Progress In Electromagnetics Research B, Vol. 91, 9–17, 2021

Diffraction Radiation Generated by a Density-Modulated Electron Beam Flying over the Periodic Boundary of the Medium Section. II. Impact of True Eigen Waves

Yuriy Sirenko[1, 2], Seil Sautbekov[3, *], Nataliya Yashina[1], and Kostyantyn Sirenko[1]

Abstract—This paper is the continuation and development of the discussion started in our previous work with the same title. For the first time, eigen waves of the plane boundary separating vacuum and an artificial plasma-like medium are considered in reasonably substantiated way and in a sufficiently extensive and profound volume. The possibility of extending the results obtained for a plane boundary to the case of a weakly profiled periodically uneven boundary is shown. This paper demonstrates the potential and urge to use the analytical results in the studies of the resonant transformation of the field of a plane, density modulated electron beam flying over a periodically uneven boundary of a natural or artificial medium in the field of bulk outgoing waves.

1. INTRODUCTION

In [1] it was shown that calculation of the key energy characteristics

$$W_n^+(k) = |R_n|^2 \frac{\mathrm{Re}\Gamma_n^+}{|\Gamma_0^+|}, \quad W_n^-(k) = \varepsilon^{-1}(k)\,|T_n|^2 \frac{\mathrm{Re}\Gamma_n^-}{|\Gamma_0^+|} \tag{1}$$

of diffraction radiation (Vavilov-Cherenkov radiation [2] or Smith-Purcell radiation [3]), which is generated by a plane density-modulated electron beam flying over a periodically uneven boundary $\Sigma_x^{\varepsilon,\mu}$ (see Fig. 1 in [1]) separating vacuum ($\varepsilon = \mu = 1.0$) and a dispersive medium with material parameters $\varepsilon(k)$, $\mu(k)$, is reduced to solving the boundary value problem

$$\left\{\begin{array}{l} \left[\partial_y^2 + \partial_z^2 + \varepsilon(g,k)\,\mu(g,k)\,k^2\right] U(g,k) = 0;\ g = \{y,z\} \in \Omega_{\mathrm{int}} \\ \mathbf{E}_{\mathrm{tg}}(q,k),\ \mathbf{H}_{\mathrm{tg}}(q,k)\ \text{are continuous when crossing}\ \Sigma^{\varepsilon,\mu} = \Sigma_x^{\varepsilon,\mu} \times (-\infty < x < \infty) \\ \text{and virtual boundaries}\ y = 0,\ y = -h;\ q = \{x,y,z\} \\ U\{\partial_z U\}(y,l,k) = \exp(2\pi i\zeta)\,U\{\partial_z U\}(y,0,k)\ \text{for}\ -h \le y \le 0 \end{array}\right., \tag{2a}$$

$$\begin{aligned} U(g,k) &= V_0(g,k) + U^+(g,k) = V_0(g,k) + \sum_{n=-\infty}^{\infty} U_n^+(g,k) \\ &= \exp\left(-i\Gamma_0^+ y\right)\varphi_0(z) + \sum_{n=-\infty}^{\infty} R_n(k)\exp\left(i\Gamma_n^+ y\right)\varphi_n(z); \quad g \in \bar{\mathrm{A}}, \end{aligned} \tag{2b}$$

$$U(g,k) = U^-(g,k) = \sum_{n=-\infty}^{\infty} U_n^-(g,k) = \sum_{n=-\infty}^{\infty} T_n(k)\exp\left(-i\Gamma_n^-(y+h)\right)\varphi_n(z); \quad g \in \bar{B}. \tag{2c}$$

Received 1 November 2020, Accepted 2 February 2021, Scheduled 6 February 2021
* Corresponding author: Seil S. Sautbekov (sautbek@mail.ru).
[1] Department of Diffraction Theory and Diffraction Electronics, O.Ya. Usikov Institute for Radio Physics and Electronics, National Academy of Sciences of Ukraine, 12 Academician Proskura St., Kharkiv 61085, Ukraine. [2] Department of Applied Mathematics, V.M. Karazin Kharkiv National University, 4 Svobody Sq., Kharkiv 61022, Ukraine. [3] Department of Physics and Technology, al-Farabi Kazakh National University, 71 al-Farabi Ave., Almaty 050040, Republic of Kazakhstan.

Here, $U(g,k)$ is H_x-component of complete H-polarized ($E_x = H_y = H_z = 0$, $\partial_x = 0$) electromagnetic field $\{\mathbf{E}(g,k),\mathbf{H}(g,k)\}$, $g = \{y,z\}$ in the 'beam-boundary' system; $V_0(g,k)$ is H_x-component of a beam own field; $W_n^+(k)$ and $W_n^-(k)$ are functions determining the efficiency of diffraction radiation at spatial harmonics $U_n^+(g,k)$ and $U_n^-(g,k)$, which are outgoing upward (in half-space occupied by vacuum) and downward (in half-space occupied by a dispersive medium) from the boundary, respectively; $\Gamma_n^+ = \sqrt{k^2-\Phi_n^2}$, $\mathrm{Re}\Gamma_n^+ \geq 0$, $\mathrm{Im}\Gamma_n^+ \geq 0$ and $\Gamma_n^- = \sqrt{k^2\varepsilon(k)\mu(k)-\Phi_n^2}$, $\varepsilon^{-1}(k)\mathrm{Re}\Gamma_n^- \geq 0$, $\mathrm{Im}\Gamma_n^- \geq 0$ are vertical propagation constants of these harmonics; $\Phi_n = (n+\zeta)2\pi/l$, $\zeta 2\pi/l = \Phi_0 = k/\beta$ (with this value of Φ_0, $\mathrm{Re}\Gamma_0^+ = 0$ and $\mathrm{Im}\Gamma_0^+ > 0$, $V_0(g,k)$ is an inhomogeneous plane wave component); k and $0 < \beta < 1$ are modulation frequency and relative beam velocity; $k = 2\pi/\lambda$ is the frequency parameter, which is set by the modulation frequency; λ is the wavelength of the radiation field in free space, l and h are the period and height of the boundary $\Sigma_x^{\varepsilon,\mu} = \{g:\ y = f(z),\ -h \leq f(z) \leq 0\}$. A more detailed description of the problem (2) is given in [1]. The choice of branches of the two-valued functions $\Gamma_n^{\pm}(k,\zeta)$ has been made and justified ibid.

In present work, we focus on the normal (or eigen) modes of the periodic interface between the media [1, 4, 5] (non-trivial solutions of the problem (2) for $V_0(g,k) \equiv 0$), which are supposed to be responsible for the anomalously high levels of coherent diffraction radiation generated by a density-modulated electron beam moving over a periodic boundary separating an ordinary and artificial medium with a specific frequency dispersion of permittivity and permeability. It was observed in a number of computational experiments.

2. PLANE BOUNDARY. VAVILOV-CHERENKOV RADIATION

Obviously, in the case of a plane (non-transforming) boundary $y = f(z) \equiv 0$, the homogeneous problem (2) (problem (2) with $V_0(g,k) \equiv 0$) is reduced to an infinite set of independent homogeneous systems of linear algebraic equations

$$\left\{ \begin{array}{l} R_n = T_n \\ R_n \Gamma_n^+(\zeta) = -T_n \Gamma_n^-(\zeta)\,\varepsilon^{-1}(k) \end{array} \right. ; \quad n = 0, \pm 1, \pm 2, \ldots \tag{3}$$

with respect to unknown complex amplitudes R_n and T_n.

The equations of systems in Eq. (3) are obtained by 'sewing', in the plane $y = 0$, the tangential components of the field $\{\mathbf{E}(g,k),\mathbf{H}(g,k)\}$, it is H_x-component equal to $U(g,k)$, and E_z-component, which is connected with $U(g,k)$ by the relation (2) from [1].

Non-trivial solutions of the systems (3) for each fixed value $k > 0$ determine an infinite set of practically identical eigen waves of a 'periodic' structure:

$$U\left(g,\bar{\zeta}_n^{\pm}\right) = \left\{ \begin{array}{ll} U_n^+\left(g,\bar{\zeta}_n^{\pm}\right) = l^{-1/2} R_n \exp\left[i\left(\Gamma_n^+\left(\bar{\zeta}_n^{\pm}\right)y + \Phi_n\left(\bar{\zeta}_n^{\pm}\right)z\right)\right]; & y > 0 \\ U_n^-\left(g,\bar{\zeta}_n^{\pm}\right) = l^{-1/2} T_n \exp\left[i\left(-\Gamma_n^-\left(\bar{\zeta}_n^{\pm}\right)y + \Phi_n\left(\bar{\zeta}_n^{\pm}\right)z\right)\right]; & y < 0 \end{array} \right. ;$$

$$\Gamma_n^+\left(\bar{\zeta}_n^{\pm}\right) = \sqrt{k^2 - \Phi_n^2\left(\bar{\zeta}_n^{\pm}\right)}, \quad \Gamma_n^-\left(\bar{\zeta}_n^{\pm}\right) = \sqrt{k^2\varepsilon(k)\,\mu(k) - \Phi_n^2\left(\bar{\zeta}_n^{\pm}\right)}, \tag{4}$$

$$\Phi_n\left(\bar{\zeta}_n^{\pm}\right) = \left(n + \bar{\zeta}_n^{\pm}\right)2\pi/l = \pm k\sqrt{\frac{\varepsilon(k)\left(\mu(k) - \varepsilon(k)\right)}{1 - \varepsilon^2(k)}}, \quad n = 0, \pm 1, \pm 2, \ldots$$

Here,

$$\bar{\zeta}_n^{\pm} = -n \pm k\frac{l}{2\pi}\sqrt{\frac{\varepsilon(k)\left(\mu(k) - \varepsilon(k)\right)}{1 - \varepsilon^2(k)}} \tag{5}$$

are propagation constants $\bar{\zeta}$ of the eigen waves in Eq. (4). The nature of these waves determines both the specific value of complex, in the general case, $\bar{\zeta}$, and its position on the four-sheet surface F_n of a pair of two-valued functions $\{\Gamma_n^+(\zeta), \Gamma_n^-(\zeta)\}$ (see details in [4]). On the real axis $\mathrm{Re}\zeta$ of the first, physical sheet of the surface F_n, values $\mathrm{Re}\Gamma_n^+(\zeta)$, $\mathrm{Im}\Gamma_n^+(\zeta)$, $\varepsilon^{-1}(k)\mathrm{Re}\Gamma_n^-(\zeta)$ and $\mathrm{Im}\Gamma_n^-(\zeta)$ are non-negative [1, 4]. Analyzing the physics of diffraction radiation processes, it is extremely important to know whether the propagation constant $\bar{\zeta}$, corresponding to eigen wave of a specified boundary $\Sigma_x^{\varepsilon,\mu}$ of a medium with specified constitutive parameters $\varepsilon(k)$ and $\mu(k)$, is located on the first, physical sheet of the infinite-sheeted Riemann surface F, uniting all four-sheet surfaces F_n (see Section 3 in [1]). Here we will only note that in solving this problem, the equality

$$\Gamma_n^+\left(\bar{\zeta}_n^{\pm}\right) = -\Gamma_n^-\left(\bar{\zeta}_n^{\pm}\right)\varepsilon^{-1}(k), \tag{6}$$

will be useful for us. It follows from Eq. (3) and establishes a connection between the signs of real and imaginary parts of complex, in the general case, propagation constants $\Gamma_n^+(\bar{\zeta}_n^{\pm})$ and $\Gamma_n^-(\bar{\zeta}_n^{\pm})$ of the spatial harmonics $U_n^+(g,\bar{\zeta}_n^{\pm})$ and $U_n^-(g,\bar{\zeta}_n^{\pm})$.

Suppose now that in the problem (2), the constitutive parameters of the medium filling the half-space $y < f(z)$ are given by the relations

$$\varepsilon(k) = 1 - k_\varepsilon^2/k^2 \quad \text{and} \quad \mu(k) = 1 - k_\mu^2/k^2. \tag{7}$$

Such a medium can be called 'plasma-like medium', and the real valued numbers $k_\varepsilon > 0$ and $k_\mu > 0$ are its characteristic frequencies.

From Eqs. (4) and (5), for medium of this kind, we have

$$\Phi_n\left(\bar{\zeta}_n^{\pm}\right) = \pm\frac{k}{k_\varepsilon}\sqrt{\frac{\left(k_\varepsilon^2-k^2\right)\left(k_\varepsilon^2-k_\mu^2\right)}{k_\varepsilon^2-2k^2}}, \tag{8}$$

and for $k_\varepsilon = k_\mu$, $\Phi_n(\bar{\zeta}_n^{\pm}) = 0$, if only $k \neq \sqrt{0.5}k_\varepsilon$; and $\Phi_n(\bar{\zeta}_n^{\pm}) = \pm 0.5k_\varepsilon$, in the case of $k = \sqrt{0.5}k_\varepsilon$. And in the first and second cases $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^{\pm}) = 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^{\pm}) = 0$. It follows from Eq. (6) that the values $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^{\pm})$ and $\varepsilon^{-1}(k)\mathrm{Re}\Gamma_n^-(\bar{\zeta}_n^{\pm})$ differ in signs, and therefore: (a) the propagation constant $\bar{\zeta}_n^{\pm}$ cannot belong to a physical sheet of the surface F_n; (b) the real eigen wave $U(g,\bar{\zeta}_n^{\pm})$ only for convenience may be called 'something like leaky wave'. Its partial components $U_n^+(g,\bar{\zeta}_n^{\pm})$ and $U_n^-(g,\bar{\zeta}_n^{\pm})$ above and below the interface transfer energy in the same direction; if one of these waves arrives onto the boundary $y = 0$, then the second one leaves this boundary. The property (b) of the eigen wave $U(g,\bar{\zeta}_n^{\pm})$, projected on the first sheet of F_n, allows to define sets of parameters at which the boundary is completely (without reflection) transparent for a homogeneous plane wave arriving onto it from above or below [4, 6, 7].

If $k_\varepsilon > k_\mu$ then, as it follows from Eq. (8):

- For $0 < k < \sqrt{0.5}k_\varepsilon$, $\varepsilon(k) < 0$ and $\Phi_n(\bar{\zeta}_n^{\pm}) = \pm\frac{k}{k_\varepsilon}\sqrt{\frac{(k_\varepsilon^2-k^2)(k_\varepsilon^2-k_\mu^2)}{k_\varepsilon^2-2k^2}}$. In the range $k < \frac{k_\varepsilon k_\mu}{\sqrt{k_\mu^2+k_\varepsilon^2}}$, $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^{\pm}) = 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^{\pm}) = 0$, which means that the real eigen wave $U(g,\bar{\zeta}_n^{\pm})$ is 'something like leaky wave'. In the range $k > \frac{k_\varepsilon k_\mu}{\sqrt{k_\mu^2+k_\varepsilon^2}}$, $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^{\pm}) = 0$ and $\mathrm{Re}\Gamma_n^-(\bar{\zeta}_n^{\pm}) = 0$, and magnitudes $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^{\pm})$ and $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^{\pm})$ have the same sign. This means that the propagation constant $\bar{\zeta}_n^{\pm}$ can fall both on a physical sheet of the surface F_n (here $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^{\pm}) > 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^{\pm}) > 0$, and then $U(g,\bar{\zeta}_n^{\pm})$ is the real surface eigen wave (or 'true eigen wave'), which propagates near and along the media boundary without attenuation), and on one of non-physical sheets F_n (here $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^{\pm}) < 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^{\pm}) < 0$, and then $U(g,\bar{\zeta}_n^{\pm})$ is the real eigen wave, whose partial components $U_n^+(g,\bar{\zeta}_n^{\pm})$ and $U_n^-(g,\bar{\zeta}_n^{\pm})$ grow exponentially with the distance from the interface).
  In the range $k < k_\varepsilon$, $\varepsilon(k) < 0$ holds, and partial components $U_n^+(g,\bar{\zeta}_n^{\pm})$, $U_n^-(g,\bar{\zeta}_n^{\pm})$ of the true eigen waves $U(g,\bar{\zeta}_n^{\pm})$ transfer the energy in opposite along the axis z directions when the directionality $\Phi_n(\bar{\zeta}_n^{\pm})\mathbf{z}$ of their phase velocities coincides (see relation (7) in [1]). Let's call these waves 'unusual true eigen waves' in contrast to the usual 'true eigen waves' $U(g,\bar{\zeta}_n^{\pm})$, whose region of existence is limited by the frequencies $k > k_\varepsilon$ for which $\varepsilon(k) > 0$.
- For $k \to \sqrt{0.5}k_\varepsilon$ (left limit), $\Phi_n(\bar{\zeta}_n^{\pm}) \to \pm\infty$ and $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^{\pm}) = 0$, $\mathrm{Re}\Gamma_n^-(\bar{\zeta}_n^{\pm}) = 0$, $|\mathrm{Im}\Gamma_n^{\pm}(\bar{\zeta}_n^{\pm})| \to \infty$. The propagation constant $\bar{\zeta}_n^{\pm}$ can be located on both physical and non-physical sheets of the surface F_n. The exotic characteristics of the respective real eigen wave $U(g,\bar{\zeta}_n^{\pm})$ hardly deserve to be discussed, but we note that for $\bar{\zeta}_n^{\pm}$ from the first sheet of F_n, $U(g,\bar{\zeta}_n^{\pm})$ is a wave, whose field occupies practically zero volume (plane $y = 0$), and whose phase velocity is practically zero.
- For $\sqrt{0.5}k_\varepsilon < k < k_\varepsilon$, $\Phi_n(\bar{\zeta}_n^{\pm}) = \pm i\frac{k}{k_\varepsilon}\sqrt{\frac{(k_\varepsilon^2-k^2)(k_\varepsilon^2-k_\mu^2)}{2k^2-k_\varepsilon^2}}$ and $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^{\pm}) = 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^{\pm}) = 0$, which means that the imaginary eigen wave $U(g,\bar{\zeta}_n^{\pm})$ is 'something like leaky wave', but now

this property is not connected with the condition of complete transparency of the boundary $y = 0$ for homogeneous plane waves which was discussed above.

- For $k = k_\varepsilon$, $\varepsilon(k) = 0$ and $\Phi_n(\bar{\zeta}_n^\pm) = 0$, $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) = 0$, $\Gamma_n^-(\bar{\zeta}_n^\pm) = 0$. Exotic configuration of the real eigen wave $U(g, \bar{\zeta}_n^\pm)$ with the propagation constant $\bar{\zeta}_n^\pm$ from the first sheet of the surface F_n (here $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^\pm) = k$) and from another sheet (here $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^\pm) = -k$) can be easily depicted using the representation (4).
- For $k > k_\varepsilon$, $\varepsilon(k) > 0$ and $\Phi_n(\bar{\zeta}_n^\pm) = \pm\frac{k}{k_\varepsilon}\sqrt{\frac{(k^2-k_\varepsilon^2)(k_\varepsilon^2-k_\mu^2)}{2k^2-k_\varepsilon^2}}$. $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) = 0$ and $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^\pm) = 0$, and the signs $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^\pm)$ and $\varepsilon^{-1}(k)\mathrm{Re}\Gamma_n^-(\bar{\zeta}_n^\pm)$ are opposite: the propagation constant $\bar{\zeta}_n^\pm$ cannot belong to a physical sheet of the surface F_n. Real eigen wave $U(g, \bar{\zeta}_n^\pm)$ is 'something like leaky wave', which means that its partial components $U_n^+(g, \bar{\zeta}_n^\pm)$ and $U_n^-(g, \bar{\zeta}_n^\pm)$ transfer energy in the same direction above and below the boundary. Namely, if one of these waves arrives onto the boundary $y = 0$, then the second one leaves this boundary.

If $k_\varepsilon < k_\mu$ then, as it follows from Eq. (8):

- For $0 < k < \sqrt{0.5}k_\varepsilon$, $\varepsilon(k) < 0$ and $\Phi_n(\bar{\zeta}_n^\pm) = \pm i\frac{k}{k_\varepsilon}\sqrt{\frac{(k_\varepsilon^2-k^2)(k_\mu^2-k_\varepsilon^2)}{k_\varepsilon^2-2k^2}}$, $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) = 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^\pm) = 0$, which means that the imaginary eigen wave $U(g, \bar{\zeta}_n^\pm)$ is 'something like leaky wave', but now this property has nothing to do with the condition of complete transparency of the boundary $y = 0$ for homogeneous plane waves which was discussed above.
- For $k \to \sqrt{0.5}k_\varepsilon$ (left limit), $\Phi_n(\bar{\zeta}_n^\pm) \to \pm i\infty$ and $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) = 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^\pm) = 0$, $|\mathrm{Re}\Gamma_n^\pm(\bar{\zeta}_n^\pm)| \to \infty$. It is possible to somehow imagine an eigen wave existing here only with the corresponding limiting passage and considering the wave $U(g, \bar{\zeta}_n^\pm)$ from the previous point.
- For $\sqrt{0.5}k_\varepsilon < k < k_\varepsilon$, $\varepsilon(k) < 0$ and $\Phi_n(\bar{\zeta}_n^\pm) = \pm\frac{k}{k_\varepsilon}\sqrt{\frac{(k_\varepsilon^2-k^2)(k_\mu^2-k_\varepsilon^2)}{2k^2-k_\varepsilon^2}}$. In the range $k < \frac{k_\varepsilon k_\mu}{\sqrt{k_\mu^2+k_\varepsilon^2}}$, $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^\pm) = 0$, $\mathrm{Re}\Gamma_n^-(\bar{\zeta}_n^\pm) = 0$, $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm)$ and $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^\pm)$ have the same sign. This means that the propagation constant $\bar{\zeta}_n^\pm$ can lie both on a physical sheet of the surface F_n (here $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) > 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^\pm) > 0$, and $U(g, \bar{\zeta}_n^\pm)$ is the real surface eigen wave (or 'true eigen wave', more precisely, it is 'unusual true eigen wave'), which propagates near and along the medium boundary without attenuation), and on one of non-physical sheets F_n (here $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) < 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^\pm) < 0$, and $U(g, \bar{\zeta}_n^\pm)$ is the real eigen wave, whose partial components $U_n^+(g, \bar{\zeta}_n^\pm)$ and $U_n^-(g, \bar{\zeta}_n^\pm)$ grow exponentially with the distance from the boundary). In the range $k > \frac{k_\varepsilon k_\mu}{\sqrt{k_\mu^2+k_\varepsilon^2}}$, $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) = 0$ and $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^\pm) = 0$, and $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^\pm)$ and $\varepsilon^{-1}(k)\mathrm{Re}\Gamma_n^-(\bar{\zeta}_n^\pm)$ have different signs, which means that the real eigen wave $U(g, \bar{\zeta}_n^\pm)$ is 'something like leaky wave'.
- For $k = k_\varepsilon$, $\varepsilon(k) = 0$ and $\Phi_n(\bar{\zeta}_n^\pm) = 0$, $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) = 0$, $\Gamma_n^-(\bar{\zeta}_n^\pm) = 0$. Exotic configuration of the real eigen wave $U(g, \bar{\zeta}_n^\pm)$ for the propagation constant $\bar{\zeta}_n^\pm$ from the first sheet of the surface F_n (here $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^\pm) = k$) and from another sheet (here $\mathrm{Re}\Gamma_n^+(\bar{\zeta}_n^\pm) = -k$) can be easily depicted using the representation (4).
- For $k > k_\varepsilon$, $\varepsilon(k) > 0$ and $\Phi_n(\bar{\zeta}_n^\pm) = \pm i\frac{k}{k_\varepsilon}\sqrt{\frac{(k^2-k_\varepsilon^2)(k_\mu^2-k_\varepsilon^2)}{2k^2-k_\varepsilon^2}}$. $\mathrm{Im}\Gamma_n^+(\bar{\zeta}_n^\pm) = 0$, $\mathrm{Im}\Gamma_n^-(\bar{\zeta}_n^\pm) = 0$, which means that the imaginary eigen wave $U(g, \bar{\zeta}_n^\pm)$ is 'something like leaky wave', but there is no connection between this property and the condition of complete transparency of the boundary $y = 0$ for homogeneous plane waves which was discussed above

The analysis shows that the point $k = \sqrt{0.5}k_\varepsilon = k^{\mathrm{sing}}$ is singular when the propagation constants $\bar{\zeta}(k)$ of the eigen waves $U(g, \bar{\zeta}(k))$ are defined in it: infinite limits of the functions $\Phi_n(\bar{\zeta}_n^\pm(k))$ at k do not coincide when tending to k^{sing} from left and right. In addition, in the case of $k_\varepsilon > k_\mu$ and k approaching to k^{sing} from the left, and in the case of $k_\varepsilon < k_\mu$ and k approaching to k^{sing} from the right, we are increasingly faced with the real propagation constants $\bar{\zeta}_n^\pm(k)$ of 'unusual true eigen waves' $U(g, \bar{\zeta}_n^\pm)$ (see, for example, Fig. 1); and k^{sing} is such point of accumulation of singularities that, as shown by

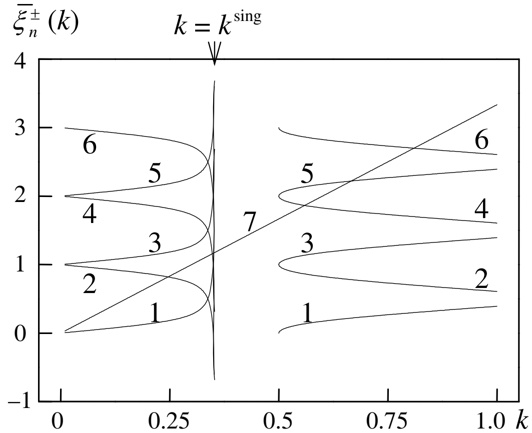

Figure 1. Propagation constants of real eigen waves $U(g, \bar{\zeta}_n^{\pm})$ ($1 - \bar{\zeta}_0^+$, $2 - \bar{\zeta}_{-1}^-$, $3 - \bar{\zeta}_{-1}^+$, $4 - \bar{\zeta}_{-2}^-$, $5 - \bar{\zeta}_{-2}^+$, $6 - \bar{\zeta}_{-3}^-$) and value of $\zeta = kl/2\pi\beta$ (7) for $l = 2\pi$, $k_\varepsilon = 0.5$, $k_\mu = 0.4$, $\beta = 0.3$.

computational experiments in [8], significantly affect the electrodynamic characteristics of a periodically uneven interface.

A plane interface between the media (boundary $y = 0$) is not transforming. This means that an electron beam flying over a plane interface between the media (we associate its own field with the field of wave $V_0(g, k)$) generates in half-spaces $y > 0$ (vacuum) and $y < 0$ (dispersive medium) only the basic spatial harmonics $U_0^{\pm}(g, k)$ of the secondary field. $\mathrm{Re}\Gamma_0^+(\zeta) = 0$ ($V_0(g, k)$ is inhomogeneous plane wave), and therefore, diffraction radiation does not occur in the upper half-space. Vavilov-Cherenkov radiation (radiation into the lower half-space at the only harmonic $U_0^-(g, k)$ arising here) is possible only if (see Eq. (1))

$$\mathrm{Im}\Gamma_0^-(k) = \mathrm{Im}\sqrt{k^2\varepsilon(k)\,\mu(k) - \Phi_0^2} = \mathrm{Im}\sqrt{k^2\varepsilon(k)\,\mu(k) - k^2/\beta^2} = 0. \tag{9}$$

It is true only in the case of binegative ($\varepsilon(k) < 0$ and $\mu(k) < 0$) or bipositive (conventional) medium. From Eq. (9), for the dispersion law in Eq. (7), we obtain the following restriction on the frequencies at which the propagating in the half-space $y < 0$ plane wave $U_0^-(g, k)$ is able to take away some part of the electron beam energy:

$$k^2 < \frac{-\beta^2\left(k_\varepsilon^2 + k_\mu^2\right) + \sqrt{\beta^4\left(k_\varepsilon^2 + k_\mu^2\right)^2 + 4\left(1 - \beta^2\right)k_\varepsilon^2 k_\mu^2 \beta^2}}{2\left(1 - \beta^2\right)}. \tag{10}$$

Figures 2 and 3 present the results which allow to estimate the efficiency of this taking away or, in other words, the intensity of diffraction radiation (Vavilov-Cherenkov radiation) into a plasma-like medium. The curves $\mathrm{G}_0^- = \{k, \beta\} : \beta = [\varepsilon(k)\,\mu(k)]^{-1/2}$ limit the ranges of parameter values where $W_0^-(\ldots, \ldots)$ is nonzero. In the case $k_\varepsilon < k_\mu$, to achieve high radiation intensity, the parameters k and β should be $0.3 < \beta < 0.6$ and $k < 0.25$ (Fig. 2(a)), and in the case $k_\varepsilon > k_\mu$ (Fig. 2(b)), it is shifted towards large β. In these regions, the radiation intensity at a constant velocity of the beam is practically independent of its modulation frequency.

The lines $W_0^-(k_\varepsilon, k_\mu) = \text{const}$ presented in Fig. 3 give a fairly complete picture of the influence of constitutive parameters of a dispersive medium with plane boundary on the energy characteristics of radiation. Interestingly, in the case of large k and β (Fig. 3(b)), the lines $W_0^-(k_\varepsilon, k_\mu) = \text{const}$ intersect with a straight line (diagonal dashed line) at an angle close to the right one (which means a rapid change in the radiation intensity), while for smaller k and β (Fig. 3(a)) and $k_\varepsilon > 0.8$, movement along the line $k_\varepsilon = k_\mu$ does not lead to any noticeable change in the radiation intensity.

Generally, in the case of a plane interface between the media, the energy characteristics of diffraction radiation change very smoothly when moving from the limiting boundary G_0^- (beyond which this radiation is possible) and within a not very large interval. This is because at the given frequency

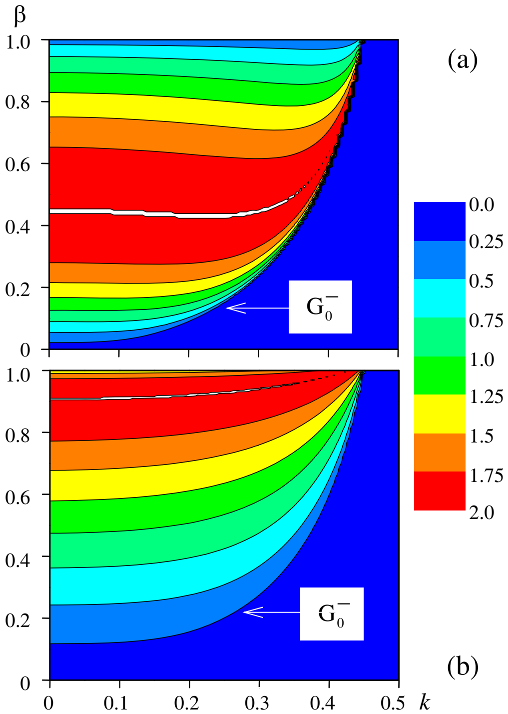

Figure 2. Lines $W_0^-(k,\beta) = \text{const}$, characterizing the intensity of diffraction radiation into the half-space $y < 0$: $a - k_\varepsilon = 0.5$, $k_\mu = 1.0$; $b - k_\varepsilon = 1.0$, $k_\mu = 0.5$.

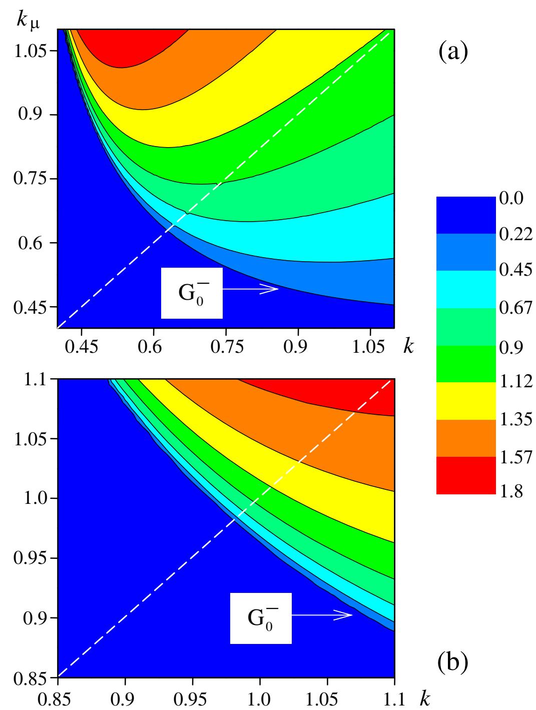

Figure 3. Lines $W_0^-(k_\varepsilon, k_\mu) = \text{const}$: $a - k = 0.3$, $\beta = 0.3$; $b - k = 0.6$, $\beta = 0.6$.

k, the electron beam can excite only one eigen wave, $U(g, \bar{\zeta}_0^{\pm})$. And the excitation of this wave can lead to a rather sharp and strong rise in the field amplitude $U(g,k)$ only at frequencies k close to k^{sing}, for which the values of $\zeta = kl/2\pi\beta$ and $\bar{\zeta}_0^{+}(k)$ practically coincide (see, for example, Fig. 1, the area of intersection of the curves 1 and 7). But these frequencies, obviously, lie outside the range of k, where Vavilov-Cherenkov radiation (VChR) is possible. There, $\text{Re}\Gamma_0^-(k,\zeta) = \text{Re}\Gamma_0^-(k,\bar{\zeta}_0^+) = 0$, $U_0^-(g,k)$ is an inhomogeneous plane wave, and the wave $U(g,\bar{\zeta}_0^+)$ is 'unusual true eigen wave'.

3. PERIODIC INTERFACE. SMITH-PURCELL RADIATION

The electrodynamic characteristics of a periodically uneven boundary and a plane boundary differ fundamentally. In particular, all the spectrum of eigen waves is excited in the case $h > 0$, and thus continues (with certain distortions) eigen waves $U(g,\bar{\zeta}_n^{\pm})$, $n = 0, \pm 1, \pm 2, \ldots$, which correspond to $h = 0$. For small values of $h > 0$ and sufficiently smooth functions $f(z)$, the spectral characteristics of periodically uneven and plane interfaces differ insignificantly [4]. We keep the same designations for them as in the case $h = 0$. With growing h, the magnitudes of $\bar{\zeta}_n^{\pm}$ deform significantly, but their behavior and the behavior of the corresponding eigen waves obey common regularities for periodic structures established in [4, 5, 9]. An important fact is that with smooth variations of any parameter τ of the problem (2), the existing propagation constants $\bar{\zeta}(\tau)$ of eigen waves $U(g,\bar{\zeta}(\tau))$, which move along the sheets of the surface F, cannot disappear anywhere in their finite part [4, 10].

It is very important in the study of diffraction radiation to determine correctly the limits of parameters variation which allow implementation of the given regime of electron beam field transformation into the field of waves outgoing infinitely far from the periodic interface between the media. The regime identifier $\{N^+, N^-\}$ is set by N^+ and N^- which are the numbers of harmonics propagating without attenuation in the reflection ($y > 0$) and transmission ($y < -h$) zones of the periodic structure. The limits of the domains corresponding to this regime in the plane of the variables

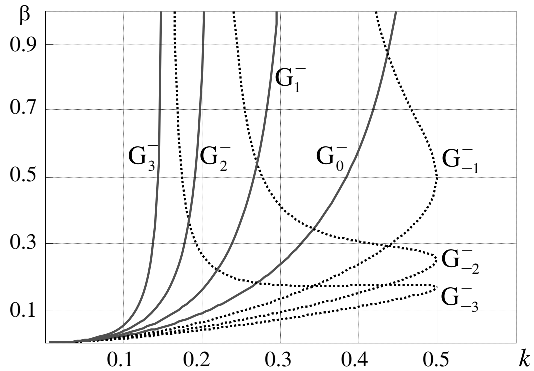

Figure 4. Configuration of limiting curves G_n^- for $l = 2\pi, k_\varepsilon = 0.5$ and $k_\mu = 1.0$.

k and β are determined by the curves $\mathrm{G}_n^\pm = \{k, \beta\} : \Gamma_n^\pm(k, \beta) = 0$ (see, for example, Fig. 4). Obviously, for $\Phi_0 = k/\beta$, only harmonics with negative indexes n can propagate without attenuation in the domain $y > 0$.

In [1], we have already noted the impact of the propagation constants $\bar{\zeta}$, which are located near the real axis of the first, physical sheet of the surface F, onto the formation of resonant response of the periodic boundary $\Sigma_x^{\varepsilon,\mu}$ when it is excited by a homogeneous or inhomogeneous plane wave $V_0(g, k)$. Now we shall briefly illustrate this thesis by calculating the efficiency of Smith-Purcell radiation (SPR) in one particular case. This result, by the way, can be considered as a prelude to a detailed analysis of anomalously high levels of coherent diffraction radiation generated by a density-modulated electron beam moving over a periodic boundary separating an ordinary medium and an artificial medium with specific frequency dispersion of permittivity and permeability. We are going to focus on this analysis in our next works.

VChR is radiation at the spatial harmonic $U_0^-(g, k)$ of the periodic boundary into an optically denser medium occupying the half-space $y < f(z)$ [2, 11–13]. Radiation at harmonics $U_n^\pm(g, k)$, $n \neq 0$, which propagate without attenuation in the domains $y > 0$ and $y < -h$, is already Smith-Purcell radiation (SPR) [3, 4, 14, 15]. The result below is about SPR.

Let us plot, on the frequency interval $0.01 \leq k \leq 1.0$ (with the sampling step of 0.001), the solution to the problem (2) for the values $\zeta(k) = kl/2\pi\beta$, $l = 2\pi$, $f(z) = 0.5h(\cos z - 1)$, $h = 0.01,\ 0.05,\ 0.1$, $\beta = 0.3$, $k_\varepsilon = 0.5$, $k_\mu = 0.4$; and let us use the data presented in Fig. 2 for its analyzing. The equality $\bar{\zeta}_n^\pm(k) \approx \zeta(k)$ required for the realization of resonant scattering modes is fulfilled at the points: (i) $k \approx 0.246$ (here $\bar{\zeta}_{-1}^-(k) \approx \zeta(k)$ and, in the field $U(g, k)$, the harmonics $U_{-1}^\pm(g, k)$ are propagating as $\mathrm{Im}\Gamma_{-1}^\pm = 0$); (ii) $k^{\text{sing}} - 0.05 < k < k^{\text{sing}}$ (here $\zeta(k) \approx \bar{\zeta}_0^+(k) \approx \bar{\zeta}_{-2}^-(k) \approx \ldots$ and $\mathrm{Im}\Gamma_{-1}^\pm = 0$); (iii) $k \approx 0.559$ (here $\bar{\zeta}_{-2}^-(k) \approx \zeta(k)$ and $\mathrm{Im}\Gamma_{-2}^\pm = \mathrm{Im}\Gamma_{-3}^+ = 0$); (iv) $k \approx 0.665$ (here $\bar{\zeta}_{-2}^+ \approx \zeta(k)$ and $\mathrm{Im}\Gamma_{-2}^\pm = \mathrm{Im}\Gamma_{-3}^+ = 0$); (v) $k \approx 0.81$ (here $\bar{\zeta}_{-3}^- \approx \zeta(k)$ and $\mathrm{Im}\Gamma_{-2}^+ = \mathrm{Im}\Gamma_{-3}^\pm = 0$). But only in case (ii), when $U(g, \bar{\zeta}_n^\pm)$ are 'unusual true eigen waves', the propagating harmonics of the field $U(g, k)$ (amplitudes of these harmonics determine the intensity of SPR) respond in an expected manner to the fulfillment of resonance conditions (Fig. 5). When they are fulfilled, we obtain very high values of $W_{-1}^\pm(k)$ against the background of completely insignificant values $W_{-1}^\pm(k)$ in all other points of the ranges where harmonics propagate without attenuation in the half-spaces $y > 0$ and $y < -h$.

It is worth to point out the practical significance of results associated with the detection and detailed analysis of such effects. They can serve as a basis for development of fundamentally new, accurate measuring schemes, e.g., diagnostic schemes and determination of intrinsic parameters of plasma-like media and charged particle beams.

In cases (i), (iii)–(v), where $U(g, \bar{\zeta}_n^\pm)$ are real eigen 'something like leaky waves', the propagating harmonics of the field $U(g, k)$ do not react in any way to the fulfillment of resonance conditions. The functions $W_n^\pm(k)$ change insignificantly for all points of the ranges, where the corresponding harmonics are propagating and remain here at a level not exceeding 10^{-1}. The reason behind this huge difference between case (ii) and cases (i), (iii)–(v) is practically obvious — the propagation constants $\bar{\zeta}$ of the

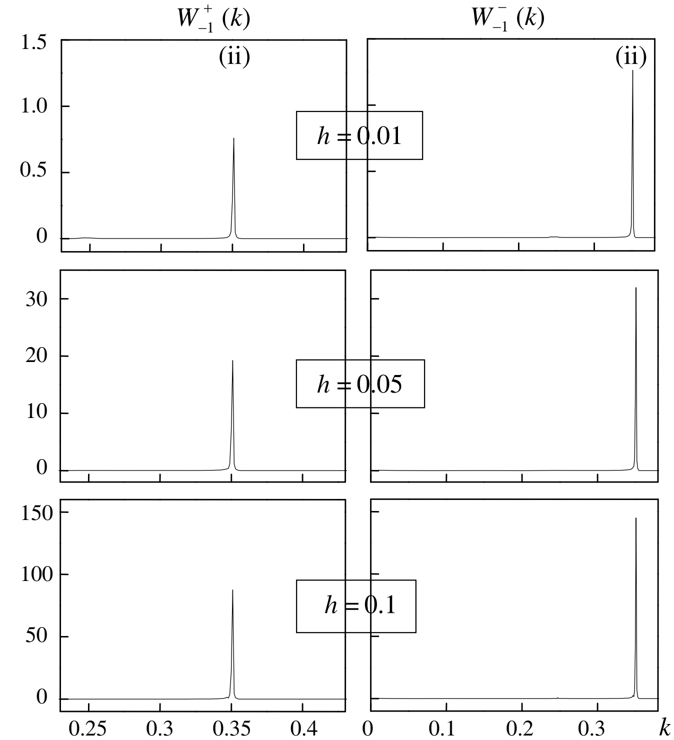

Figure 5. SPR intensity in resonant wave scattering modes. Frequency bands k where spatial harmonics $U^{\pm}_{-1}(g,k)$ propagate without attenuation are covered.

normal modes $U(g,\bar{\zeta}_n^{\pm})$ in cases (i), (iii)–(v) fall on higher, non-physical sheets of the surface F, and only in case (ii) the relevant propagation constant turns out to be really close to the parameter $\zeta = kl/2\pi\beta$ of the excitation wave, which is measured on the axis Reζ of the first, physical sheet of F.

4. CONCLUSION

In this work, we dwelt in detail on the problem of eigen waves of a plane boundary separating vacuum and an artificial plasma-like medium. The propagation constants $\bar{\zeta}_n^{\pm}$ of the normal modes $U(g,\bar{\zeta}_n^{\pm})$ are determined, and they are located both on the first and higher sheets of the Riemann surface, which is the natural region of variation of the spectral parameter ζ. The characteristic features of eigen waves are described, and analytical results for a plane boundary $y = 0$ are extended to the case of a periodically uneven boundary $\Sigma_x^{\varepsilon,\mu}$ with a small profiling depth h. The results of numerical experiments, indicating the direction of further development of these problems, are presented.

ACKNOWLEDGMENT

This research has been partially funded by the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan (Grant No. AP08855557).

REFERENCES

1. Sirenko, Y., S. Sautbekov, N. Yashina, and K. Sirenko, "Diffraction radiation generated by a density-modulated electron beam flying over the periodic boundary of the medium Section. I. Analytical basis," *Progress In Electromagnetics Research B*, Vol. 91, 1–8, 2021.
2. Cherenkov, P., "Visible radiation produced by electrons moving in a medium with velocities exceeding that of light," *Physical Review*, Vol. 52, 378–379, 1937.

3. Smith, S. and E. Purcell, "Visible light from localized surface charges moving across a grating," *Physical Review*, Vol. 92, No. 4, 1069–1073, 1953.
4. Shestopalov, V. and Y. Sirenko, *Dynamic Theory of Gratings*, Naukova Dumka, Kiev, 1989 (in Russian).
5. Sirenko, Y. and S. Strom (eds.), *Modern Theory of Gratings. Resonant Scattering: Analysis Techniques and Phenomena*, Springer, New York, 2010.
6. Rud', L., Y. Sirenko, V. Yatsik, and N. Yashina, "A spectral method of analyzing the effects of total wave transformation by open-periodic and waveguide resonators," *Radiophysics and Quantum Electronics*, Vol. 31, No. 10, 894–899, 1988.
7. Sirenko, Y., N. Yashina, and K. Schuenemann, "Synthesis of mode converters in waveguides and gratings based on spectral theory," *Journal of Electromagnetic Waves and Applications*, Vol. 16, No. 5, 611–628, 2002.
8. Sirenko, Y., P. Melezhik, A. Poyedinchuk, S. Sautbekov, A. Shmat'ko, K. Sirenko, A. Vertiy, and N. Yashina, "Radiation of electromagnetic waves induced by electron beam passage over artificial material periodic interfaces," *An Essential Guide to Electrodynamics*, N. Brewer (ed.), Chap. 5, 169–206, Nova Science Publishers, New York, 2019.
9. Sirenko, Y., "A grating in the field of a compact monochromatic source," *Electromagnetics*, Vol. 13, No. 3, 255–272, 1993.
10. Sanchez-Palencia, E., *Nonhomogeneous Media and Vibration Theory*, Springer-Verlag, New York, 1980.
11. Jelley, J., *Cherenkov Radiation and its Applications*, Pergamon Press, London, 1958.
12. Frank, I., *Vavilov-Cherenkov Radiation*, Nauka, Moscow, 1988 (in Russian).
13. Granet, G., P. Melezhik, A. Poyedinchuk, S. Sautbekov, Y. Sirenko, and N. Yashina, "Resonances in reverse Vavilov-Cherenkov radiation produced by electron beam passage over periodic interface," *International Journal of Antennas and Propagation*, Vol. 2015, 10p, 2015.
14. Bolotovskiy, B. and G. Voskresenskiy, "Emission from charged particles in periodic structures," *Soviet Physics-Uspekhi*, Vol. 11, No. 2, 143–162, 1968.
15. Shestopalov, V., *The Smith-Purcell Effect*, Nova Science Publishes, New York, 1998.