Progress In Electromagnetics Research B, Vol. 92, 91–107, 2021

Orthogonal System of Eigenwaves of an Open Cylindrical Gyrotropic Waveguide Located in Free Space

Vasiliy A. Es'kin*, **Alexander V. Kudrin, and Nadezhda V. Yurasova**

Abstract—A new method for obtaining an orthogonal system of eigenwaves of an open cylindrical waveguide filled with a gyrotropic medium and located in free space is presented. The advantage of the method is that it enables one to explicitly represent the fields of eigenwaves, which correspond to the discrete and continuous parts of the eigenvalue spectrum of such a guiding structure. Orthogonality relations for the eigenwaves and the procedure of expanding an electromagnetic field in terms of these modal solutions are discussed. The limiting transition from the case of a closed cylindrical waveguide with a perfectly conducting wall and a coaxial cylindrical gyrotropic core to the case of an open waveguide is considered. To illustrate the completeness of the obtained system of eigenwaves, a given field is expanded in terms of the found discrete- and continuous-spectrum waves and then resynthesized by evaluating the corresponding expansion numerically. Perfect coincidence between the initially specified field and the result yielded by this evaluation is demonstrated.

1. INTRODUCTION

Expansion of the field in a closed waveguide in terms of eigenwaves was applied for the first time by Lord Rayleigh to the sound waves [1]. Then this approach was extended to the case of propagation of electromagnetic waves in waveguides with perfectly conducting walls [2] and along dielectric rods [3]. As is known, to represent the total electromagnetic field in the presence of an open guiding structure, it is necessary to use a complete set of eigenwaves, which comprises the terms corresponding to the discrete and continuous parts of the eigenvalue spectrum for such a waveguide [4–7]. Most works on the subject deal with isotropic [5–9] or anisotropic [10–13] cylindrical waveguides located in free space. It has long been known that in this case, orthogonalization of the continuous-spectrum waves meets certain difficulties related to polarization degeneration of waves in an isotropic outer medium (see, e.g., [10–16] and references therein). At the same time, the required orthogonalization relations are readily established in the case of an anisotropic outer medium [17–20] where the polarization degeneration does not occur.

It is the purpose of this work to develop a method that enables one to readily obtain an orthogonal system of eigenwaves of an open gyrotropic cylindrical waveguide located in free space. Such guiding structures are often encountered in actual practice, especially when dealing with excitation and propagation of electromagnetic waves in magneto-optical fibers [10–12], magnetized plasma columns [13], helicon plasma sources [21–23], and other types of open gyrotropic waveguides [18, 24, 25]. The feature of the proposed method is that it consistently employs two different polarization types of the continuous-spectrum waves, along with the conventional consideration of the discrete-spectrum waves, for constructing a complete set of eigenwaves. Although some aspects of the method presented here

Received 10 March 2021, Accepted 22 April 2021, Scheduled 26 April 2021
* Corresponding author: Vasiliy Alekseevich Es'kin (vasiliy.eskin@gmail.com).
The authors are with the Department of Radiophysics, University of Nizhny Novgorod, 23 Gagarin Avenue, Nizhny Novgorod 603022, Russia.

have already been considered with applications to particular diffraction problems [26, 27], those works describe this method and its use very succinctly and do not provide analysis as thorough as desired.

Our work is organized as follows. In Section 2, we formulate the studied problem. Section 3 presents the derivation of the fields of eigenwaves, along with the orthogonality relations for these modal solutions and their expansion coefficients. The limiting transition from the case of a closed circular waveguide with a coaxial cylindrical gyrotropic core to the case of an open waveguide is discussed in Section 4. In Section 5, the completeness of the obtained eigenwave system is demonstrated numerically. Our conclusions are summarized in Section 6. Some auxiliary mathematical derivations are given in appendixes.

2. BASIC FORMULATION

Consider a uniform cylinder of radius a which is located in free space and aligned with the z axis of a cylindrical coordinate system (ρ, ϕ, z). It is assumed that the cylinder is filled with a gyrotropic medium that may be frequency-dispersive and is described by the permittivity tensor

$$\boldsymbol{\varepsilon} = \epsilon_0 \begin{pmatrix} \varepsilon & -ig & 0 \\ ig & \varepsilon & 0 \\ 0 & 0 & \eta \end{pmatrix}, \tag{1}$$

where ϵ_0 is the permittivity of free space. Note that such a tensor is typical of gyroelectric media of natural [28] or artificial [29] origin.

Our main task in this work is to obtain an orthogonal system of eigenwaves for representing the electromagnetic field in the presence of the considered open system. Such waves are described by cylindrical vector eigenfunctions which are solutions of the source-free Maxwell equations. In a gyrotropic medium with the tensor of Eq. (1), these equations for an electromagnetic field with $\exp(i\omega t)$ time dependence have the form

$$\nabla \times \mathbf{E} = -i\omega\mu_0 \mathbf{H}, \quad \nabla \times \mathbf{H} = i\omega\boldsymbol{\varepsilon} \cdot \mathbf{E}, \tag{2}$$

where μ_0 is the permeability of free space. In free space outside the gyrotropic cylinder, the tensor $\boldsymbol{\varepsilon}$ should be replaced by ϵ_0.

The transverse (with respect to the z axis) components of the desired solutions of the Maxwell equations can be expressed in terms of the longitudinal components, which are represented in the modal form

$$\begin{bmatrix} E_{z;m,s}(\mathbf{r}, q) \\ H_{z;m,s}(\mathbf{r}, q) \end{bmatrix} = \begin{bmatrix} E_{z;m,s}(\rho, q) \\ H_{z;m,s}(\rho, q) \end{bmatrix} \exp[-im\phi - ik_0 p_s(q) z], \tag{3}$$

where m is the azimuthal index ($m = 0, \pm 1, \pm 2, \ldots$); k_0 is the free-space wave number; q is the normalized (to k_0) transverse wave number in free space; the function $p_s(q)$ gives the dependence of the normalized longitudinal wave number p on the transverse wave number q in free space; the subscript s denotes the waves transferring energy in the positive ($s = +$) and negative ($s = -$) directions of the z axis; and $E_{z;m,s}(\rho, q)$ and $H_{z;m,s}(\rho, q)$ are the scalar functions describing the radial distributions of the longitudinal field components of a wave with the transverse wave number q and the indices m and s. The quantities $p_+(q)$ and $p_-(q)$ satisfy the relation $p_\pm(q) = \pm p(q)$, where

$$p(q) = (1 - q^2)^{1/2}. \tag{4}$$

It is assumed that at least in the limit of vanishing losses in the outer medium, $\mathrm{Im}\, p(q) < 0$.

It can be shown from the Maxwell equations that in the region $\rho < a$, the transverse components $\mathbf{E}_{\perp;m,s}(\mathbf{r}, q)$ and $\mathbf{H}_{\perp;m,s}(\mathbf{r}, q)$ of the modal solutions are expressed via the longitudinal components $E_{z;m,s}(\mathbf{r}, q)$ and $H_{z;m,s}(\mathbf{r}, q)$ as [18]

$$\begin{aligned} \mathbf{E}_{\perp;m,s} = \frac{1}{k_0 W} \Big\{ & i p_s (\varepsilon - p_s^2) \nabla_\perp E_{z;m,s} + p_s g \big(\mathbf{z}_0 \times \nabla_\perp E_{z;m,s}\big) \\ & + Z_0 g \nabla_\perp H_{z;m,s} - i Z_0 (\varepsilon - p_s^2) \big(\mathbf{z}_0 \times \nabla_\perp H_{z;m,s}\big) \Big\}, \end{aligned} \tag{5}$$

$$\begin{aligned}\mathbf{H}_{\perp;m,s} \;=\; & \frac{1}{k_0 W}\Big\{-Z_0^{-1}p_s^2 g\nabla_\perp E_{z;m,s} - iZ_0^{-1}\left[g^2-\varepsilon(\varepsilon-p_s^2)\right]\big(\mathbf{z}_0\times\nabla_\perp E_{z;m,s}\big)\\ & +ip_s(\varepsilon-p_s^2)\nabla_\perp H_{z;m,s} + p_s g\big(\mathbf{z}_0\times\nabla_\perp H_{z;m,s}\big)\Big\},\end{aligned} \tag{6}$$

where $W = g^2-(\varepsilon-p_s^2)^2$; $\nabla_\perp$ is the transverse (with respect to the z axis) part of the del operator; $\mathbf{z}_0$ is the unit vector aligned with the z axis; and $Z_0=(\mu_0/\epsilon_0)^{1/2}$ is the impedance of free space.

In turn, the quantities $E_{z;m,s}(\rho,q)$ and $H_{z;m,s}(\rho,q)$ satisfy the following equations in the inner region of the cylinder [18]:

$$\hat{L}_m E_{z;m,s} - k_0^2\frac{\eta}{\varepsilon}(p_s^2-\varepsilon)E_{z;m,s} \;=\; -ik_0^2\frac{g}{\varepsilon}p_s Z_0 H_{z;m,s}, \tag{7}$$

$$\hat{L}_m H_{z;m,s} - k_0^2\left(p_s^2+\frac{g^2-\varepsilon^2}{\varepsilon}\right)H_{z;m,s} \;=\; ik_0^2\frac{g}{\varepsilon}\eta p_s Z_0^{-1}E_{z;m,s}, \tag{8}$$

where

$$\hat{L}_m = \frac{\partial^2}{\partial\rho^2}+\frac{1}{\rho}\frac{\partial}{\partial\rho}-\frac{m^2}{\rho^2}.$$

Equations for the longitudinal field components and expressions for the corresponding transverse field components outside the cylinder are obtained from Eqs. (5)–(8) by putting $\varepsilon=1$, $g=0$, and $\eta=1$.

Solution of the equations for $E_{z;\,m,s}(\rho,q)$ and $H_{z;,m,s}(\rho,q)$ should be regular on the z-axis, ensure the boundary conditions for the tangential field components at $\rho=a$, and satisfy the following boundedness conditions at $\rho\to\infty$ [6]:

$$\rho^{1/2}|E_{z;m,s}(\rho,q)| < R_m^{(1)}, \quad \rho^{1/2}|H_{z;m,s}(\rho,q)| < R_m^{(2)}, \tag{9}$$

where $R_m^{(1)}$ and $R_m^{(2)}$ are finite constants. Equations (7) and (8), along with the above-mentioned conditions, constitute the boundary value problem on finding the eigenvalues q and the corresponding eigenwaves, in terms of which the electromagnetic field can be expanded. It is known that the resulting source-excited field given by such an expansion satisfies the radiation condition at infinity [6].

3. EIGENWAVES OF AN OPEN CYLINDRICAL WAVEGUIDE

3.1. General Solution

Inside the gyrotropic cylinder ($\rho < a$), the solution of Eqs. (7) and (8) is represented in terms of cylindrical functions as [18]

$$E_{z;m,s}(\rho,q) \;=\; i\eta^{-1}\sum_{k=1}^{2}B_{m,s}^{(k)}(q)n_{k,s}q_k J_m(k_0q_k\rho), \tag{10}$$

$$H_{z;m,s}(\rho,q) \;=\; -Z_0^{-1}\sum_{k=1}^{2}B_{m,s}^{(k)}(q)q_k J_m(k_0q_k\rho). \tag{11}$$

Here, J_m is the Bessel function of the first kind of order m, and $B_{m,s}^{(1)}$ and $B_{m,s}^{(2)}$ are the amplitude coefficients of the mth azimuthal field harmonic inside the cylinder. Other notations in Eqs. (10) and (11) are given by

$$n_{k,s} \;=\; -\varepsilon\left(q_k^2+p_s^2+\frac{g^2}{\varepsilon}-\varepsilon\right)(p_s g)^{-1}, \quad k=1,2, \tag{12}$$

$$q_k \;=\; \left\{[\varepsilon^2-g^2+\varepsilon\eta-(\varepsilon+\eta)p_s^2+(-1)^k R]/(2\varepsilon)\right\}^{1/2}, \tag{13}$$

where

$$R=\left\{(\varepsilon-\eta)^2p_s^4+2\left[g^2(\varepsilon+\eta)-\varepsilon(\varepsilon-\eta)^2\right]p_s^2+\left(\varepsilon^2-g^2-\varepsilon\eta\right)^2\right\}^{1/2}. \tag{14}$$

It should be noted that q_1 and q_2 are the normalized (to k_0) transverse wave numbers corresponding to two normal waves of a gyrotropic medium with the longitudinal wave number equal to $p = p_s(q)$.

Expressions that describe the dependence of the transverse field components on ρ in the region $\rho < a$, with allowance for Eqs. (5) and (6), are written as

$$E_{\rho;m,s}(\rho,q) = \sum_{k=1}^{2} B_{m,s}^{(k)}(q)\left[(1+u_k)J_{m+1}(k_0q_k\rho) - u_k m\frac{J_m(k_0q_k\rho)}{k_0q_k\rho}\right], \tag{15}$$

$$E_{\phi;m,s}(\rho,q) = i\sum_{k=1}^{2} B_{m,s}^{(k)}(q)\left[J_{m+1}(k_0q_k\rho) + u_k m\frac{J_m(k_0q_k\rho)}{k_0q_k\rho}\right], \tag{16}$$

$$H_{\rho;m,s}(\rho,q) = -iZ_0^{-1}\sum_{k=1}^{2} B_{m,s}^{(k)}(q)\left[p_s J_{m+1}(k_0q_k\rho) - n_{k,s}v_k m\frac{J_m(k_0q_k\rho)}{k_0q_k\rho}\right], \tag{17}$$

$$H_{\phi;m,s}(\rho,q) = -Z_0^{-1}\sum_{k=1}^{2} B_{m,s}^{(k)}(q)n_{k,s}\left[J_{m+1}(k_0q_k\rho) - v_k m\frac{J_m(k_0q_k\rho)}{k_0q_k\rho}\right], \tag{18}$$

where

$$u_k = g^{-1}[q_k^2 + p_s^2 - \varepsilon] - 1, \quad v_k = p_s n_{k,s}^{-1} + 1. \tag{19}$$

Outside the cylinder ($\rho > a$), the solution for the longitudinal field components can be written as follows:

$$E_{z;m,s}(\rho,q) = q\sum_{k=1}^{2} C_{m,s}^{(k)}(q)H_m^{(k)}(k_0q\rho), \tag{20}$$

$$H_{z;m,s}(\rho,q) = Z_0^{-1}q\sum_{k=1}^{2} D_{m,s}^{(k)}(q)H_m^{(k)}(k_0q\rho). \tag{21}$$

Here, $H_m^{(1)}$ and $H_m^{(2)}$ are the mth-order Hankel functions of the first and second kinds, respectively, and $C_{m,s}^{(1,2)}$ and $D_{m,s}^{(1,2)}$ are the amplitude coefficients. Then for the transverse field components in the outer region $\rho > a$, we have

$$E_{\rho;m,s}(\rho,q) = -\sum_{k=1}^{2}\left[iC_{m,s}^{(k)}(q)p_s H_m^{(k)\prime}(k_0q\rho) + D_{m,s}^{(k)}(q)m\frac{H_m^{(k)}(k_0q\rho)}{k_0q\rho}\right], \tag{22}$$

$$E_{\phi;m,s}(\rho,q) = -\sum_{k=1}^{2}\left[C_{m,s}^{(k)}(q)p_s m\frac{H_m^{(k)}(k_0q\rho)}{k_0q\rho} - iD_{m,s}^{(k)}(q)H_m^{(k)\prime}(k_0q\rho)\right], \tag{23}$$

$$H_{\rho;m,s}(\rho,q) = Z_0^{-1}\sum_{k=1}^{2}\left[C_{m,s}^{(k)}(q)m\frac{H_m^{(k)}(k_0q\rho)}{k_0q\rho} - iD_{m,s}^{(k)}(q)p_s H_m^{(k)\prime}(k_0q\rho)\right], \tag{24}$$

$$H_{\phi;m,s}(\rho,q) = -Z_0^{-1}\sum_{k=1}^{2}\left[iC_{m,s}^{(k)}(q)H_m^{(k)\prime}(k_0q\rho) + D_{m,s}^{(k)}(q)p_s m\frac{H_m^{(k)}(k_0q\rho)}{k_0q\rho}\right]. \tag{25}$$

Hereafter, the prime indicates the derivative with respect to the argument.

As is known [6], the boundedness conditions in Eq. (9) can be satisfied for some discrete complex values of q, which give the discrete part of the eigenvalue spectrum and correspond to the transversely localized discrete-spectrum waves, i.e., eigenmodes of the gyrotropic cylinder. In addition, these boundedness conditions are also satisfied for all real values of q, which constitute the continuous part of the eigenvalue spectrum and correspond to the continuous-spectrum waves. Before proceeding to obtaining eigenwaves of the discrete and continuous spectrum, we introduce some notations, which will be used in what follows.

Let $\mathbf{S}_{m,s}(Z_m^{(\mathrm{I})}, Z_m^{(\mathrm{II})})$ denote the matrix

$$\mathbf{S}_{m,s}(Z_m^{(\mathrm{I})}, Z_m^{(\mathrm{II})}) = \begin{pmatrix} \frac{i}{\eta} n_{1,s} Q_1 J_m(Q_1) & \frac{i}{\eta} n_{2,s} Q_2 J_m(Q_2) & -Q Z_m^{(\mathrm{I})}(Q) & 0 \\ Q_1 J_m(Q_1) & Q_2 J_m(Q_2) & 0 & Q Z_m^{(\mathrm{II})}(Q) \\ i \hat{J}_m^{(1)} & i \hat{J}_m^{(2)} & p_s m \frac{Z_m^{(\mathrm{I})}(Q)}{Q} & -i Z_m^{(\mathrm{II})\prime}(Q) \\ n_{1,s} \tilde{J}_m^{(1)} & n_{2,s} \tilde{J}_m^{(2)} & -i Z_m^{(\mathrm{I})\prime}(Q) & -p_s m \frac{Z_m^{(\mathrm{II})}(Q)}{Q} \end{pmatrix}, \tag{26}$$

where

$$\hat{J}_m^{(k)} = J_{m+1}(Q_k) + u_k m \frac{J_m(Q_k)}{Q_k}, \quad \tilde{J}_m^{(k)} = J_{m+1}(Q_k) - v_k m \frac{J_m(Q_k)}{Q_k}, \tag{27}$$

$Q_k = k_0 q_k a$, $Q = k_0 q a$, and $Z_m^{(\mathrm{I})}$ and $Z_m^{(\mathrm{II})}$ are cylindrical functions of order m. Other notations in Eqs. (26) and (27) are defined in Eqs. (4), (12), (13), and (19). Using the matrix given by Eq. (26), we introduce the following determinants:

$$\begin{aligned} &\Delta_{m,s}^{(1)}(q) = \det \mathbf{S}_{m,s}\left(H_m^{(2)}, H_m^{(2)}\right), \quad \Delta_{m,s}^{(2)}(q) = \det \mathbf{S}_{m,s}\left(H_m^{(1)}, H_m^{(1)}\right), \\ &\Delta_{m,s}^{(3)}(q) = \det \mathbf{S}_{m,s}\left(H_m^{(2)}, H_m^{(1)}\right) - \det \mathbf{S}_{m,s}\left(H_m^{(1)}, H_m^{(2)}\right). \end{aligned} \tag{28}$$

Expressions for $\Delta_{m,s}^{(1)}(q)$ and $\Delta_{m,s}^{(2)}(q)$ are cumbersome and will not be presented here. An expression for $\Delta_{m,s}^{(3)}(q)$ can be written as

$$\Delta_{m,s}^{(3)}(q) = \frac{4}{\pi} \left\{ Q_1 J_m(Q_1) \left[n_{2,s} \tilde{J}_m^{(2)} + \frac{n_{1,s}}{\eta} \hat{J}_m^{(2)} \right] - Q_2 J_m(Q_2) \left[n_{1,s} \tilde{J}_m^{(1)} + \frac{n_{2,s}}{\eta} \hat{J}_m^{(1)} \right] \right\}. \tag{29}$$

Note that in deriving Eq. (29), use was made of the Wronskian of the Hankel functions $H_m^{(1)}(Q)$ and $H_m^{(2)}(Q)$ [30].

3.2. Discrete-Spectrum Waves

To obtain the discrete-spectrum waves, which are transversely localized, we should put $C_{m,s}^{(1)}(q) = 0$ and $D_{m,s}^{(1)}(q) = 0$ in Eqs. (20)–(25) and then satisfy the boundary conditions for the tangential field components, i.e., $E_{\phi;m,s}$, $E_{z;m,s}$, $H_{\phi;m,s}$, and $H_{z;m,s}$, at $\rho = a$. This yields a system of four linear homogeneous equations for the coefficients $B_{m,s}^{(1)}(q)$, $B_{m,s}^{(2)}(q)$, $C_{m,s}^{(2)}(q)$, and $D_{m,s}^{(2)}(q)$. To obtain a nontrivial solution for these coefficients, we should require that the determinant of such a system is zero, which gives the eigenvalue equation

$$\Delta_{m,s}^{(1)}(q) = 0, \tag{30}$$

where $\Delta_{m,s}^{(1)}$ is defined in Eq. (28). It is evident that roots $q = q_{m,n}$ of Eq. (30), where $n = 1, 2, \ldots$, correspond to the transversely localized eigenmodes, i.e., the discrete-spectrum waves, if $\operatorname{Im} q_{m,n} < 0$. The propagation constants of such modes are given by $p_{m,s,n} = p_s(q_{m,n})$. Substituting $q = q_{m,n}$ into Eqs. (10), (11), and (15)–(18) for $\rho < a$ and into Eqs. (20)–(25) for $\rho > a$, with allowance for the condition $C_{m,s}^{(1)}(q) = D_{m,s}^{(1)}(q) = 0$, we obtain the quantities which describe the components of the discrete-spectrum waves as functions of ρ. Multiplying these quantities by $\exp(-im\phi - ik_0 p_{m,s,n})$, we arrive at the resulting expressions for the components of the fields $\mathbf{E}_{m,s,n}(\mathbf{r})$ and $\mathbf{H}_{m,s,n}(\mathbf{r})$ of the discrete-spectrum waves. The amplitude coefficients for these wave can now be denoted as

$$B_{m,s,n}^{(1,2)} = B_{m,s}^{(1,2)}(q_{m,n}), \quad C_{m,s,n}^{(2)} = C_{m,s}^{(2)}(q_{m,n}), \quad D_{m,s,n}^{(2)} = D_{m,s}^{(2)}(q_{m,n}). \tag{31}$$

A procedure for obtaining explicit expressions of these coefficients is described in Appendix A. Note that this procedure yields the coefficients in Eq. (31) in such a form that the eigenfunctions determining the components of $\mathbf{E}_{m,s,n}(\mathbf{r})$ and $\mathbf{H}_{m,s,n}(\mathbf{r})$ satisfy the relations

$$\begin{aligned} &E_{\rho,\phi;m,-s,n}(\rho) = E_{\rho,\phi;m,s,n}(\rho), \quad E_{z;m,-s,n}(\rho) = -E_{z;m,s,n}(\rho), \\ &H_{\rho,\phi;m,-s,n}(\rho) = -H_{\rho,\phi;m,s,n}(\rho), \quad H_{z;m,-s,n}(\rho) = H_{z;m,s,n}(\rho), \end{aligned} \tag{32}$$

which will be useful in the forthcoming derivations.

Alternatively, one may put $C^{(2)}_{m,s}(q) = 0$ and $D^{(2)}_{m,s}(q) = 0$ in Eqs. (20)–(25) and again satisfy the boundary conditions for the tangential field components. This gives another eigenvalue equation

$$\Delta^{(2)}_{m,s}(q) = 0, \tag{33}$$

where $\Delta^{(2)}_{m,s}(q)$ is also defined in Eq. (28). The roots $q = \tilde{q}_{m,n}$ of Eq. (33), such that $\operatorname{Im} \tilde{q}_{m,n} > 0$, where $n = 1, 2, \ldots$, yield localized eigenmodes as well. However, the solutions of Eq. (33) do not give new discrete-spectrum waves compared with those yielded by the solutions of Eq. (30) and can therefore be rejected.

3.3. Continuous-Spectrum Waves

We now consider the real q values which constitute the continuous part of the eigenvalue spectrum. To construct the corresponding continuous-spectrum waves in a form that automatically ensures the fulfillment of the required orthogonality relations, we note that according to [6], an open waveguide can be regarded the limiting case of a closed waveguide with the perfectly conducting boundary of radius R that tends to infinity. In the region $a \le \rho \le R$ of such a waveguide, the field is described by Eqs. (20)–(25). Since $E_{\phi;m,s}(R, q) = 0$ and $E_{z;m,s}(R, q) = 0$, one obtains from Eqs. (20) and (23) for $R \to \infty$ that

$$C^{(1)}_{m,s}(q) H^{(1)}_m(k_0 qR) + C^{(2)}_{m,s}(q) H^{(2)}_m(k_0 qR) = 0, \tag{34}$$

$$D^{(1)}_{m,s}(q) H^{(1)\prime}_m(k_0 qR) + D^{(2)}_{m,s}(q) H^{(2)\prime}_m(k_0 qR) = 0. \tag{35}$$

Using the relation

$$H^{(1)}_m(k_0 qR)/H^{(2)}_m(k_0 qR) = -H^{(1)\prime}_m(k_0 qR)/H^{(2)\prime}_m(k_0 qR),$$

which is valid for $R \to \infty$, we can deduce from Eqs. (34) and (35) that in this limit,

$$C^{(1)}_{m,s}(q)/C^{(2)}_{m,s}(q) = -D^{(1)}_{m,s}(q)/D^{(2)}_{m,s}(q). \tag{36}$$

Equation (36) allows us to assume that the following ansatz can be used to relate $C^{(1)}_{m,s}(q)$ and $D^{(1)}_{m,s}(q)$ to $C^{(2)}_{m,s}(q)$ and $D^{(2)}_{m,s}(q)$, respectively, in the case of an open waveguide:

$$C^{(1)}_{m,s}(q) = \psi_m(q) C^{(2)}_{m,s}(q), \quad D^{(1)}_{m,s}(q) = -\psi_m(q) D^{(2)}_{m,s}(q). \tag{37}$$

Here, $\psi_m(q)$ is a certain function of q, which is to be determined. It will be shown below that this ansatz makes it possible to readily obtain the continuous-spectrum waves satisfying the orthogonality relations.

To determine $\psi_m(q)$, we substitute Eq. (37) into Eqs. (20)–(25) and then satisfy the boundary conditions for the tangential field components at $\rho = a$. This yields a system of four linear homogeneous equations for the coefficients $B^{(1)}_{m,s}(q)$, $B^{(2)}_{m,s}(q)$, $C^{(2)}_{m,s}(q)$, and $D^{(2)}_{m,s}(q)$. Requiring that the determinant of such a system be zero gives the following quadratic equation for $\psi_m(q)$:

$$\Delta^{(2)}_{m,s}(q) \psi^2_m(q) + \Delta^{(3)}_{m,s}(q) \psi_m(q) - \Delta^{(1)}_{m,s}(q) = 0. \tag{38}$$

Then it follows that

$$\psi_{m,\gamma}(q) = \frac{\Delta^{(3)}_{m,s}(q)}{2\Delta^{(2)}_{m,s}(q)} \left\{ (-1)^{\gamma-1} \left[1 + 4\Delta^{(1)}_{m,s}(q) \Delta^{(2)}_{m,s}(q) / \left(\Delta^{(3)}_{m,s}(q) \right)^2 \right]^{1/2} - 1 \right\}, \tag{39}$$

where the subscripts $\gamma = 1$ and $\gamma = 2$ denote the polarization type of the continuous-spectrum waves, and the square root is defined so as to have the positive real part. Note that the quantities $\Delta^{(1)}_{m,s}(q)$, $\Delta^{(2)}_{m,s}(q)$, and $\Delta^{(3)}_{m,s}(q)$ reverse their sign at the replacement $s \to -s$. Therefore, the functions $\psi_{m,1}(q)$ and $\psi_{m,2}(q)$ are independent of s, as expected. The existence of two solutions $\psi_{m,1}(q)$ and $\psi_{m,2}(q)$ implies that we have two types of the continuous-spectrum waves, whose amplitude coefficients will be marked using the additional subscript γ. The longitudinal components for the corresponding continuous-spectrum waves in Eq. (3) are then denoted as $E_{z;m,s,\gamma}(\mathbf{r}, q)$ and $H_{z;m,s,\gamma}(\mathbf{r}, q)$ and can be found using Eqs. (10), (11), (20), and (21). Calculating the functions in Eqs. (15)–(18) and (22)–(25) for $\gamma = 1, 2$ and multiplying them by the same exponential function as in Eq. (3), we obtain the transverse components of the fields $\mathbf{E}_{m,s,\gamma}(\mathbf{r}, q)$ and $\mathbf{H}_{m,s,\gamma}(\mathbf{r}, q)$ of the continuous-spectrum waves. The coefficients $B^{(1)}_{m,s,\gamma}(q)$, $B^{(2)}_{m,s,\gamma}(q)$, $C^{(2)}_{m,s,\gamma}(q)$, and $D^{(2)}_{m,s,\gamma}(q)$ for such waves are calculated as described in Appendix A. With these coefficients, one can verify that similar to Eq. (32), the following relations take place:

$$\begin{aligned} E_{\rho,\phi;m,-s,\gamma}(\rho, q) &= E_{\rho,\phi;m,s,\gamma}(\rho, q), \quad E_{z;m,-s,\gamma}(\rho, q) = -E_{z;m,s,\gamma}(\rho, q), \\ H_{\rho,\phi;m,-s,\gamma}(\rho, q) &= -H_{\rho,\phi;m,s,\gamma}(\rho, q), \quad H_{z;m,-s,\gamma}(\rho, q) = H_{z;m,s,\gamma}(\rho, q). \end{aligned} \tag{40}$$

In what follows, we will use such eigenwaves only for the positive real q values because, as a careful inspection shows, the negative real q values do not give new solutions for the continuous-spectrum waves.

Interestingly, substituting $q = q_{m,n}$ for q into $\psi_{m,1}(q)$ yields $\psi_{m,1}(q_{m,n}) = 0$, so that $C^{(1)}_{m,s,\gamma}(q_{m,n}) = 0$ and $D^{(1)}_{m,s,\gamma}(q_{m,n}) = 0$ for $\gamma = 1$. Similarly, $\psi^{-1}_{m,2}(\tilde{q}_{m,n}) = 0$, so that $C^{(2)}_{m,s,\gamma}(\tilde{q}_{m,n}) = 0$ and $D^{(2)}_{m,s,\gamma}(\tilde{q}_{m,n}) = 0$ for $\gamma = 2$.

In addition, we note that there exists an alternative way for deriving the continuous-spectrum waves. To this end, one may consider a closed waveguide with the perfect magnetic boundary at $\rho = R \to \infty$ instead of the perfect electric boundary used in the preceding analysis. Then one should employ the conditions $H_{\phi;m,s}(R, q) = 0$ and $H_{z;m,s}(R, q) = 0$ and repeat steps similar to those described above.

It should also be mentioned that an analogous set of eigenwaves of an open cylindrical waveguide filled with a gyrotropic medium can be constructed using the scattering matrix method [7, 10, 11], in which two kinds of continuous-spectrum waves correspond to two eigenvalues of the scattering matrix that can be introduced for cylindrical waves according to a special procedure. The approach presented here differs from the scattering matrix method in that the relations of the quantities $C^{(1)}_{m,s,\gamma}(q)$ and $D^{(1)}_{m,s,\gamma}(q)$ to the quantities $C^{(2)}_{m,s,\gamma}(q)$ and $D^{(2)}_{m,s,\gamma}(q)$, which follow from Eq. (37), are determined by considering the reflection of the continuous-spectrum waves from an infinitely distant, perfectly conducting cylindrical surface. In the scattering matrix method, the corresponding relations are derived using the eigenvalues of the scattering matrix for cylindrical waves, which is obtained by solving a more complicated problem of reflection of such waves from an open gyrotropic waveguide. It is important to note that our approach gives closed-form representations for the eigenwaves which automatically satisfy the required orthogonality relations discussed in the forthcoming section. Moreover, it will be shown in what follows that the use of both types of the continuous-spectrum waves is generally necessary for rigorous representation of an arbitrary field. At the same time, in works [7, 10, 11], in which the scattering matrix method was developed, only one kind of the continuous-spectrum waves was employed by default. However, this may generally be insufficient since an accurate field representation should involve the use of both kinds of such eigenwaves.

3.4. Orthogonality Relations for Eigenwaves and the Field Expansion

As is known, orthogonality relations for guiding systems with gyrotropic filling are established using the transposed formulation of Lorentz's theorem [18]. According to it, the fields in an auxiliary ("transposed") medium described by the transposed dielectric tensor $\boldsymbol{\varepsilon}^{\mathrm{T}}$ are involved in the analysis, along with the fields in a medium described by the tensor $\boldsymbol{\varepsilon}$. Based on Eqs. (5)–(8), it is easily verified that the fields of the discrete- and continuous-spectrum waves can always be defined so as to satisfy the relations

$$
\begin{aligned}
&E^{(\mathrm{T})}_{\rho;-m,-s,n}(\rho) = -E_{\rho;m,s,n}(\rho), \quad E^{(\mathrm{T})}_{\phi,z;-m,-s,n}(\rho) = E_{\phi,z;m,s,n}(\rho),\\
&H^{(\mathrm{T})}_{\rho;-m,-s,n}(\rho) = -H_{\rho;m,s,n}(\rho), \quad H^{(\mathrm{T})}_{\phi,z;-m,-s,n}(\rho) = H_{\phi,z;m,s,n}(\rho)
\end{aligned} \tag{41}
$$

and

$$
\begin{aligned}
&E^{(\mathrm{T})}_{\rho;-m,-s,\gamma}(\rho,q) = -E_{\rho;m,s,\gamma}(\rho,q), \quad E^{(\mathrm{T})}_{\phi,z;-m,-s,\gamma}(\rho,q) = E_{\phi,z;m,s,\gamma}(\rho,q),\\
&H^{(\mathrm{T})}_{\rho;-m,-s,\gamma}(\rho,q) = -H_{\rho;m,s,\gamma}(\rho,q), \quad H^{(\mathrm{T})}_{\phi,z;-m,-s,\gamma}(\rho,q) = H_{\phi,z;m,s,\gamma}(\rho,q).
\end{aligned} \tag{42}
$$

Hereafter, the superscript (T) denotes field quantities taken in the "transposed" medium. Moreover, on close examination, one can find that Eqs. (41) and (42) are fulfilled automatically when using the field coefficients determined according to Appendix A. Note that the derivation of the orthogonality relations for eigenwaves is significantly simplified with the help of Eqs. (41) and (42), although they are not a prerequisite for this procedure.

The orthogonality relations for the discrete- and continuous-spectrum waves are written as

$$
\int_0^{2\pi} d\phi \int_0^{\infty} \Big[\mathbf{E}_{m,s,n}(\mathbf{r}) \times \mathbf{H}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{n}}(\mathbf{r}) - \mathbf{E}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{n}}(\mathbf{r}) \times \mathbf{H}_{m,s,n}(\mathbf{r})\Big] \cdot \mathbf{z}_0 \rho d\rho = N_{m,s,n}\delta_{m,-\tilde{m}}\delta_{s,-\tilde{s}}\delta_{n,\tilde{n}}, \tag{43}
$$

$$
\int_0^{2\pi} d\phi \int_0^{\infty} \Big[\mathbf{E}_{m,s,\gamma}(\mathbf{r},q) \times \mathbf{H}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{n}}(\mathbf{r}) - \mathbf{E}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{n}}(\mathbf{r}) \times \mathbf{H}_{m,s,\gamma}(\mathbf{r},q)\Big] \cdot \mathbf{z}_0 \rho d\rho = 0, \tag{44}
$$

$$
\begin{aligned}
&\int_0^{2\pi} d\phi \int_0^{\infty} \Big[\mathbf{E}_{m,s,\gamma}(\mathbf{r},q) \times \mathbf{H}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{\gamma}}(\mathbf{r},\tilde{q}) - \mathbf{E}^{(\mathrm{T})}_{\tilde{m},\tilde{s},\tilde{\gamma}}(\mathbf{r},\tilde{q}) \times \mathbf{H}_{m,s,\gamma}(\mathbf{r},q)\Big] \cdot \mathbf{z}_0 \rho d\rho\\
&= N_{m,s,\gamma}(q)\delta(q-\tilde{q})\delta_{m,-\tilde{m}}\delta_{s,-\tilde{s}}\delta_{\gamma,\tilde{\gamma}},
\end{aligned} \tag{45}
$$

where $\delta_{\alpha,\beta}$ is the Kronecker delta, and $\delta(q)$ is the Dirac function. The norms of eigenwaves obey the relations $N_{m,+,\gamma}(q) = -N_{m,-,\gamma}(q) = N_{m,\gamma}(q)$ and $N_{m,+,n} = -N_{m,-,n} = N_{m,n}$ and are given by the formulas

$$
N_{m,\gamma}(q) = -\frac{16\pi}{Z_0 k_0^2}\frac{p}{q}\left[\left(C^{(2)}_{m,+,\gamma}(q)\right)^2 + \left(D^{(2)}_{m,+,\gamma}(q)\right)^2\right]\psi_{m,\gamma}(q), \tag{46}
$$

$$
N_{m,n} = \frac{1}{2\pi i}\left.\frac{dN_{m,1}(q)}{dq}\right|_{q=q_{m,n}}. \tag{47}
$$

Equations (43)–(47) are derived in Appendix B with allowance for Eqs. (41) and (42).

Using the above orthogonality relations, one can immediately calculate the total field due to given sources [25], which can be specified as electric and magnetic currents with the densities $\mathbf{J}(\mathbf{r})$ and $\mathbf{M}(\mathbf{r})$, respectively, on the right-hand sides of the Maxwell curl equations. For example, outside the source region, the total field can be expanded in the form

$$
\begin{bmatrix}\mathbf{E}(\mathbf{r})\\ \mathbf{H}(\mathbf{r})\end{bmatrix} = \sum_{m=-\infty}^{\infty}\left(\sum_n a_{m,s,n}\begin{bmatrix}\mathbf{E}_{m,s,n}(\mathbf{r})\\ \mathbf{H}_{m,s,n}(\mathbf{r})\end{bmatrix} + \sum_{\gamma=1}^{2}\int_0^{\infty} a_{m,s,\gamma}(q)\begin{bmatrix}\mathbf{E}_{m,s,\gamma}(\mathbf{r},q)\\ \mathbf{H}_{m,s,\gamma}(\mathbf{r},q)\end{bmatrix} dq\right), \tag{48}
$$

where one should take $s = -$ or $s = +$ for the field quantities taken on the left or right of the source region along the z axis, respectively, and the expansion coefficients $a_{m,s,n}$ and $a_{m,s,\gamma}$ are given by

$$
a_{m,\pm,n} = \frac{1}{N_{m,n}}\int_V \Big[\mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^{(\mathrm{T})}_{-m,\mp,n}(\mathbf{r}) - \mathbf{M}(\mathbf{r}) \cdot \mathbf{H}^{(\mathrm{T})}_{-m,\mp,n}(\mathbf{r})\Big]\, d\mathbf{r}, \tag{49}
$$

$$
a_{m,\pm,\gamma}(q) = \frac{1}{N_{m,\gamma}(q)}\int_V \Big[\mathbf{J}(\mathbf{r}) \cdot \mathbf{E}^{(\mathrm{T})}_{-m,\mp,\gamma}(\mathbf{r},q) - \mathbf{M}(\mathbf{r}) \cdot \mathbf{H}^{(\mathrm{T})}_{-m,\mp,\gamma}(\mathbf{r},q)\Big]\, d\mathbf{r}, \tag{50}
$$

with integration over the volume V occupied by the source currents. It is evident that Eqs. (48)–(50) differ from the corresponding formulas for an anisotropic outer medium [25] in that here two kinds of the continuous-spectrum waves satisfying the free-space dispersion relation of Eq. (4) are used instead of the ordinary and extraordinary waves of an anisotropic outer medium. In the interests of brevity, we do not present the field expansion inside the source region since such an expansion can also be obtained by analogy with [25].

3.5. Limiting Transition to the Case of Free Space

It is interesting to consider the transition from the eigenfunctions obtained for an open gyrotropic cylindrical waveguide to those in the case of free space. To do this, we should consider either the limiting transitions $\varepsilon \to 1$, $g \to 0$, and $\eta \to 1$ or the transition $a \to 0$ in the preceding formulas. The latter case is somewhat simpler for consideration. With decreasing radius a, the discrete-spectrum modes eventually disappear, while $\Delta^{(1)}_{m,s}(q)/\Delta^{(2)}_{m,s}(q) \to 1$ and $\Delta^{(3)}_{m,s}(q)/\Delta^{(1,2)}_{m,s}(q) \to 0$. For one value of γ, we then have

$$\psi_{m,\gamma}(q) = 1, \quad D^{(2)}_{m,s,\gamma}(q) = 0, \tag{51}$$

$$E_{z;m,s,\gamma}(\rho, q) = 2C^{(2)}_{m,s,\gamma}(q) q J_m(k_0 q\rho), \quad H_{z;m,s,\gamma}(\rho, q) = 0. \tag{52}$$

For the other value $\tilde{\gamma} \neq \gamma$, we get

$$\psi_{m,\tilde{\gamma}}(q) = -1, \quad C^{(2)}_{m,s,\tilde{\gamma}}(q) = 0, \tag{53}$$

$$E_{z;m,s,\tilde{\gamma}}(\rho, q) = 0, \quad H_{z;m,s,\tilde{\gamma}}(\rho, q) = 2Z_0^{-1} D^{(2)}_{m,s,\tilde{\gamma}}(q) q J_m(k_0 q\rho). \tag{54}$$

It follows from Eqs. (52) and (54) that in the considered limiting transition, the continuous-spectrum waves of the two types transform to the E and H waves of free space. Note that without loss of generality, the quantities $C^{(2)}_{m,s,\gamma}(q)$ and $D^{(2)}_{m,s,\tilde{\gamma}}(q)$ in these formulas can be replaced by an arbitrary constant.

4. TRANSITION TO AN OPEN CYLINDRICAL WAVEGUIDE FROM A CLOSED WAVEGUIDE

It is instructive to compare the properties of eigenmodes of a closed waveguide with indefinitely increasing radius, which was discussed above, and those of the continuous-spectrum waves of an open waveguide. Bearing in mind that the eigenvalue spectrum of the closed waveguide is discrete, we will analyze the behavior of the ratio $C^{(1)}_{m,s,n}/C^{(2)}_{m,s,n}$ of the field coefficients for the eigenmodes of the closed waveguide and the dependences of the quantities $\psi_{m,1}$ and $\psi_{m,2}$ for the continuous-spectrum waves of the open waveguide. The parameters of the gyrotropic cylinder of radius a in both waveguides are assumed identical.

As a gyrotropic medium in the region $\rho < a$, we take a cold collisionless electron magnetoplasma with the superimposed static magnetic field $\mathbf{B}_0 = B_0\mathbf{z}_0$. Then the elements of the tensor in Eq. (1) are given by the expressions

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2 - \omega_H^2}, \quad g = \frac{\omega_p^2 \omega_H}{(\omega^2 - \omega_H^2)\omega}, \quad \eta = 1 - \frac{\omega_p^2}{\omega^2}, \tag{55}$$

where ω_H and ω_p are the gyrofrequency and the plasma frequency of electrons, respectively.

Figure 1 shows the real parts of the quantities $\psi_{m,1}(q)$ and $\psi_{m,2}(q)$ as functions of q for an open waveguide and the values of the quantities $\mathrm{Re}\,(C^{(1)}_{m,s,n}/C^{(2)}_{m,s,n})$ for eigenmodes (with the transverse wave numbers $q = q_{m,n}$) of a closed waveguide. For this figure, we chose the following parameters: the azimuthal index $m = 1$, $k_0 a = 0.585$, $\omega/\omega_H = 0.5$, and $\omega_p/\omega_H = 6.95$. With these values, $\varepsilon = 65.3$, $g = -128.6$, and $\eta = -192$. The radius of the closed waveguide is varied from $R = 2a$ to $R = 50a$. It is seen in Fig. 1 that the green and red lines, referring to $\psi_{m,1}$ and $\psi_{m,2}$, respectively, transform continuously to each other at their coalescence points. The existence of such points is related to the jumps of $\psi_{m,1}$ and $\psi_{m,2}$ at the corresponding values of q due to the adopted convention for choosing the branch of the square root in Eq. (39). We observe that the eigenmodes of the closed waveguide are eventually divided into two families with increasing R, so that the value of the ratio $C^{(1)}_{m,s,n}/C^{(2)}_{m,s,n}$ in each family tends to approach either $\psi_{m,1}(q)$ or $\psi_{m,2}(q)$. Obviously, for $R \to \infty$, the continuous eigenvalue spectrum of the open waveguide forms from the discrete spectrum of the closed waveguide, and the corresponding continuous-spectrum solutions become either $\gamma = 1$ or $\gamma = 2$ waves. Note that we limit ourselves to analysis of the real parts of the corresponding quantities in Fig. 1 for brevity. Analogous results can be obtained for the imaginary parts of these quantities as well.

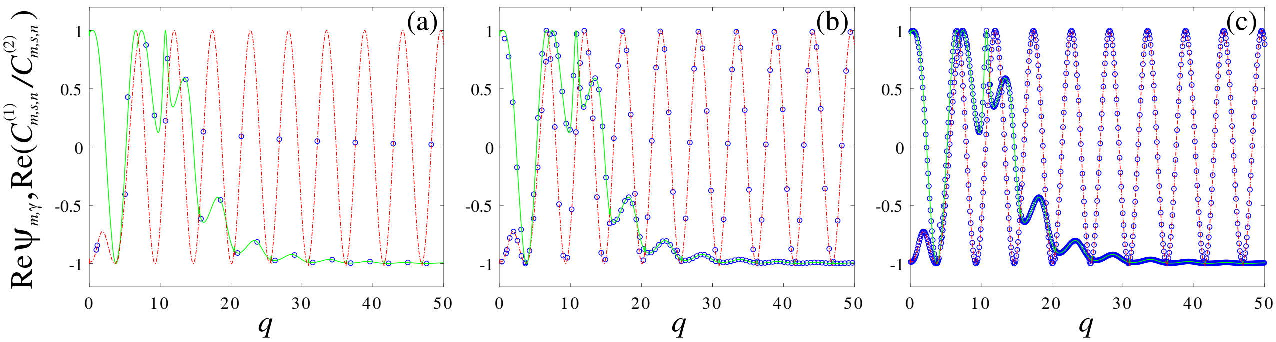

Figure 1. The real parts of the quantities $\psi_{m,1}$ (green solid lines) and $\psi_{m,2}$ (red dash-dotted lines) as functions of q for an open waveguide and the quantities $\mathrm{Re}\,(C^{(1)}_{m,s,n}/C^{(2)}_{m,s,n})$ (circles) for eigenmodes of a closed waveguide with radius (a) $R = 2a$, (b) $R = 10a$, and (c) $R = 50a$. See text for discussion.

5. TEST OF COMPLETENESS

Although the property of completeness has not been established rigorously for a system of the discrete and continuous-spectrum waves of an open waveguide that is considered in this work, this property can be shown to take place at the physical level of rigor if an arbitrary field expanded in terms of the found eigenwaves can be resynthesized by evaluating the corresponding expansion [31].

Let us specify a certain field $(\mathbf{E}^{(0)}, \mathbf{H}^{(0)})$ in the cross section $z = 0$ in which $\mathbf{r} = \mathbf{r}_\perp$, where $\mathbf{r}_\perp$ is the transverse part of the radius vector with respect to the z axis. The transverse parts $\mathbf{E}^{(0)}_\perp$ and $\mathbf{H}^{(0)}_\perp$ of this field in the chosen cross section can be expanded in terms of the above-described eigenwaves as

$$\mathbf{E}^{(0)}_\perp(\mathbf{r}_\perp) = \sum_{m=-\infty}^{\infty}\left(\sum_n b_{m,s,n}\mathbf{E}_{\perp;m,s,n}(\mathbf{r}_\perp) + \sum_{\gamma=1}^{2}\int_0^\infty b_{m,s,\gamma}(q)\mathbf{E}_{\perp;m,s,\gamma}(\mathbf{r}_\perp, q)\, dq\right), \tag{56}$$

$$\mathbf{H}^{(0)}_\perp(\mathbf{r}_\perp) = \sum_{m=-\infty}^{\infty}\left(\sum_n c_{m,s,n}\mathbf{H}_{\perp;m,s,n}(\mathbf{r}_\perp) + \sum_{\gamma=1}^{2}\int_0^\infty c_{m,s,\gamma}(q)\mathbf{H}_{\perp;m,s,\gamma}(\mathbf{r}_\perp, q)\, dq\right), \tag{57}$$

where the eigenwaves propagating only in one direction, i.e., either $s = -$ or $s = +$, are employed. Note that the expansion coefficients $b_{m,s,n}$ and $b_{m,s,\gamma}(q)$ for the electric field in Eq. (56) and the analogous coefficients $c_{m,s,n}$ and $c_{m,s,\gamma}(q)$ for the magnetic field in Eq. (57) are generally different for an arbitrary field that does not satisfy the same boundary conditions as those valid for the eigenwaves. Introducing new coefficients according to the relations

$$d_{m,s,n} = (b_{m,s,n} + c_{m,s,n})/2, \quad d_{m,-s,n} = (b_{m,s,n} - c_{m,s,n})/2, \tag{58}$$

$$d_{m,s,\gamma}(q) = [b_{m,s,\gamma}(q) + c_{m,s,\gamma}(q)]/2, \quad d_{m,-s,\gamma}(q) = [b_{m,s,\gamma}(q) - c_{m,s,\gamma}(q)]/2 \tag{59}$$

and allowing for Eqs. (32) and (40), one can rewrite the expansions of Eqs. (56) and (57) in the form

$$\begin{bmatrix}\mathbf{E}^{(0)}_\perp(\mathbf{r}_\perp)\\ \mathbf{H}^{(0)}_\perp(\mathbf{r}_\perp)\end{bmatrix} = \sum_{s=-}^{+}\sum_{m=-\infty}^{\infty}\left(\sum_n d_{m,s,n}\begin{bmatrix}\mathbf{E}_{\perp;m,s,n}(\mathbf{r}_\perp)\\ \mathbf{H}_{\perp;m,s,n}(\mathbf{r}_\perp)\end{bmatrix} + \sum_{\gamma=1}^{2}\int_0^\infty d_{m,s,\gamma}(q)\begin{bmatrix}\mathbf{E}_{\perp;m,s,\gamma}(\mathbf{r}_\perp, q)\\ \mathbf{H}_{\perp;m,s,\gamma}(\mathbf{r}_\perp, q)\end{bmatrix} dq\right). \tag{60}$$

To obtain the longitudinal components of the field, it is necessary to use the Maxwell equations projected onto the z-axis direction. Then, from Eqs. (1) and (2), one obtains

$$\mathbf{z}_0 \cdot (\nabla_\perp \times \mathbf{E}_\perp) = -i\omega\mu_0 H_z, \quad \mathbf{z}_0 \cdot (\nabla_\perp \times \mathbf{H}_\perp) = i\omega\epsilon_0\eta E_z. \tag{61}$$

Since the fields of eigenwaves satisfy these equations, from Eqs. (60) and (61) we get

$$\begin{bmatrix}E^{(0)}_z(\mathbf{r}_\perp)\\ H^{(0)}_z(\mathbf{r}_\perp)\end{bmatrix} = \sum_{s=-}^{+}\sum_{m=-\infty}^{\infty}\left(\sum_n d_{m,s,n}\begin{bmatrix}E_{z;m,s,n}(\mathbf{r}_\perp)\\ H_{z;m,s,n}(\mathbf{r}_\perp)\end{bmatrix} + \sum_{\gamma=1}^{2}\int_0^\infty d_{m,s,\gamma}(q)\begin{bmatrix}E_{z;m,s,\gamma}(\mathbf{r}_\perp, q)\\ H_{z;m,s,\gamma}(\mathbf{r}_\perp, q)\end{bmatrix} dq\right). \tag{62}$$

Thus, the field in the chosen cross section is represented using the common expansion coefficients for the transverse and longitudinal field components of eigenwaves of both propagation directions. In turn, these expansion coefficients are derived with the help of Eqs. (43)–(45), which express the property of orthogonality of the eigenwaves. As a result, we have

$$d_{m,\pm,n} = \frac{1}{N_{m,n}} \int_0^{2\pi} d\phi \int_0^{\infty} \left[\mathbf{E}^{(0)}(\mathbf{r}_\perp) \times \mathbf{H}^{(\mathrm{T})}_{-m,\mp,n}(\mathbf{r}_\perp) - \mathbf{E}^{(\mathrm{T})}_{-m,\mp,n}(\mathbf{r}_\perp) \times \mathbf{H}^{(0)}(\mathbf{r}_\perp)\right] \cdot \mathbf{z}_0 \rho d\rho, \quad (63)$$

$$\begin{aligned} d_{m,\pm,\gamma}(q) = {} & \frac{1}{N_{m,\gamma}(q)} \int_0^{2\pi} d\phi \int_0^{\infty} \Big[\mathbf{E}^{(0)}(\mathbf{r}_\perp) \times \mathbf{H}^{(\mathrm{T})}_{-m,\mp,\gamma}(\mathbf{r}_\perp, q) \\ & - \mathbf{E}^{(\mathrm{T})}_{-m,\mp,\gamma}(\mathbf{r}_\perp, q) \times \mathbf{H}^{(0)}(\mathbf{r}_\perp)\Big] \cdot \mathbf{z}_0 \rho d\rho. \end{aligned} \quad (64)$$

It is worth noting that only the transverse field components determine the right-hand sides of Eqs. (63) and (64).

As the test field ($\mathbf{E}^{(0)}$, $\mathbf{H}^{(0)}$), we choose the field of an E-polarized vortex Bessel beam propagating in free space and possessing the transverse wave number q_0 and azimuthal index m_0. The longitudinal components of the field of such a beam have the form

$$E_z^{(0)}(\mathbf{r}) = J_{m_0}(k_0 q_0 \rho) \exp(-im_0\phi - ik_0 p_0 z), \quad H_z^{(0)}(\mathbf{r}) = 0, \quad (65)$$

where $p_0 = (1 - q_0^2)^{1/2}$. The transverse field components for the beam can be obtained by substituting the longitudinal components of Eq. (65) into Eqs. (5) and (6), in which one should put $\varepsilon = \eta = 1$, $g = 0$, and $p_s = p_0$. It can be easily verified that the expansion coefficients are nonzero only for $m = m_0$. In addition, it is found straightforwardly that the coefficients $d_{m_0,s,\gamma}(q)$ are recast as

$$d_{m_0,s,\gamma}(q) = \tilde{d}_{m_0,s,\gamma}(q) + d^{(0)}_{m_0,\gamma} \delta\left(q - q_0\right) \delta_{+,s}, \quad (66)$$

where

$$d^{(0)}_{m_0,\gamma} = -\frac{4\pi}{Z_0 k_0^2} \frac{p_0}{q_0^2} \frac{C^{(2)}_{m_0,+,\gamma}(q_0)}{N_{m_0,\gamma}(q_0)} \left[1 + \psi_{m_0,\gamma}(q_0)\right]. \quad (67)$$

In Eq. (66), the singular term with the Dirac function is isolated. Due to such isolation, the resulting integrands in the field expressions will contain only the regular functions of the variable q, which significantly facilitates the numerical integration over this variable.

Let us present the results of evaluation of the azimuthal field components yielded by the expansion of Eq. (60) at $z = 0$. These components can be written as follows:

$$\begin{bmatrix} E_\phi^{(0)}(\mathbf{r}_\perp) \\ H_\phi^{(0)}(\mathbf{r}_\perp) \end{bmatrix} = \begin{bmatrix} E_\phi^{(\mathrm{I})}(\mathbf{r}_\perp) \\ H_\phi^{(\mathrm{I})}(\mathbf{r}_\perp) \end{bmatrix} + \begin{bmatrix} E_\phi^{(\mathrm{II})}(\mathbf{r}_\perp) \\ H_\phi^{(\mathrm{II})}(\mathbf{r}_\perp) \end{bmatrix} + \begin{bmatrix} E_\phi^{(\mathrm{III})}(\mathbf{r}_\perp) \\ H_\phi^{(\mathrm{III})}(\mathbf{r}_\perp) \end{bmatrix}, \quad (68)$$

where

$$\begin{bmatrix} E_\phi^{(\mathrm{I})}(\mathbf{r}_\perp) \\ H_\phi^{(\mathrm{I})}(\mathbf{r}_\perp) \end{bmatrix} = \sum_{s=-}^{+} \sum_{n} d_{m_0,s,n} \begin{bmatrix} E_{\phi;m_0,s,n}(\rho) \\ H_{\phi;m_0,s,n}(\rho) \end{bmatrix} \exp(-im_0\phi), \quad (69)$$

$$\begin{bmatrix} E_\phi^{(\mathrm{II})}(\mathbf{r}_\perp) \\ H_\phi^{(\mathrm{II})}(\mathbf{r}_\perp) \end{bmatrix} = \sum_{s=-}^{+} \sum_{\gamma=1}^{2} \int_0^{\infty} \tilde{d}_{m_0,s,\gamma}(q) \begin{bmatrix} E_{\phi;m_0,s,\gamma}(\rho, q) \\ H_{\phi;m_0,s,\gamma}(\rho, q) \end{bmatrix} dq \, \exp(-im_0\phi), \quad (70)$$

$$\begin{bmatrix} E_\phi^{(\mathrm{III})}(\mathbf{r}_\perp) \\ H_\phi^{(\mathrm{III})}(\mathbf{r}_\perp) \end{bmatrix} = \sum_{\gamma=1}^{2} d^{(0)}_{m_0,\gamma} \begin{bmatrix} E_{\phi;m_0,+,\gamma}(\rho, q_0) \\ H_{\phi;m_0,+,\gamma}(\rho, q_0) \end{bmatrix} \exp(-im_0\phi). \quad (71)$$

Figure 2 shows the azimuthal components of the test field, the synthesized field yielded by Eq. (68), and the partial contributions given by Eqs. (69)–(71) to the synthesized field as functions of the coordinate ρ in the plane $z = 0$ at $\phi = 0$. The calculations were performed for the following parameters of the test field: $m_0 = 1$, $q_0 = 1/\sqrt{2}$, and $\omega/\omega_H = 6.98$. The dimensionless parameters of the gyrotropic cylinder were equal to $\omega_p/\omega_H = 6.95$ and $k_0 a = 8.167$. In this case, $\varepsilon\eta < 0$, so that the waveguide can support an infinite number of eigenmodes of the quasielectrostatic type [13, 21, 22]. For calculations, we took into account the first 56 eigenmodes, and the infinite upper integration limit in Eq. (70) was replaced by a finite quantity equal to 40. This turned out to be sufficient to ensure the required accuracy of evaluating the field. Indeed, it is seen in Fig. 2 that the spatial distributions of the initial field and the synthesized field coincide with graphical accuracy. It should be emphasized that such agreement can only be reached if both the $\gamma = 1$ and $\gamma = 2$ continuous-spectrum waves are taken into account, along with the eigenmodes.

To illustrate the last statement, Fig. 3 shows the azimuthal components of the total field, which have been synthesized using Eq. (68), and the same quantities calculated without allowance for the $\gamma = 1$ term or $\gamma = 2$ term in Eq. (70). Note that when Fig. 3 was plotted, the same values of all parameters as in Fig. 2 were used. Under such conditions, $\operatorname{Im} E_\phi = 0$ and $\operatorname{Re} H_\phi = 0$ in the total field in the cross section $z = 0$. Therefore, the red solid line is absent in Fig. 3(a), whereas the blue solid line is absent in Fig. 3(b). It is evident that neglecting either the $\gamma = 1$ or $\gamma = 2$ term in Eq. (70) makes the field representation inaccurate. A similar situation takes place if we omit any of these terms in Eq. (71) or do so in Eqs. (70) and (71) simultaneously. Thus, rejecting one of two kinds of the continuous-spectrum

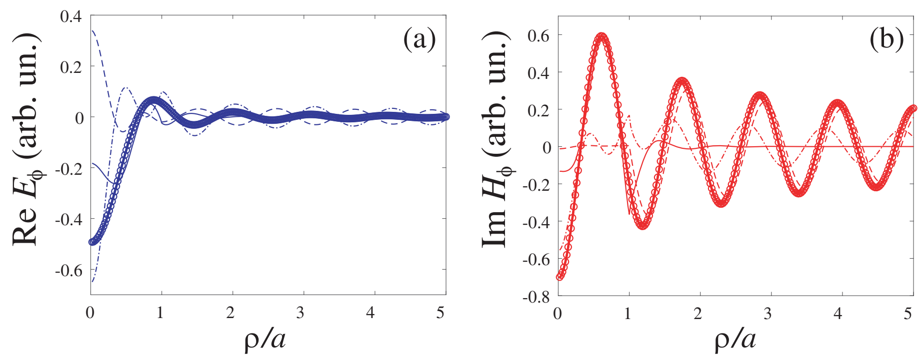

Figure 2. Azimuthal components of the test field (bold solid line), the synthesized field yielded by Eq. (68) (circles), and the partial contributions given by Eqs. (69), (70), and (71) and denoted by the thin solid, dash-dotted, and dashed lines, respectively, to the synthesized field as functions of the coordinate ρ in the plane $z = 0$ at $\phi = 0$ for $m_0 = 1$, $q_0 = 1/\sqrt{2}$, $\omega/\omega_H = 6.98$, $\omega_p/\omega_H = 6.95$, and $k_0 a = 8.167$. The real and imaginary parts of the field components are shown by the blue and red lines, respectively.

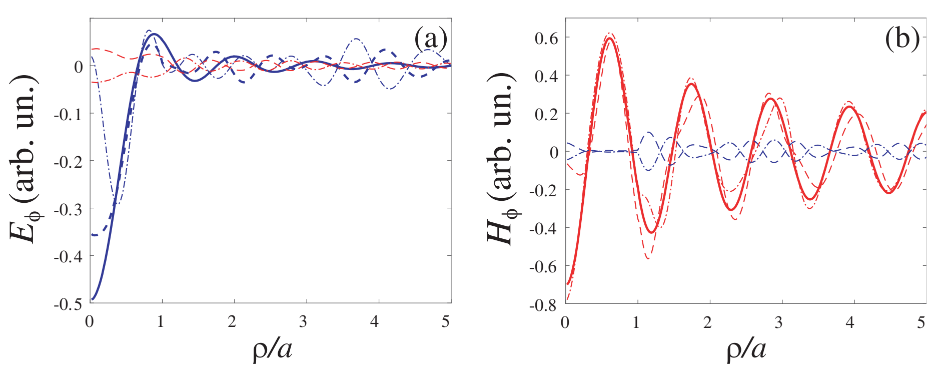

Figure 3. Azimuthal components of the total field (bold solid lines) synthesized using Eq. (68) and the same quantities calculated without allowance for the $\gamma = 1$ and $\gamma = 2$ terms (dashed and dash-dotted lines, respectively) in Eq. (70). Same values of parameters as in Fig. 2. The real and imaginary parts of the field quantities are shown by the blue and red lines, respectively.

waves makes the set of eigenwaves incomplete and, hence, incapable of representing the specified field correctly. Therefore, only the complete system of eigenwaves should be generally employed for expanding the electromagnetic field in the presence of an open gyrotropic cylindrical waveguide.

6. CONCLUSION

In this work, we have presented a method for obtaining a complete orthogonal system of the discrete- and continuous-spectrum waves of an open cylindrical waveguide filled with a gyrotropic medium and located in free space. We have derived closed-form expressions for the fields of the eigenwaves and their norms. It is shown that there exist two kinds of the continuous-spectrum waves which should be taken into account to correctly describe the total electromagnetic field in the presence of such a waveguide. In applying the resultant formulation, a given test field has been expanded in terms of the found eigenwaves. Then this field has been resynthesized by evaluating the obtained expansion and perfect coincidence between the field yielded by this procedure and the initially specified test field has been demonstrated numerically.

The results obtained can be useful in studying the excitation and propagation of electromagnetic waves in various open systems, including magneto-optical fibers, magnetized plasma waveguides, etc. Moreover, the proposed method can be extended to the case of open guiding structures filled with a medium having more complicated properties than those considered in this work. Future work should also include an application of the developed method of field expansion to solving diffraction problem in the presence of semi-infinite open gyrotropic waveguides.

ACKNOWLEDGMENT

This work was supported by the Russian Science Foundation (project No. 18–72–10046).

APPENDIX A. DERIVATION OF THE FIELD COEFFICIENTS

The coefficients $B^{(1)}_{m,s,n}$, $B^{(2)}_{m,s,n}$, $C^{(2)}_{m,s,n}$, and $D^{(2)}_{m,s,n}$ are related by a system of four linear homogeneous equations which ensure the fulfillment of the boundary conditions for the eigenmodes at $\rho = a$. Since the determinant of this system is zero, one of the coefficients can be arbitrary, and it is sufficient to take any three equations of the system for obtaining the remaining coefficients:

$$\begin{pmatrix} \dfrac{i}{\eta} n_{1,s} Q_1 J_m(Q_1) & \dfrac{i}{\eta} n_{2,s} Q_2 J_m(Q_2) & -Q H^{(2)}_m(Q) \\ Q_1 J_m(Q_1) & Q_2 J_m(Q_2) & 0 \\ i\hat{J}^{(1)}_m & i\hat{J}^{(2)}_m & p_s m \dfrac{H^{(2)}_m(Q)}{Q} \end{pmatrix} \begin{pmatrix} B^{(1)}_{m,s,n} \\ B^{(2)}_{m,s,n} \\ C^{(2)}_{m,s,n} \end{pmatrix} = \begin{pmatrix} 0 \\ -Q H^{(2)}_m(Q) \\ i H^{(2)\prime}_m(Q) \end{pmatrix} D^{(2)}_{m,s,n}. \tag{A1}$$

Here, all terms depending on q are taken at $q = q_{m,n}$. It is convenient to choose the coefficient $D^{(2)}_{m,s,n}$ equal to the determinant of the matrix on the left-hand side of Eq. (A1). The other coefficients in Eq. (A1) are then expressed via $D^{(2)}_{m,s,n}$.

The coefficients for the continuous-spectrum waves can also be found using any three equations of a system of four linear equations that are obtained for $B^{(1)}_{m,s,\gamma}(q)$, $B^{(2)}_{m,s,\gamma}(q)$, $C^{(2)}_{m,s,\gamma}(q)$, and $D^{(2)}_{m,s,\gamma}(q)$ from the boundary conditions for these waves at $\rho = a$. Thus, the coefficients for the continuous-spectrum waves are determined from the system

$$\begin{pmatrix} \dfrac{i}{\eta} n_{1,s} Q_1 J_m(Q_1) & \dfrac{i}{\eta} n_{2,s} Q_2 J_m(Q_2) & -Q \tilde{\mathcal{H}}_{m,\gamma} \\ Q_1 J_m(Q_1) & Q_2 J_m(Q_2) & 0 \\ i\hat{J}^{(1)}_m & i\hat{J}^{(2)}_m & p_s m \dfrac{\tilde{\mathcal{H}}_{m,\gamma}}{Q} \end{pmatrix} \begin{pmatrix} B^{(1)}_{m,s,\gamma} \\ B^{(2)}_{m,s,\gamma} \\ C^{(2)}_{m,s,\gamma} \end{pmatrix} = \begin{pmatrix} 0 \\ -Q \mathcal{H}_{m,\gamma} \\ i\mathcal{H}'_{m,\gamma} \end{pmatrix} D^{(2)}_{m,s,\gamma}, \tag{A2}$$

where $\mathcal{H}_{m,\gamma} = H_m^{(2)}(Q) - \psi_{m,\gamma}(q)H_m^{(1)}(Q)$, $\tilde{\mathcal{H}}_{m,\gamma} = H_m^{(2)}(Q) + \psi_{m,\gamma}(q)H_m^{(1)}(Q)$, and the quantity q takes real values. The coefficient $D_{m,s,\gamma}^{(2)}$ for each γ may be determined as the determinant of the matrix on the left-hand side of Eq. (A2). Then the remaining coefficients are readily expressed in terms of $D_{m,s,\gamma}^{(2)}$ by solving Eq. (A2).

It can also be verified straightforwardly that in the case of $\tilde{\gamma} \neq \gamma$, the following relation takes place:

$$C_{m,s,\gamma}^{(2)}(q)C_{m,s,\tilde{\gamma}}^{(2)}(q) + D_{m,s,\gamma}^{(2)}(q)D_{m,s,\tilde{\gamma}}^{(2)}(q) = 0. \tag{A3}$$

This result turns out to be useful when deriving the orthogonality relations for the continuous-spectrum waves.

APPENDIX B. DERIVATION OF THE ORTHOGONALITY RELATIONS

To derive Eqs. (43)–(47), we consider two fields, ($\mathbf{E}_{\text{I}}$, $\mathbf{H}_{\text{I}}$) and ($\mathbf{E}_{\text{II}}^{(\text{T})}$, $\mathbf{H}_{\text{II}}^{(\text{T})}$), which have the same frequency and satisfy the homogeneous Maxwell equations in different media that are described by the tensors $\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}^{\text{T}}$, respectively. Then it follows from the Maxwell equations that

$$\nabla \cdot \left(\mathbf{E}_{\text{I}} \times \mathbf{H}_{\text{II}}^{(\text{T})} - \mathbf{E}_{\text{II}}^{(\text{T})} \times \mathbf{H}_{\text{I}}\right) = 0. \tag{B1}$$

It is worth noting that Eq. (B1) represents a transposed formulation of Lorentz's theorem in a gyrotropic medium in the absence of sources. Applying the divergence theorem in a two-dimensional form to Eq. (B1) yields

$$\frac{\partial}{\partial z}\int_{S_\perp}\left(\mathbf{E}_{\text{I}} \times \mathbf{H}_{\text{II}}^{(\text{T})} - \mathbf{E}_{\text{II}}^{(\text{T})} \times \mathbf{H}_{\text{I}}\right)\cdot \mathbf{z}_0 dS_\perp = -\oint_L \left(\mathbf{E}_{\text{I}} \times \mathbf{H}_{\text{II}}^{(\text{T})} - \mathbf{E}_{\text{II}}^{(\text{T})} \times \mathbf{H}_{\text{I}}\right)\cdot \mathbf{n}_0 dl. \tag{B2}$$

Here, $S_\perp$ is an arbitrary cross-sectional area with respect to the z axis; the line integral is along the boundary L of $S_\perp$; and $\mathbf{n}_0$ is the unit outward normal on L in the plane of $S_\perp$.

To obtain the relations given by Eqs. (43) and (44), we should substitute the fields occurring in the integrands of these relations into Eq. (B2) with allowance for Eqs. (41) and (42) and then tend the contour L to infinity where the fields of the discrete-spectrum waves are zero. The above-described procedure makes it possible to immediately arrive at Eqs. (43) and (44). During this derivation, use should be made of the identity

$$p_{-m,-s,n}^{(\text{T})} = -p_{m,s,n}, \tag{B3}$$

which follows from the dispersion relation in the form of Eq. (30) for the discrete-spectrum waves, and the formula

$$\int_0^{2\pi} \exp\left[-i(m+\tilde{m})\phi\right] d\phi = 2\pi\delta_{m,-\tilde{m}}. \tag{B4}$$

Derivation of Eqs. (45) and (46) is more involved. Denoting the left-hand side of Eq. (45) as $J_{\tilde{m},\tilde{s},\tilde{\gamma}}^{m,s,\gamma}$, we now substitute the fields occurring in the integrand of Eq. (45) into Eq. (B2) and, after some algebra, obtain

$$\begin{aligned} J_{\tilde{m},\tilde{s},\tilde{\gamma}}^{m,s,\gamma} &= \frac{\exp\left\{-ik_0\left[p_s(q)+p_{\tilde{s}}(\tilde{q})\right]z\right\}}{ik_0\left[p_s(q)+p_{\tilde{s}}(\tilde{q})\right]}\int_0^{2\pi}\exp\left[-i(m+\tilde{m})\phi\right]d\phi \\ &\times \lim_{\rho\to\infty}\rho\left[E_{\phi;m,s,\gamma}(\rho,q)H_{z;\tilde{m},\tilde{s},\tilde{\gamma}}^{(\text{T})}(\rho,\tilde{q}) - E_{\phi;\tilde{m},\tilde{s},\tilde{\gamma}}^{(\text{T})}(\rho,\tilde{q})H_{z;m,s,\gamma}(\rho,q)\right. \\ &\left. + E_{z;\tilde{m},\tilde{s},\tilde{\gamma}}^{(\text{T})}(\rho,\tilde{q})H_{\phi;m,s,\gamma}(\rho,q) - E_{z;m,s,\gamma}(\rho,q)H_{\phi;\tilde{m},\tilde{s},\tilde{\gamma}}^{(\text{T})}(\rho,\tilde{q})\right]. \end{aligned} \tag{B5}$$

Making use of the fact that $J_{\tilde{m},\tilde{s},\tilde{\gamma}}^{m,s,\gamma} = 0$ for $\tilde{m} \neq -m$ by virtue of Eq. (B4), we can further consider only the case of $\tilde{m} = -m$. Then we should analyze the cases where $\tilde{s} = -s$ and $\tilde{s} \neq -s$.

For $\tilde{s} = -s$, taking into account the relations in Eq. (42) and the large-argument approximations of the cylindrical functions determining the fields in Eq. (B5), we arrive at the following expression:

$$\begin{aligned} J^{m,s,\gamma}_{-m,-s,\tilde{\gamma}} = & \frac{4(q-\tilde{q})\exp\{-ik_0[p_s(q)-p_s(\tilde{q})]z\}}{Z_0k_0^2[p_s(q)-p_s(\tilde{q})](q\tilde{q})^{1/2}} \\ & \times \lim_{\rho\to\infty}\left\{(-1)^m\left(C^{(2)}_{m,s,\gamma}(q)C^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})-D^{(2)}_{m,s,\gamma}(q)D^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})\right)\right. \\ & \times\left(\psi_{m,\gamma}(q)\psi_{m,\tilde{\gamma}}(\tilde{q})e^{ik_0\rho(q+\tilde{q})}+e^{-ik_0\rho(q+\tilde{q})}\right) \\ & -i\,(q+\tilde{q})\left(\psi_{m,\gamma}(q)C^{(2)}_{m,s,\gamma}(q)C^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})-\psi_{m,\tilde{\gamma}}(\tilde{q})C^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})C^{(2)}_{m,s,\gamma}(q)\right. \\ & \left.+\,\psi_{m,\gamma}(q)D^{(2)}_{m,s,\gamma}(q)D^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})-\psi_{m,\tilde{\gamma}}(\tilde{q})D^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})D^{(2)}_{m,s,\gamma}(q)\right)\frac{\cos\left[k_0\rho\,(q-\tilde{q})\right]}{q-\tilde{q}} \\ & +(q+\tilde{q})\left(\psi_{m,\gamma}(q)C^{(2)}_{m,s,\gamma}(q)C^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})+\psi_{m,\tilde{\gamma}}(\tilde{q})C^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})C^{(2)}_{m,s,\gamma}(q)\right. \\ & \left.\left.+\,\psi_{m,\gamma}(q)D^{(2)}_{m,s,\gamma}(q)D^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})+\psi_{m,\tilde{\gamma}}(\tilde{q})D^{(2)}_{m,s,\tilde{\gamma}}(\tilde{q})D^{(2)}_{m,s,\gamma}(q)\right)\frac{\sin\left[k_0\rho\,(q-\tilde{q})\right]}{q-\tilde{q}}\right\}. \end{aligned} \tag{B6}$$

Calculating the limit $\rho\to\infty$ in Eq. (B6) within the framework of the theory of distributions and taking into account the well-known formula

$$\delta(\xi) = \lim_{R\to\infty}\frac{\sin(R\xi)}{\pi\xi}, \tag{B7}$$

we obtain the expression

$$J^{m,s,\gamma}_{-m,-s,\tilde{\gamma}} = \frac{16\pi}{Z_0k_0^2}\left(\frac{\mathrm{d}p_s}{\mathrm{d}q}\right)^{-1}\left[\left(C^{(2)}_{m,s,\gamma}(q)\right)^2+\left(D^{(2)}_{m,s,\gamma}(q)\right)^2\right]\psi_{m,\gamma}(q)\delta\left(q-\tilde{q}\right)\delta_{\gamma,\tilde{\gamma}}, \tag{B8}$$

which corresponds to the orthogonality relation in Eq. (45) for $\tilde{m} = -m$ and $\tilde{s} = -s$, with the norm $N_{m,\gamma}(q)$ given by Eq. (46), if one takes into account that $p_s'(q) = -q/p_s(q)$. Note that in deriving Eq. (B8), use was made of Eq. (A3) for $\tilde{\gamma} \neq \gamma$.

In the case where $\tilde{s} \neq -s$, i.e., $\tilde{s} = s$, we should perform similar calculations with allowance for the relation $p_s(q) + p_{\tilde{s}}(\tilde{q}) = p_s(q) + p_s(\tilde{q})$. Then it turns out that the expression for $J^{m,s,\gamma}_{-m,\tilde{s},\tilde{\gamma}}$ contains the quantity

$$\frac{q-\tilde{q}}{p_s(q)+p_s(\tilde{q})}\delta\left(q-\tilde{q}\right)=0,$$

which leads to a zero result for $J^{m,s,\gamma}_{-m,\tilde{s},\tilde{\gamma}}$ in this case.

To derive Eq. (47) for $N_{m,n}$, we note that this quantity can be represented as

$$N_{m,n} = 2\pi\lim_{R\to\infty}\lim_{q\to q_{m,n}}\int_0^R\left[\mathbf{E}_{m,+,n}(\mathbf{r})\times\mathbf{H}^{(\mathrm{T})}_{-m,-,1}(\mathbf{r},q)-\mathbf{E}^{(\mathrm{T})}_{-m,-,1}(\mathbf{r},q)\times\mathbf{H}_{m,+,n}(\mathbf{r})\right]\cdot\mathbf{z}_0\rho d\rho. \tag{B9}$$

Using Eq. (B2), the integral over the cross-sectional area in Eq. (B9) can be transformed to the line integral over the circular boundary of this area by analogy with that made in deriving Eq. (B5). Applying this procedure and making use of the relations in Eq. (42) for $\gamma = 1$, we rewrite Eq. (B9) as

$$\begin{aligned} N_{m,n} = & \ 2\pi\lim_{R\to\infty}\lim_{q\to q_{m,n}}\frac{\exp\left\{-ik_0\left[p(q_{m,n})-p(q)\right]z\right\}}{ik_0\left[p(q_{m,n})-p(q)\right]} \\ & \times\rho\left[E_{\phi;m,+,n}(\rho)H^{(\mathrm{T})}_{z;-m,-,1}(\rho,q)-E^{(\mathrm{T})}_{\phi;-m,-,1}(\rho,q)H_{z;m,+,n}(\rho)\right. \\ & \left.\left.+\ E^{(\mathrm{T})}_{z;-m,-,1}(\rho,q)H_{\phi;m,+,n}(\rho)-E_{z;m,+,n}(\rho)H^{(\mathrm{T})}_{\phi;-m,-,1}(\rho,q)\right]\right|_{\rho=R}. \end{aligned} \tag{B10}$$

Performing the operations indicated here and taking into account the inequality $\operatorname{Im} q_{m,n} < 0$ and the relation $\psi_{m,1}(q_{m,n}) = 0$, we find

$$N_{m,n} = \frac{8i}{Z_0k_0^2}\frac{p}{q}\left[\left(C^{(2)}_{m,s,1}(q)\right)^2+\left(D^{(2)}_{m,s,1}(q)\right)^2\right]\left.\frac{\mathrm{d}\psi_{m,1}(q)}{\mathrm{d}q}\right|_{q=q_{m,n}}. \tag{B11}$$

Bearing in mind Eq. (46), we come from Eq. (B11) to Eq. (47) for $N_{m,n}$.

REFERENCES

1. Lord Rayleigh, J. W., *The Theory of Sound*, Macmillan, London, 1894.
2. Sommerfeld, A., "Ueber die fortpflanzung elektrodynamischer wellen längs eines drahtes," *Ann. Phys. Chem.*, Vol. 303, No. 2, 233–290, 1899.
3. Hondros, D. and P. Debye, "Elektromagnetische wellen an dielektrischen drähten," *Ann. Phys.*, Vol. 337, No. 8, 465–476, 1910.
4. Marcuvitz, N., "On field representations in terms of leaky modes or eigenmodes," *IRE Trans. Antennas Propag.*, Vol. 4, No. 3, 192–194, 1956.
5. Angulo, C. and W. Chang, "A variational expression for the terminal admittance of a semi-infinite dielectric rod," *IRE Trans. Antennas Propag.*, Vol. 7, No. 3, 207–212, 1959.
6. Shevchenko, V. V., *Continuous Transitions in Open Waveguides*, Golem Press, Boulder, 1971.
7. Manenkov, A. B., "The excitation of open homogeneous waveguides," *Radiophys. Quantum Electron.*, Vol. 13, No. 5, 578–586, 1970.
8. Polat, B., "High-frequency scattering from a semi-infinite cylindrical dielectric rod," *ZAMM — J. Appl. Math. Mech.*, Vol. 80, No. 6, 389–398, 2000.
9. Collin, R. E., *Field Theory of Guided Waves*, IEEE Press, New York, 1990.
10. Manenkov, A. B., "Propagation of waves in open waveguides with anisotropic dielectrics," *Radiophys. Quantum Electron.*, Vol. 24, No. 1, 60–69, 1981.
11. Manenkov, A. B., "Irregular magneto-optical waveguides," *IEEE Trans. Microw. Theory Tech.*, Vol. MTT-29, No. 9, 906–910, 1981.
12. Uzunoglu, N. K., P. G. Cottis, and J. G. Fikioris, "Excitation of electromagnetic waves in a gyroelectric cylinder," *IEEE Trans. Antennas Propagat.*, Vol. 33, No. 1, 90–95, 1985.
13. Kudrin, A. V., V. A. Es'kin, C. Krafft, and T. M. Zaboronkova, "Whistler wave excitation by a loop antenna in a bounded collisional magnetoplasma," *Phys. Scr.*, Vol. 77, No. 5, 055501, 2008.
14. Marcuse, D., "Radiation losses of the dominant mode in round dielectric waveguides," *Bell Syst. Tech. J.*, Vol. 49, No. 8, 1665–1693, 1970.
15. Snyder, A. W., "Continuous mode spectrum of a circular dielectric rod," *IEEE Trans. Microw. Theory Tech.*, Vol. MTT-19, No. 8, 720–727, 1971.
16. Tigelis, I., C. N. Capsalis, and N. K. Uzunoglu, "Computation of the dielectric rod waveguide radiation modes," *Int. J. Infrared Millim. Waves*, Vol. 8, No. 9, 1053–1068, 1987.
17. Manenkov, A. B., "Orthogonality relations for the eigenmodes of lossy anisotropic waveguides (fibres)," *IEE Proc. J*, Vol. 140, No. 3, 206–212, 1993.
18. Kondrat'ev, I. G., A. V. Kudrin, and T. M. Zaboronkova, *Electrodynamics of Density Ducts in Magnetized Plasmas*, Gordon and Breach, Amsterdam, 1999.
19. Kostrov, A. V., A. V. Kudrin, L. E. Kurina, G. A. Luchinin, A. A. Shaykin, and T. M. Zaboronkova, "Whistlers in thermally generated ducts with enhanced plasma density: Excitation and propagation," *Phys. Scr.*, Vol. 62, No. 1, 51–65, 2000.
20. Zaboronkova, T. M., A. V. Kudrin, and M. Yu. Lyakh, "Excitation of nonsymmetric waves by given sources in a magnetoplasma in the presence of a cylindrical plasma channel," *Radiophys. Quantum Electron.*, Vol. 46, Nos. 5–6, 407–424, 2003.
21. Boswell, R. W. and F. F. Chen, "Helicons — The early years," *IEEE Trans. Plasma Sci.*, Vol. 25, No. 6, 1229–1244, 1997.
22. Chen, F. F. and R. W. Boswell, "Helicons — The past decade," *IEEE Trans. Plasma Sci.*, Vol. 25, No. 6, 1245–1257, 1997.
23. Arnush, D., "The role of Trivelpiece-Gould waves in antenna coupling to helicon waves," *Phys. Plasmas*, Vol. 7, No. 7, 3042–3050, 2000.

24. Gurevich, A. G. and G. A. Melkov, *Magnetization Oscillations and Waves*, CRC Press, Boca Raton, 1996.
25. Kudrin, A. V., E. Yu. Petrov, G. A. Kyriacou, and T. M. Zaboronkova, "Electromagnetic radiation from sources embedded in a cylindrically stratified unbounded gyrotropic medium," *Progress In Electromagnetics Research B*, Vol. 12, 297–331, 2009.
26. Es'kin, V. A. and A. V. Kudrin, "Diffraction of an electromagnetic vortex Bessel beam by the end of a semi-infinite magnetized plasma cylinder," *2017 Progress In Electromagnetics Research Symposium — Spring (PIERS)*, 1372–1379, St. Petersburg, Russia, May 22–25, 2017.
27. Es'kin, V. A. and A. V. Kudrin, "Diffraction of an electromagnetic Laguerre-Gaussian beam by the end of a semi-infinite gyrotropic cylinder," *2019 International Conference on Electromagnetics in Advanced Applications (ICEAA)*, 133–138, Granada, Spain, September 9–13, 2019.
28. Ginzburg, V. L., *The Propagation of Electromagnetic Waves in Plasmas*, Pergamon Press, Oxford, 1970.
29. Capolino, F., *Theory and Phenomena of Metamaterials*, CRC Press, Boca Raton, 2017.
30. Abramowitz, M. and I. A. Stegun, eds., *Handbook of Mathematical Functions, with Formulas, Graphs, and Mathematical Tables*, Dover, Mineola, 1974.
31. Shevchenko, V. V., "On the completeness of spectral expansion of the electromagnetic field in the set of dielectric circular rod waveguide eigenwaves," *Radio Sci.*, Vol. 17, No. 1, 229–231, 1982.