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New Theoretical Floquet Modal Analysis to Study 3-D Finite Almost Periodic Structures with Coupled Cells

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Abstract—This research letter offers a generalization character to our previous work [4, 5] to examine a 3-D almost periodic phased array antenna excited by arbitrarily located sources. An original modal formulation based on the Floquet analysis procedure is proposed utilizing the periodic walls along x, y, and z-axes, where the analysis region in the spectral domain is reduced to the Brillouin zone. Here, a good idea is provided to enforce the given boundary conditions for obtaining an integral equation formalism developed through Galerkin's method for solving periodic volumic structure (e.g., 3-D regular structure in the cubic grid). The interaction between cells in 3-D geometry (lattice) could be deduced using a novel expression of mutual coupling. However, it is possible to obtain it explicitly by the mean of Fourier transform and its inverse. Then, it is proven how Floquet analysis can be employed to study a 3D-finite array configuration with arbitrary amplitude and linear phase distribution along x, y, and z directions, including mutual interaction effects. To deal with the real hole 3D array configuration, a superposition theorem is suggested to describe the electromagnetic behavior in the spatial domain. For modeling the given 3D antenna array, one numerical method is adopted: The moment method combined with an equivalent circuit (MoM-GEC). An important gain in the running time and memory used would be achieved using Floquet analysis in comparison with other spatial conventional methods (especially, when the number of cells increases by adding the second and third directions).

1. INTRODUCTION

Nowadays, almost periodic structures in 3D-dimensional arrangement become the subject of extensive scientific research, especially in defense and space applications, communication systems, and electronic devices [1–3]. Common numerical techniques have been employed in this way, and they intend to solve partial differential equations with 3D periodic boundary conditions. In this work, we suggest calculating the mutual coupling parameters between antennas array sources in 3D-dimensional configuration (lattice). To consider coupling effects, a new Floquet modal analysis is required to decrease the difficulty of the reviewed problem [4–7]. Then, the field components can be therefore expressed in the generalized Fourier series expansions, and the analysis region can be reduced to simple unity periodicity cell. For solving this problem, we propose adopting an integral method based on the generalized equivalent model and using the Floquet modes which are written as excitation sources [4, 5]. The transition to the real spatial domain is provided through a simple superposition theorem that is established using a Fourier transform's inverse [6, 7]. The proposed paper is structured into four sections as follows. The first step is to remind the essential theoretical Floquet modal analysis. To start with, the Finite Fourier Transform (FFT) and spectral decomposition are given. Next, in Section 3 the studied problem is formulated by an integral equation based on a new operator formalism deduced using the Generalized Equivalent Circuit (GEC) approach. Then, in Section 4 the principal advantages in terms of the memory used and numerical execution time are presented and discussed. Finally, in the last section, some conclusions are drawn.

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2. FLOQUET MODAL ANALYSIS

In this section, we propose to tackle the elements in a 3D configuration to generalize the preceding case with more complex periodicities [4, 5]. Consider a 3-D structure, a linear array of point sources on the x-y plane with its (i, s, k)th element placed at $\vec{r_{isk}} = (id_x, sd_y, kd_z)$ with $-\frac{N_x}{2} \leq i \leq \frac{N_x}{2} - 1$, $-\frac{N_y}{2} \leq s \leq \frac{N_y}{2} - 1$, $-\frac{N_z}{2} \leq k \leq \frac{N_z}{2} - 1$, where d_x, d_y, and d_z are the inter-element periods in the x, y, and z coordinates, respectively [11–14]. Adding the third direction allows three Floquet modes $(\alpha_p, \beta_q, \psi_t)$ in the modal space when each element is surrounded by a suitable periodic wall along (ox), (oy), and (oz) directions [1–3]. Using the modal analysis, the periodic symmetry of the structure forces us to focus on one cell of the array. The unit cell can be defined as the basic building block (can be an arbitrary metallic shape) of the array that repeats itself infinitely obtained by the periodicity (d_x, d_y, d_z), as mentioned in Figure 1. As we have already seen, this efficient original modal analysis introduces a new excitation source decomposition in spectral-domain according to a finite (respectively infinite) periodic structure that removes the complexity of the problem under consideration to model and analyze the periodic structure when motifs are strongly or weakly coupled [9]. As in 1-D and 2-D cases, the electric source field for a 3-D grid $E(i, s, k)$ is decomposed as

$$E(id_x, sd_y, kd_z) = \frac{1}{\sqrt{N_x N_y N_z}} \sum_{p=-\frac{N_x}{2}, q=-\frac{N_y}{2}, t=-\frac{N_z}{2}}^{\frac{N_x}{2}-1, \frac{N_y}{2}-1, \frac{N_z}{2}-1} \tilde{E}_{\alpha_p, \beta_q, \psi_t} e^{j\alpha_p(id_x)} e^{j\beta_q(sd_y)} e^{j\psi_t(kd_z)} \tag{1}$$

with $\alpha_p = \frac{2\pi p}{N_x d_x}$, $\beta_q = \frac{2\pi q}{N_y d_y}$, and $\psi_t = \frac{2\pi t}{N_z d_z}$, where $-\frac{N_x}{2} \leq i \leq \frac{N_x}{2} - 1$, $-\frac{N_y}{2} \leq s \leq \frac{N_y}{2} - 1$, $-\frac{N_z}{2} \leq k \leq \frac{N_z}{2} - 1$.

Like in 1-D and 2-D cases, we rewrite the IFFT as follows:

$$\tilde{E}_{\alpha_p, \beta_q, \psi_t} = \frac{1}{\sqrt{N_x N_y N_z}} \sum_{i=-\frac{N_x}{2}, s=-\frac{N_y}{2}, k=-\frac{N_z}{2}}^{\frac{N_x}{2}-1, \frac{N_y}{2}-1, \frac{N_z}{2}-1} E(id_x, sd_y, kd_z) e^{j\alpha_p(id_x)} e^{j\beta_q(sd_y)} e^{j\psi_t(kd_z)} \tag{2}$$

Identically, the current distributions $J(i, s, k)$ and $\tilde{J}_{\alpha_p, \beta_q, \psi_t}$ are expressed in the same way as the electric source fields. This Floquet principle requires to define 3D periodic walls for Maxwell equations.

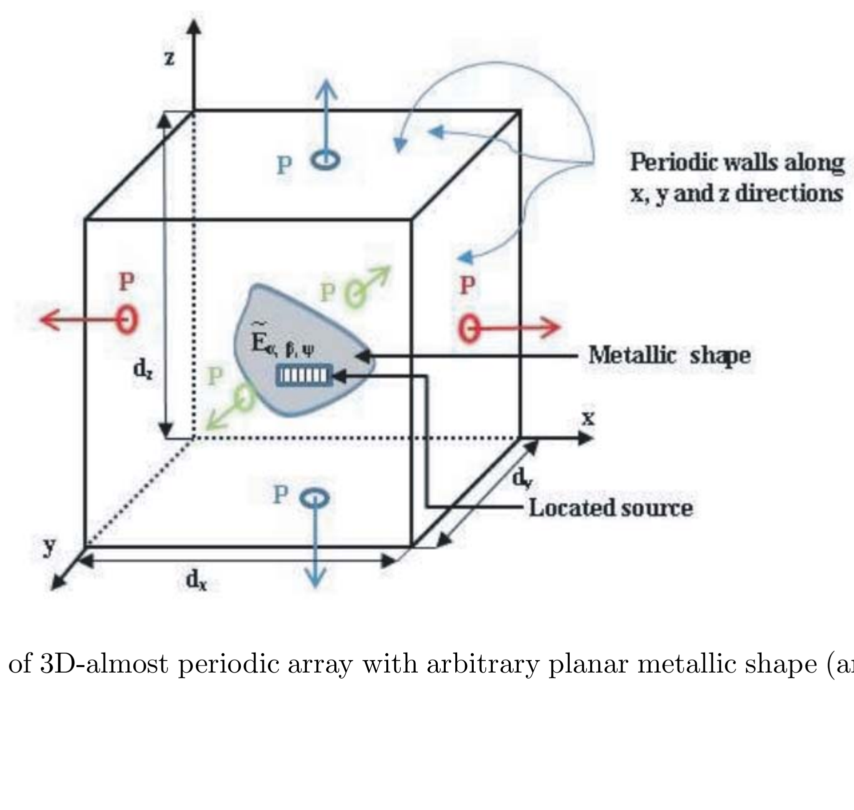

Figure 1. Unit cell of 3D-almost periodic array with arbitrary planar metallic shape (arbitrary motifs).

The obtained Floquet modal electric and magnetic fields, E and H, are governed by the differential form of Maxwell's relations. Chapter three of ARUN K. BHATTACHARYYA's book explains how to demonstrate the normalized Floquet modal functions [8]. Here, we will describe a new expression $|F_{mn,\alpha}^{TE,TM}\rangle$ of higher-order Floquet modes, which represents **3D periodic walls**,

$\diamondsuit\ TE_z - Fields:\ (i.e., E_z = 0)$

$$\vec{E_{mn}^{TE}} = \frac{E_{mn}^{TE}}{\sqrt{K_{xm,\alpha}^2 + K_{yn,\beta}^2}} \exp\left(-j(K_{xm,\alpha} + K_{yn,\beta}y + K_{zt,\psi}z)\right)\left(-K_{yn,\beta}\hat{x} + K_{xm,\alpha}\hat{y}\right) \tag{3}$$

$$\vec{H_{mn}^{TE}} = \frac{E_{mn}^{TE}\left(\frac{\mu\omega}{K_{zt,\psi}}\right)^{-1}}{\sqrt{K_{xm,\alpha}^2 + K_{yn,\beta}^2}} \exp\left(-j(K_{xm,\alpha} + K_{yn,\beta}y + K_{zt,\psi}z)\right)\left(K_{xm,\alpha}\hat{x} + K_{yn,\beta}\hat{y} - \frac{K_{xm,\alpha}^2 + K_{yn,\beta}^2}{K_{zt,\psi}}\hat{z}\right) \tag{4}$$

$\diamondsuit\ TM_z - Fields:\ (i.e., H_z = 0)$

$$\vec{E_{mn}^{TM}} = \frac{E_{mn}^{TM}\left(\frac{K_{zt,\psi}}{\epsilon\omega}\right)}{\eta\sqrt{K_{xm,\alpha}^2 + K_{yn,\beta}^2}} \exp(-j(K_{xm,\alpha} + K_{yn,\beta}y + K_{zt,\psi}z))\left(K_{xm,\alpha}\hat{x} + K_{yn,\beta}\hat{y} - \frac{K_{xm,\alpha}^2 + K_{yn,\beta}^2}{K_{zt,\psi,(n=0,m=0)}}\hat{z}\right) \tag{5}$$

$$\vec{H_{mn}^{TM}} = \frac{E_{mn}^{TM}}{\eta\sqrt{K_{xm,\alpha}^2 + K_{yn,\beta}^2}} \exp\left(-j\left(K_{xm,\alpha} + K_{yn,\beta}y + K_{zt,\psi}z\right)\right)\left(-K_{yn,\beta}\hat{x} + K_{xm,\alpha}\hat{y}\right) \tag{6}$$

Similarly, a particular fundamental Floquet mode is given when $m = 0,\ n = 0$ and $t = 0$. The wave number along z is obtained as: $\gamma_{z,mn} = k_{zt,\psi} = \sqrt{K_0^2 - K_{xm,\alpha}^2 - K_{yn,\beta}^2}$. For a general periodic grid structure, the wavenumbers constants are given by: $K_{xm,\alpha} = \frac{2m\pi}{d_x} + \alpha$, $K_{yn,\beta} = \frac{2n\pi}{d_y} + \beta$, and $K_{zt,\psi} = \frac{2t\pi}{d_z} + \psi$ [11–14]. Referring to our previous work [4, 5], a new mathematical mutual coupling formula is given, which is absolutely different from the other literature examples [10]. To understand the idea better, it is sufficient to follow our transformation process detailed in [4, 5]. In this paper, we prefer to generalize the mutual coupling expression, considering the 3D almost periodic grid. Based on both 1D and 2D almost periodic arrays, it is possible to construct a new Fourier matrix representation that follows the 3D periodic grid [1] (Like the examples shown in [5], and it may be important to obtain the Matlab source code that corresponds to a 3D Fourier matrix). The Fourier matrix TF has elements $TF_{3D} = w_x^{ip} w_y^{sq} w_z^{kt}$, and w is a complex nth root of unity: $w_x = e^{-j\frac{2\pi d_x}{L_x}}$, $w_y = e^{-j\frac{2\pi d_y}{L_y}}$, and $w_z = e^{-j\frac{2\pi d_z}{L_z}}$. The 3D almost periodic structure's Fourier matrix form keeps the same properties of the 1D and 2D Fourier matrices explained in [4, 5]. Generally, the use of every proper Floquet wave vector (wave state) obtained in the spectral domain comes to define a diagonal operator successfully which contains all possible modal input impedances $\tilde{z}_{\alpha_p,\beta_q,\psi_t}$. Finally, the mainly coupling expression is given below:

$$[Z_{i,j}] = TF_{3D}^{-1}\tilde{z}_{\alpha_p,\beta_q,\psi_t}TF_{3D} \tag{7}$$

The consequence of this latter transformation is entirely to deduce the mutual admittance and scattering parameters between periodic elements in an array environment which may define as:

$$[Y_{i,j}] = TF_{3D}^{-1}\tilde{y}_{\alpha_p,\beta_q,\psi_t}TF_{3D} \tag{8}$$

$$[S_{i,j}] = TF_{3D}^{-1}\tilde{s}_{\alpha_p,\beta_q,\psi_t}TF_{3D} \tag{9}$$

To produce the interaction in the spatial domain, a novel matrix representation is proposed where a division by submatrix is considered: The interaction established between the elements belonging to the same plane is given by submatrix disposed diagonally along the hole spatial mutual impedance matrix. The coupling between elements placed in different planes is given by an upper triangle of the hole spatial coupling matrix (also called interaction submatrices). According to the reciprocity theorem, the lower triangular part of the spatial coupling matrix is obtained identically to the upper values ($Z_{ij} = Z_{ji}$).

3. PROBLEM FORMULATION: UNIT CELL OF 3D ALMOST PERIODIC ANTENNA ARRAY

This part introduces a distinct formulation to investigate the theoretical improvement appreciably for resolving the planar structure in 3D periodic arrays: The modal formulation restraint to modelize the unit structure that is designed to support the dependence on Floquet modes. In more details, we assum a novel schema of an equivalent circuit proposed to examine the latter 3D-unit structure, as shown in Figure 2. Next, this circuit leads to an equations system, given as:

$$\begin{pmatrix} \tilde{E}_{1,\psi} \\ \tilde{J}_{1,\psi} \end{pmatrix} = \begin{pmatrix} & \hat{C_{i,j}} & \end{pmatrix} \begin{pmatrix} \tilde{E}_{2,\alpha\beta} \\ \tilde{J}_{2,\alpha\beta} \end{pmatrix} \tag{10}$$

$$\begin{cases} \tilde{E}_{2,\alpha\beta} = \tilde{E}'_{2,\alpha\beta} = E_{0,\alpha\beta} + E_{e,\alpha\beta} \\ \tilde{J}_{2,\alpha\beta} = \tilde{J}_{0,\alpha\beta} + \tilde{J}'_{2,\alpha\beta} \quad (\tilde{J}_{0,\alpha\beta} = \tilde{J}_{e,\alpha\beta}) \end{cases} \tag{11}$$

$$\begin{pmatrix} \tilde{E}'_{2,\alpha\beta} \\ \tilde{J}'_{2,\alpha\beta} \end{pmatrix} = \begin{pmatrix} & \hat{C'_{i,j}} & \end{pmatrix} \begin{pmatrix} \tilde{E'_{3,\psi}} \\ \tilde{J'_{3,\psi}} \end{pmatrix} \tag{12}$$

Following the Floquet theorem (and the passage from each element to the other along the z-axis), we can note

$$\begin{cases} \tilde{E}_{1,\psi} = \tilde{E}'_{3,\psi} e^{j\psi d_z} \\ \tilde{J}_{1,\psi} = \tilde{J}'_{3,\psi} e^{j\psi d_z} \end{cases} \tag{13}$$

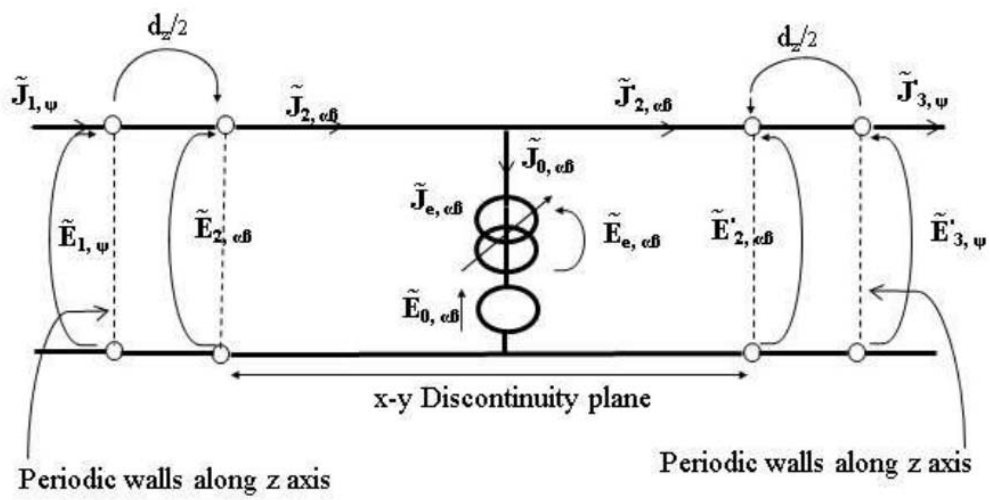

Figure 2. Equivalent circuit of the unit cell in 3D almost periodic array.

How $\hat{C_{i,j}} = \hat{C'_{i,j}}$ matrix is calculated.

Normally, the elements of $\hat{C_{i,j}}$ operator representation can be defined as:

$$\hat{C_{1,1}} = \sum_{mn} |f_{mn}\rangle \cosh\left(\gamma_{mn}^{TE,TM}\frac{d_z}{2}\right)\langle f_{mn}|$$

$$\hat{C_{1,2}} = \sum_{mn} |f_{mn}\rangle \sinh\left(\gamma_{mn}^{TE,TM}\frac{d_z}{2} z_{mn,TE,TM}^{upper,down}\right)\langle f_{mn}|$$

$$\hat{C_{2,1}} = \sum_{mn} |f_{mn}\rangle \sinh\left(\gamma_{mn}^{TE,TM}\frac{d_z}{2} y_{mn,TE,TM}^{upper,down}\right)\langle f_{mn}|$$

$$\hat{C_{2,2}} = \sum_{mn} |f_{mn}\rangle \cosh\left(\gamma_{mn}^{TE,TM}\frac{d_z}{2}\right)\langle f_{mn}|$$

Owing to the generalized equivalent approach, the matrix representation employing the Kirchhoff laws can be expressed as below:

$$\begin{pmatrix} \tilde{J}_{0,\alpha\beta} \\ \tilde{E}_{e,\alpha\beta} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \hat{G_{pq,st}} \end{pmatrix} \begin{pmatrix} \tilde{E}_{0,\alpha\beta} \\ \tilde{J}_{e,\alpha\beta} \end{pmatrix} \tag{14}$$

As a result, through analogical reasoning from the matrix chain combined with the Floquet condition (in terms of current and electric fields) to Galerkin matrix representation, by considering the Kirchhoff laws, it drives to obtain a novel operator expression $\hat{G_{pq,st}}$ written as:

$$\begin{aligned} \hat{G_{pq,st}} = & (1 - \hat{C_{1,1}})^{-1}\hat{C_{1,2}} \Big[1 + \hat{C'_{1,1}} \left(\hat{C_{1,1}}(1-\hat{C_{1,1}})^{-1}\hat{C_{1,2}} + \hat{C_{1,2}}\right) e^{j\psi d_z} \\ & + \hat{C'_{2,1}} \left(\hat{C_{2,1}}(1-\hat{C_{1,1}})^{-1}\hat{C_{1,2}} + \hat{C_{2,2}})\right) e^{j\psi d_z}\Big] \end{aligned} \tag{15}$$

A similar theoretical development is shown in [4, 5], as done for the 3D-unit cell to express the Floquet modal input impedance:

$$\tilde{Z}_{in,\alpha\beta\psi} = \left({}^{t}\left[\tilde{A}_{\alpha\beta\psi}\right]\left[\hat{G_{pq,st,\alpha\beta\psi}}\right]^{-1}\tilde{A}_{\alpha\beta\psi}\right)^{-1} \tag{16}$$

where $[\tilde{A}_{\alpha\beta\psi}] = [\langle f|g_{pq,st,\alpha\beta\psi}\rangle] = [\langle \frac{1}{\delta}|g_{pq,st,\alpha\beta\psi}\rangle]$. We should bear in mind that this latest Floquet modal input impedance relation is used in the previous section to calculate the interaction between cells, as given in Equation (7). Getting back to Kirchhoff's laws (current and tension), we can deduce:

$$\begin{cases} \tilde{J}_{0,\alpha\beta\psi} = \tilde{J}_{e,\alpha\beta\psi} \\ \tilde{E}_{e,\alpha\beta\psi} = -\tilde{E}_{0,\alpha\beta\psi} + \hat{G}\tilde{J}_{0,\alpha\beta\psi} \end{cases} \tag{17}$$

Then, it is possible to apply the superposition theorem (Fourier transform) with a triply periodic grid by adding the z-direction and its corresponding ψ Floquet dependence.

4. STORAGE MEMORY AND TIME COMPUTATION

In this section, we appreciate the Floquet modal formulation compared to the old spatial formulation in requirement memory cost and reducing computational time: As we have always explained in our past work [4, 5], we show how Floquet analysis is more important in terms of the total needed time and memory resources. Physically, we obtain the same observation concerning the operation number, storage memory, and time-consuming, as given in [4, 5, 15–17], when the third direction will be considered. In more details, we must take into account that the addition of z-direction permits modification of the impedance matrix dimension in both spatial and spectral formulations. However, the growing of the number N_z according to the third axis makes the electromagnetic calculation heavier in terms of memory resource and time execution than 1-D and 2-D almost periodic structures. Table 1 shows how the memory space behavior, the number of operations, and execution time increase versus the sizes of 1D, 2D, and 3D quasi-periodic arrangements, using the Floquet's modal formulation.

Table 1. A comparison table to compute the storage memory and the execution time consuming based on the Floquet modal analysis.

Floquet modal analysis			
Configurations	Memory size (Bytes)	Number of operation	T_{MoM} (total needed time) (second)
1D ($N_x = 3$)	153600	64000	1296.30
2D ($N_x = N_y = 3$)	460800	192000	3888.76
3D ($N_x = N_y = N_z = 3$)	1382400	576000	11666.30

5. CONCLUSION

In this paper, we show an innovative modal Floquet approach to investigate the mutual coupling in finite and infinite 3-D almost periodic sources arrays. Usually, the modal representation leads to elimination of the electromagnetic complexity of the considered problem. The selected formalism of Floquet analysis combined with the equivalent circuit method reduces the electromagnetic calculation of the whole 3D almost periodic structure on one unit cell. As a result, an easy relation, based on the Fourier transform and its inverse permits a manageable calculation of the coupling terms (Z, Y, and S). The main interest of this novel modal analysis is reducing the computation time and memory storages that relatively depend on the cube and square of the number of array elements. This work is a fundamental starting point for coming research of different periodic and aperiodic 3-D arrangements with different sources amplitudes.

APPENDIX A. TABLE TO RESUME HOW TO COMPUTE STORAGE MEMORY AND TIME CONSUMING

Table A1. Table of comparison between Floquet modal formulation and spatial formulation in terms of the memory consumption and the execution time for a 1D, 2D, and 3D finite almost periodic configurations.

New Floquet modal formulation			
Configurations	Memory size [15]	Number of operation [16]	T_{MoM} (total needed time) [5]
1D	$N_x(P)^2$	$N_x\frac{P^3}{3}$	$MN_x(P)^2(T_s+T_{op})$ $+\frac{2}{3}N_x(P)^3T_{op}+\delta$
2D	$(N_xN_y)(P)^2$	$(N_xN_y)\frac{P^3}{3}$	$M(N_xN_y)(P)^2(T_s+T_{op})$ $\frac{2}{3}(N_xN_y)(P)^3T_{op}+\delta$
3D	$(N_xN_yN_z)(P)^2$	$(N_xN_yN_z)\frac{P^3}{3}$	$M(N_xN_yN_z)(P)^2(T_s+T_{op})$ $+\frac{2}{3}(N_xN_yN_z)(P)^3T_{op}+\delta$
Old spatial formulation			
Configurations	Memory size [15]	Number of operation [16]	T_{MoM} (total needed time) [5]
1D	$(N_xP)^2$	$\frac{(N_xP)^3}{3}$	$M(N_xP)^2(T_s+T_{op})$ $+\frac{2}{3}(N_xP)^3T_{op}+\delta$
2D	$(N_xN_yP)^2$	$\frac{(N_xN_yP)^3}{3}$	$M(N_xN_yP)^2(T_s+T_{op})$ $\frac{2}{3}(N_xN_yP)^3T_{op}+\delta$
3D (multidimensional matrix)	$N_z(N_xN_yP)^2$	$N_z\frac{(N_xN_yP)^3}{3}$	$MN_z(N_xN_yP)^2(T_s+T_{op})$ $+\frac{2}{3}N_z(N_xN_yP)^3T_{op}+\delta$
3D (simple matrix)	$(N_xN_yN_zP)^2$	$\frac{(N_zN_xN_yP)^3}{3}$	$M(N_xN_yN_zP)^2(T_s+T_{op})$ $+\frac{2}{3}(N_xN_yN_zP)^3T_{op}+\delta$

N_x, N_xN_y, and $N_xN_yN_z$ represent the possible phases shift number of Floquet states that correspond respectively to α_p, (α_p,β_q), and $(\alpha_p,\beta_q,\psi_t)$ (according to a finite 1D, 2D, and 3D almost periodic arrays). N_x, N_xN_y, and $N_xN_yN_z$ also represent the number of arrays elements in 1D, 2D, and 3D periodic arrays. M is the total guide's modes number, and P is the total test function number (that describe the metal part).

The other parameters are defined in detail in [4, 5].

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