  [1]. The band  $k^{\text{sing}} = \sqrt{0.5k_\varepsilon} \approx 0.354 \leq k \leq k_\varepsilon k_\mu (k_\varepsilon^2 + k_\mu^2)^{-1/2} \approx 0.447$ , where the frequencies associated with the ‘true eigen waves’ of a semi-infinite structure of the same constitutive parameters are localized, does not fall into this range either [2]. Therefore, all resonance peaks of the curves  $W_{-1}^+(k)$  presented in Fig. 5 are caused by the excitation of ‘true eigen waves’ inherent only in structures limited in thickness.

The curves in Fig. 5 are clipped off at the level  $W_{-1}^+ = 10$ . Their peak values when a frequency parameter  $k$  sampling step was chosen equal to 0.0001 are as follows: 1 —  $W_{-1}^+ \approx 45.1$ , 2 —  $W_{-1}^+ \approx 441.5$ , 3 —  $W_{-1}^+ \approx 132.6$  for  $h = 0.1$  and 4 —  $W_{-1}^+ \approx 49.3$ , 5 —  $W_{-1}^+ \approx 51.8$ , 6 —  $W_{-1}^+ \approx 12.8$  for  $h = 0.01$ . It turns out that almost flat structure (its height from the substrate to free space is  $h + d = 0.02$ , which is more than 300 times less than the period  $l = 2\pi$  of the structure and more than 600 times less than

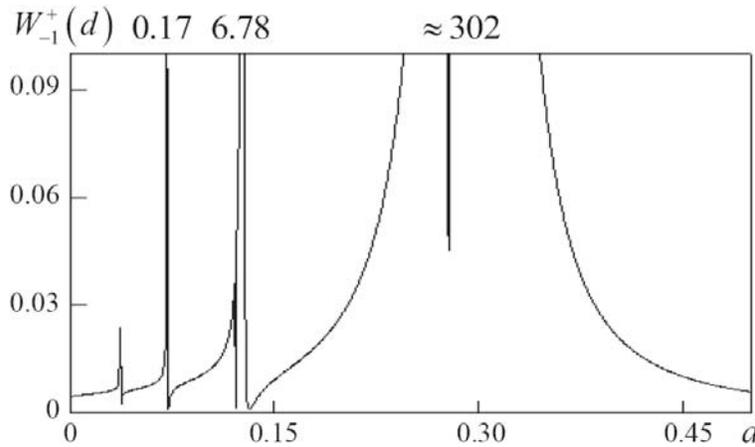


**Figure 6.** Dependences  $W_{-1}^+(d)$  and  $W_{-1}^-(d)$ , characterizing the intensity of diffraction radiation into half-spaces  $y > f(z)$  and  $y < f_1(z)$ ,  $k_\varepsilon = 0.5$ ,  $k_\mu = 1.0$ ,  $\beta = 0.2$ ,  $h = 0.128$ ,  $h_1 = 0.001$  and  $k = 0.193965$ .

the wavelength  $\lambda = 2\pi/k$ ) is able to very effectively transform the field of an electron flow into a field of a plane electromagnetic wave outgoing infinitely far into free space.

The regular sequence of very high peaks on the curves  $W_{-1}^+(d)$  and  $W_{-1}^-(d) = |T_{-1}(d)|^2 \text{Re}\Gamma_{-1}^+ / |\Gamma_0^+|$  characterizing the intensity of diffraction radiation at the minus first harmonic of the semitransparent periodic structure in the half-spaces  $y > f(z)$  and  $y < f_1(z)$  (see Fig. 6;  $f_1(z) = -h - d + 0.5h_1(\cos z - 1)$ ), allows us to associate their appearance with the excitation of ‘true eigen waves’, whose field, weakly damping, oscillates in area  $-h - d < y < -h$ . The curves  $W_{-1}^\pm(d)$  in Fig. 6 are clipped off at the level  $W_{-1}^\pm = 1.1$ , and the maximum values of their magnitude, calculated at the parameter  $d$  sampling step equal to 0.0005, are shown above the upper boundary of the corresponding fragments.

Results presented in Fig. 6 describe the process of electromagnetic wave scattering with parameters chosen in such a way that only minus first spatial harmonics  $U_{-1}^\pm(g, t)$  propagate without damping in free half-spaces  $y > f(z)$  and  $y < f_1(z)$ . At that, there are much more such harmonics in the layer. But, apparently, only one of them — the zeroth one — actively participates in the coupling between the reflection and transition zones of the periodic structure. This statement is clearly confirmed by the following evident calculation: the length of the wave inside the slab  $\lambda_{\text{layer}} = 2\pi/\Gamma_0^-$  is approximately equal to 2.97; the distance  $d$  between the peaks of the curves, the first of which we fix at a point  $d \approx 1.773$  (in Fig. 7 — at the point  $d \approx 0.296$ ), is constant and equal approximately to 1.48 — this is half of the wavelength  $\lambda_{\text{layer}}$  in the layer. Therefore, we can state that in the case under consideration the excitation of ‘true eigen waves’, ‘synchronism’ of the electron beam which leads to anomalously high levels of intensity of diffraction radiation, is associated with so-called half-wave resonance over the thickness of the layer  $d$ .



**Figure 7.** To Fig. 6. The behavior of  $W_{-1}^+(d)$  for small values of slab thickness  $d$ .

Rather special situation appears for the thin slabs, which is for small values of  $d$ . As this interval of  $d$  is not clearly seen in Fig. 6, the curve  $W_{-1}^+(d)$  in extended scale is presented in Fig. 7. For a thin slab, the origins of resonant peaks in scattering characteristics have another nature. In such a structure, not only homogeneous (propagating) plane waves but also inhomogeneous (evanescent) waves with propagation constants  $\Gamma_n^-$ ,  $n \neq 0$ , ‘trapped’ in the layer [12, 13, 16, 24], are responsible for the formation of resonance peaks on the curves  $W_{-1}^\pm(d)$ .

## 5. CONCLUSION

One common problem associated with clarification and explanation of the appearance of anomalously high levels of diffraction radiation (Vavilov-Cherenkov radiation or/and Smith-Purcell radiation) when a plane, density-modulated electron beam passes over a periodic boundary separating a vacuum and some artificial (plasma-like medium) unites four works — this one and three others published a little

bit earlier [1–3]. A universal nature underlying the implementation of the corresponding effects is found and specified: that is the ‘synchronism’ of the electron flow with one of the ‘true eigen waves’ of a structure. That can be a purely surface wave, producing the field decreasing exponentially when it leaves the periodic boundary in both directions, or a wave, producing the field oscillating in a layer of finite thickness with very weak damping. If the ‘true eigen wave’ corresponds to the eigen propagation constant  $\bar{\zeta}$ , then by ‘synchronism’ we mean the rapprochement of the values  $\text{Re}\bar{\zeta}$  and  $\zeta = kl/2\pi\beta$  — the parameters characterizing the phase shift of the field of the eigen wave and the field of the electron beam along one period of the structure. The closer these values are and the smaller the values  $\text{Im}\bar{\zeta}$  are, the greater the contributions of terms of the type  $c_m(k)(\zeta - \bar{\zeta})^m$ ,  $m = -M, -M + 1, \dots, -1$  of Laurent series are, and they represent all the electrodynamic characteristics calculated within the framework of the corresponding model problems [1, 13, 14, 22, 23]. Here  $M$  is the order of the pole of the resolvent of such problems at a point  $\bar{\zeta}$ . In this regard, it is clear why the above ‘synchronism’ leads to anomalously high levels of diffraction radiation.

## ACKNOWLEDGMENT

This research has been partially funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08855557).

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