A Simple Matrix Approach for Computing the Equivalent Resistance and Unknown Components in Resistor Networks

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Abstract—A method is presented for computing the equivalent resistance and the unknown components of simple series and parallel resistor networks. The approach consists in taking the product of a simple 2×2 matrix $(N - 1)$ times, where N is the total number of components in the network. The matrix approach originates from the study of continued fractions. Numerical computations only require an algorithm that handles matrix multiplication.

1. INTRODUCTION

Resistor networks are fundamental to many electrical and electronic devices, and a method for computing the equivalent resistance R_E or conductance $1/R_E$ is required. Finding the equivalent resistance then allows determination of other unknown quantities such as voltage or current using Ohm's law, $V = IR_E$, for example. A large number of resistor networks can be reduced to resistors in a series-parallel configuration with some exceptions such as the Wheatstone bridge circuit. For such networks there is a requirement to determine the value of one or more unknown resistors in the network when a required equivalent resistance R_E is given.

For small networks, such an analysis can be done algebraically by reducing the parallel resistors to series components and then summing all of them to obtain R_E or by solving the algebraic expressions for the unknown component. A number of interesting networks have been analyzed using simple algebraic techniques [1–3]. More complicated networks require sophisticated techniques such as Kirchhoff's approach or the simpler approach based on nodal potentials whose variables are the values of the electric potential at the circuit's nodes [4]. Other methods that have been used involve Green's function [5] for the computation of large networks, Laplace transforms [6] and variational approaches as an alternative to Kirchhoff's loop theorem [7].

The problem with these methods for computing the equivalent resistance or the value of unknown resistors in large networks is that they can be very complicated and in some cases intractable. In what follows, a new method is introduced that consists of the multiplication of a simple 2×2 matrix, $(N-1)$ times, where N is the total number of components in the network, see [8] for a good exposition of matrix algebra. This allows easy computation of R_E or the computation for the unknown resistor values of a network when R_E is given. For small resistor networks the matrices can be multiplied algebraically in order to obtain closed form solutions if required.

2. THEORETICAL DEVELOPMENT

Consider the following optimization problem. A circuit design requires that the total resistance in the circuit be equal to $R_E = 17 \, \text{k}\Omega$. Due to manufacturing costs, the cheapest resistors available that can

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be used to achieve this equivalent resistance have a value of $R = 10\,\text{k}\Omega$. How many $R = 10\,\text{k}\Omega$ resistors, which there are plenty of, can one use to produce the equivalent resistance? The answer is quite simple. The number of resistors required x is:

$$x = \frac{R_E}{R} = \frac{17}{10} = 1.7 \tag{1}$$

Thus $1.7 \times 10\,\text{k}\Omega$ resistors are sufficient provided that one is a 'whole' $10\,\text{k}\Omega$ resistor, and the other is a $0.7\,\text{k}\Omega$ resistor. This however is rather problematic as it is not possible to cut a $10\,\text{k}\Omega$ resistor to match the required fraction of $0.7\,\text{k}\Omega$. It is critical that all resistors have the value $R = 10\,\text{k}\Omega$ and no other resistor types are allowed. To resolve this problem, it is possible to use $R = 10\,\text{k}\Omega$ resistors in a network consisting of series and parallel combinations. For example, the resistors can be set out as shown in Fig. 1 and it can be shown that the equivalent resistance of this network is indeed $R_E = 17\,\text{k}\Omega$.
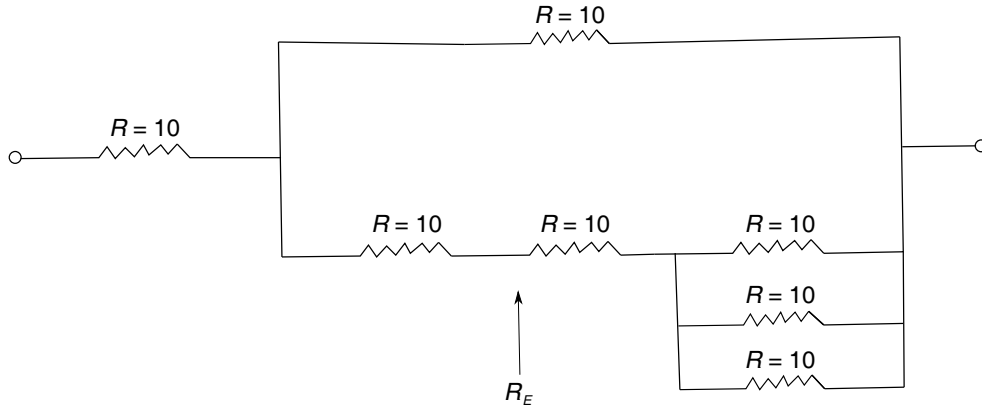

Figure 1. One way to achieve an equivalent resistance of $R_E = 17\,\text{k}\Omega$ using $R = 10\,\text{k}\Omega$ resistors is to set them up in a network consisting of series and parallel arrangements as shown.

How was this arrangement of the $R = 10\,\text{k}\Omega$ resistors made possible such that it resulted in the required equivalent resistance of $R_E = 17\,\text{k}\Omega$? To understand how this was done, suppose that any number x where $x \in R^+$, is written as two parts: $x = [x] + \{x\}$, where $[x]$ is the integer part of x and $\{x\}$ is the fractional part of x. For example, (1) can be written as $1.7 = [1.7] + \{1.7\} \equiv 1 + 0.7$. Define the fractional part of x to be $\{x_0\} = \{x\}$. Then for $n = 0, 1, 2, ...$ the following definitions are given:

$$x_{n+1} = \frac{1}{\{x_n\}} \tag{2}$$

and

$$\frac{1}{\{x_n\}} = [x_{n+1}] + \{x_{n+1}\} \tag{3}$$

The use of Eqs. (2) and (3) and why they are needed will become apparent shortly. In fact their use will reveal how many $R = 10\,\text{k}\Omega$ resistors are needed in series and how many are needed in parallel in order to facilitate the network design of Fig. 1 which achieves $R_E = 17\,\text{k}\Omega$ using only $R = 10\,\text{k}\Omega$ resistors. Since $x = 1.7$ can be written as $x = 1 + 0.7$, it is a matter of addressing the fractional part $\{x_0\} = \{x\} = 0.7$ using Eqs. (2) and (3) for $n = 0, 1, 2, ...$ Note that the number of values chosen for n is such that the nth fractional term is zero, $\{x_n\} = 0$. Alternatively n can be arbitrarily chosen to terminate the process if a good enough approximation is achieved. Using (2) the following terms are obtained:

$$n = 0: \quad x_1 = \frac{1}{\{x_0\}} = \frac{1}{0.7} = 1.428571...; \quad [x_1] = 1, \{x_1\} = 0.428571...$$

$$n = 1: \quad x_2 = \frac{1}{\{x_1\}} = \frac{1}{0.428571...} = 2.\dot{3}; \quad [x_2] = 2, \{x_2\} = 0.\dot{3} \tag{4}$$

$$n = 2: \quad x_3 = \frac{1}{\{x_2\}} = \frac{1}{0.\dot{3}} = 3; \quad [x_3] = 3, \{x_3\} = 0$$

From Eq. (4), the last fractional term is zero when $n = 2$, hence n takes the values $n = 0, 1, 2$. Using these values in Eq. (3) gives

$$\frac{1}{\{x_0\}} = [x_1] + \{x_1\}$$
$$\frac{1}{\{x_1\}} = [x_2] + \{x_2\} \tag{5}$$
$$\frac{1}{\{x_2\}} = [x_3] + \{x_3\}$$

Starting from the last term in Eq. (5) and back-substituting the reciprocal of each term into the expressions above, i.e., the last term is substituted into the second and then the second into the first gives:

$$\{x_0\} = \cfrac{1}{[x_1] + \cfrac{1}{[x_2] + \cfrac{1}{[x_3] + \{x_3\}}}} \tag{6}$$

All the values in Eq. (6) have been calculated in Eq. (4) and substituting them gives

$$0.7 = \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3 + 0}}} \tag{7}$$

Then the equivalent resistance becomes:

$$x = 1.7 = 1 + 0.7 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{3}}} \tag{8}$$

using Eq. (7) to replace 0.7. What does this all mean? To understand this expression, let all the $10\,\mathrm{k\Omega}$ resistors be represented by R, see Fig. 1. The requirement is the same as before, that is, to use the same type of resistors in some series and parallel configuration so that in the end an equivalent resistance of $R_E = 17\,\mathrm{k\Omega}$ is achieved. Thus starting with

$$\frac{R_E}{R} = 1.7 \rightarrow R_E = 1.7R = R + 0.7R \tag{9}$$

the same process is followed as before using Eq. (2):

$$n = 0: \quad x_1 = \frac{1}{\{x_0\}} = \frac{1}{0.7R} = \frac{1.428571...}{R}; \quad [x_1] = \frac{1}{R}, \{x_1\} = \frac{0.428571...}{R}$$
$$n = 1: \quad x_2 = \frac{1}{\{x_1\}} = \frac{R}{0.428571...} = 2.\dot{3}R; \quad [x_2] = 2R, \{x_2\} = 0.\dot{3}R \tag{10}$$
$$n = 2: \quad x_3 = \frac{1}{\{x_2\}} = \frac{1}{0.\dot{3}R} = \frac{3}{R}; \quad [x_3] = \frac{3}{R}, \{x_3\} = 0$$

Substituting the values in Eq. (10) into Eq. (6) gives the following result for the equivalent resistance R_E via Eq. (9):

$$R_E = R + \cfrac{1}{\cfrac{1}{R} + \cfrac{1}{2R + \cfrac{R}{3}}} \tag{11}$$

Evaluating the right-side of Eq. (11) gives

$$R_E = \frac{17R}{10} = 17\,\mathrm{k\Omega} \tag{12}$$

since all the resistors in Fig. 1 have the same value $R = 10$. The form given by Eq. (11) has a very important meaning. The series and parallel resistors alternate. To clarify this even more let series resistors be denoted by s while parallel resistors by p, then Eq. (11) becomes:

$$R_E = s + \cfrac{1}{p + \cfrac{1}{2s + \cfrac{1}{3p}}} \tag{13}$$

It should be clear now that as one moves from left to right in Fig. 1, there is one series resistor s which is in parallel with one resistor p which in turn is in parallel with two series resistors $2s$ which are themselves in series with three parallel resistors $3p$. This is how the network of Fig. 1 was created so that in the end it was possible to achieve the desired $17\,\text{k}\Omega$ using only $10\,\text{k}\Omega$ resistors.

As one moves from left to right in Fig. 1, there is an alternation between series and parallel resistors. If either a series or parallel resistor is missing set $s = 0$ or $p = 0$ in Eq. (13). For example, if there is no series resistor to begin with then the equivalent resistance becomes

$$R_E = 0 + \cfrac{1}{p + \cfrac{1}{2s + \cfrac{1}{\frac{3}{p}}}} \tag{14}$$

If instead, the first parallel resistor is removed or absent then $p = 0$ so that

$$R_E = R + \cfrac{1}{0 + \cfrac{1}{2s + \cfrac{1}{\frac{3}{p}}}} \tag{15}$$

and so on. The process that has been shown above is generalizable to resistor networks containing different values for its series and parallel components. The other important observation is that the analysis used to obtain Eq. (8) or (11) has allowed then to be written in a very familiar mathematical form. They are in fact known as continued fractions and the reader can refer to [3] for their use in circuit analysis. A 'simple' continued fraction x is defined as

$$x = a_0 + K_{n=1}^{N} \frac{1}{a_n} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ldots}}} \tag{16}$$

where $K_{n=1}^{N}$ means the fractional summation, and in this case the numbers or variables $a_0, a_1, \ldots$ replace the alternating series and parallel resistors. Whether a_n represents resistors or arbitrary numbers, they can be written in a 'pseudo-array' structure $A = [a_0, a_1, \ldots]$ that will be useful later on. The problem with using continued fractions in resistor networks should be obvious. Continued fractions can become complicated very soon especially for large resistor networks, and it can be very difficult to use them in such an analysis. For this reason a simpler approach will now be discussed which only requires the multiplication of a simple 2×2 matrix.

Let the continued fraction with components $A = [a_0, a_1, a_2]$ be expanded as

$$A = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} \tag{17}$$

Define the Q-matrix as [9, 10]:

$$\mathbf{Q}^{(a_n)} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \tag{18}$$

The premise here is that by using the Q-matrix it is possible to represent any continued fraction such as Eq. (17) as a product of the Q-matrices:

$$\mathbf{R} = \prod_{n=0}^{N-1} \mathbf{Q}^{(a_n)} \tag{19}$$

where N is the total number of elements in the array A. Then using Eq. (19), A can be obtained so that it has the exact form as the continued fraction in Eq. (17) if

$$A = \frac{R_{11}}{R_{21}} \tag{20}$$

Using the array elements and $N = 3$, Eq. (19) becomes

$$\mathbf{R} = \prod_{n=0}^{2} \mathbf{Q}^{(a_n)} = \mathbf{Q}^{(a_0)}\mathbf{Q}^{(a_1)}\mathbf{Q}^{(a_2)} \tag{21}$$

then

$$\mathbf{R} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_2 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (1 + a_0 a_1) a_2 + a_0 & 1 + a_0 a_1 \\ 1 + a_1 a_2 & a_1 \end{pmatrix} \tag{22}$$

Using Eq. (20), A becomes

$$A = \frac{a_0 + (1 + a_0 a_1) a_2}{1 + a_1 a_2} = a_0 + \frac{a_2}{a_1 a_2 + 1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} \tag{23}$$

which is equivalent to Eq. (17). By induction this can be shown to be the case for $n > 2$. Thus not only does the Q-matrix compute continued fractions, but it also computes the series and parallel components of resistor networks in order to obtain such things as the equivalent resistance R_E or other properties. These will be examined in more detail next.

2.1. The Q-Matrix and Resistor Networks

It has been established in the previous section that series-parallel resistor networks can be represented by continued fractions. In turn it was shown that such continued fractions have an equivalence or duality with the Q-matrix approach. This allows the analysis of resistor networks as follows. Let the network resistor components in series be written as elements of the array:

$$A = [a_0, a_1, a_2, \cdots, a_{N-1}] \equiv [R_0, R_1, R_2, \cdots, R_{N-1}] \tag{24}$$

where $R_0, R_1...$ are the resistor values for the network components. For resistor components in parallel (shunt) configurations, the inverse is taken for the entries of the array A,

$$A = [a_0, a_1, a_2, \cdots, a_{N-1}] \equiv [1/R_0, 1/R_1, 1/R_2, \cdots, 1/R_{N-1}] \tag{25}$$

As an example of a network consisting of both types, the component array A can be written as:

$$A = [a_0, a_1, a_2, \cdots, a_{N-1}] \equiv [R_0, R_1, 1/R_2, \cdots, R_{N-1}] \tag{26}$$

where $a_0 = R_0, a_1 = R_1$ are series resistors, and $a_2 = 1/R_2$ implies a resistor in parallel and so on, see for example Fig. 2. In all cases, N is the total number of components in the network. Recall the Q-matrix (18):

$$\mathbf{Q}^{(a_n)} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \tag{27}$$

Observe that when a component in array A is missing, a_n is set to zero so that the Q-matrix takes on a kind of 'pseudo'-identity matrix. The matrix in Eq. (27) can be used to obtain the following product for each component n of the array A as discussed in the previous section,

$$\mathbf{R} = \prod_{n=0}^{N-1} \mathbf{Q}^{(a_n)} = \mathbf{Q}^{(a_0)}\mathbf{Q}^{(a_1)}\mathbf{Q}^{(a_2)} \cdots \mathbf{Q}^{(a_{N-1})} \tag{28}$$

Then the network equivalent resistance R_E is obtained as:

$$R_E = \frac{R_{11}}{R_{21}} \tag{29}$$

Here (R_{11}, R_{21}) are obtained from the final product of all the matrices appearing in Eq. (28). It is worth pointing out that the approach discussed above also works when the number of components is as small as $N = 1$. In this case, the only contribution to $\mathbf{R}$ in Eq. (28) is $\mathbf{Q}^{(a_0)}$, and if a_0 is a series component for example, i.e., $a_0 = R_0$, then the equivalent resistance is merely $R_E = R_0$ from Eq. (29) as expected.

So far consideration has been given on how to obtain the equivalent resistance R_E when series and parallel resistors in a network are given. Another interesting problem is to determine what the values of one or more resistors should be which achieve a desired equivalent resistance R_E. This is easily done using the approach discussed above via the use of Eq. (29):

$$R_E R_{21} - R_{11} = 0 \tag{30}$$

for a given value of R_E. Suppose that a network contains N resistors in series and parallel:

$$\mathbf{R} = \prod_{n=0}^{N-1} \mathbf{Q}^{(a_n)} = \mathbf{Q}^{(a_0)} \mathbf{Q}^{(a_1)} \mathbf{Q}^{(a_2)} \cdots \mathbf{Q}^{(a_{21})} \cdots \tag{31}$$

Let the component array be given as

$$A = [a_0, a_1, a_2, ..., a_{21}, ...] = [R_0, 1/R_1, R_2, ..., 1/R_{21}, ...] \tag{32}$$

Assume that all resistors in Eq. (32) are known except R_2. For a given value of R_E, the problem requires finding the numerical value of R_2. Since R_0 and R_1 are known this implies that the product of the Q-matrices gives the final matrix

$$\mathbf{Q}^{(a_0)} \mathbf{Q}^{(a_1)} = \begin{pmatrix} R_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{1}{R_1} & 1 \\ 1 & 0 \end{pmatrix} \equiv \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{33}$$

where the elements $\alpha, \beta, \gamma, \delta$ are just numerical values. In a similar way, all matrices to the right of $\mathbf{Q}^{(a_2)}$ are multiplied out to give the final numerical matrix:

$$\cdots \quad \mathbf{Q}^{(a_{21})} \cdots \quad \equiv \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} \tag{34}$$

where once again $\epsilon, \zeta, \eta, \theta$ are numerical values. Then Eq. (31) becomes,

$$\mathbf{R} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} R_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \epsilon & \zeta \\ \eta & \theta \end{pmatrix} = \begin{pmatrix} \alpha\eta + \epsilon(\beta + \alpha R_2) & \alpha\theta + \zeta(\beta + \alpha R_2) \\ \gamma\eta + \epsilon(\delta + \gamma R_2) & \gamma\theta + \zeta(\delta + \gamma R_2) \end{pmatrix} \tag{35}$$

Obtaining R_{21} and R_{11} from Eq. (35) and using Eq. (30), the unknown resistor R_2 can be extracted from the expression

$$R_2 = \frac{\beta\epsilon + \alpha\eta - R_E(\delta\epsilon + \gamma\eta)}{\epsilon(R_E\gamma - \alpha)} \tag{36}$$

where R_E is given. The theory developed here will be applied to a couple of simple resistor networks in the next section.

3. APPLICATION TO RESISTOR NETWORKS

In order to elucidate the approach presented in the previous section, two simple resistor networks will be considered here. These networks also have the added benefit that closed form solutions can be obtained for comparison. For larger networks, closed form solutions can become horrendously complex and intractable and so numerical techniques are needed which are usually too inefficient. On the contrary, implementing the approach described here makes large network analysis rather simple and straightforward.
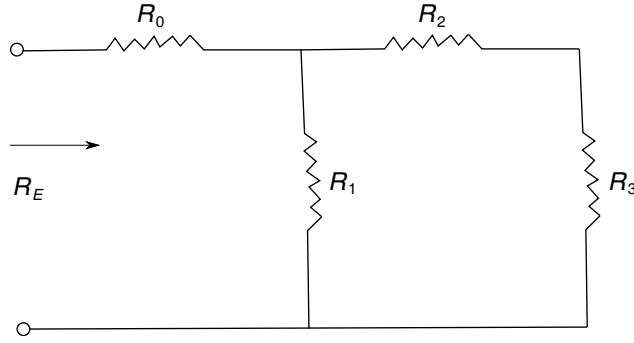
Figure 2. A simple 'ladder' resistor network consisting of $N = 4$ resistors in a series and parallel configuration.

3.1. Analysis of Network 1

Consider the 'ladder'-network shown in Fig. 2 that has $N = 4$ resistor components. A component array A is constructed by entering the resistors as they are encountered from left to right. Thus for Fig. 2, the array becomes $A = [a_0, a_1, a_2, a_3] \equiv [R_0, 1/R_1, R_2, 1/R_3]$. It is important to note that this series-parallel sequence is preserved for all computations. In other words, suppose that the parallel resistor R_1 is removed from the network, then the number of components is still $N = 4$ but we set the entry to zero for this missing component in the array: $A = [a_0, 0, a_2, a_3] \equiv [R_0, 0, R_2, 1/R_3]$. For this scenario, finding the equivalent resistance R_E is trivial since it is only the sum of the resistors: $R_E = R_0 + R_2 + R_3$. It can be shown that the matrix approach also gives the same result as follows. If $A = [R_0, 0, R_2, 1/R_3]$ then:

$$\mathbf{R} = \prod_{n=0}^{3} \mathbf{Q}^{(a_n)} = \mathbf{Q}^{(a_0)}\mathbf{Q}^{(a_1)}\mathbf{Q}^{(a_2)}\mathbf{Q}^{(a_3)} \tag{37}$$

Multiplying the Q-matrices out means that the matrix R becomes:

$$\mathbf{R} = \begin{pmatrix} R_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{R_3} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \dfrac{R_0 + R_2 + R_3}{R_3} & R_0 + R_2 \\ \dfrac{1}{R_3} & 1 \end{pmatrix} \tag{38}$$

The equivalent resistance is obtained from:

$$R_E = \frac{R_{11}}{R_{21}} = \frac{\dfrac{R_0 + R_2 + R_3}{R_3}}{\dfrac{1}{R_3}} = R_0 + R_2 + R_3 \tag{39}$$

as expected for resistors in series. Next, consider the case where R_1 is reinstated again in the network as in Fig. 2. Then the component array is $A = [a_0, a_1, a_2, a_3] \equiv [R_0, 1/R_1, R_2, 1/R_3]$. Before obtaining R_E for this network configuration using the matrix approach, a closed form solution can be derived in algebraic form for comparison. The solution for R_E becomes:

$$R_E = \frac{R_0[R_1 + R_2 + R_3] + R_1[R_2 + R_3]}{R_1 + R_2 + R_3} \tag{40}$$

The algebraic result in Eq. (40) is obtained by the reduction of parallel components to series components and then summing them starting from the right and moving to the left in order to obtain the equivalent resistor R_E. The Q-matrix approach can also be used in algebraic form to obtain the same result for verification. Using Eq. (37) gives,

$$\mathbf{R} = \begin{pmatrix} R_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{1}{R_1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} R_2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \dfrac{1}{R_3} & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \left(\begin{array}{cc} 1 + \dfrac{R_0}{R_1} + \dfrac{R_0 + R_2\left(1 + \dfrac{R_0}{R_1}\right)}{R_3} & R_0 + R_2\left[\dfrac{R_1 + R_0}{R_1}\right] \\ \dfrac{1}{R_1} + \dfrac{R_1 + R_2}{R_1 R_3} & 1 + \dfrac{R_2}{R_1} \end{array} \right) \tag{41}$$

where the second matrix is no longer the identity because the component R_1 is now included as shown in Fig. 2. The equivalent resistance is given by:

$$R_E = \frac{R_{11}}{R_{21}} = \frac{1 + \dfrac{R_0}{R_1} + \dfrac{R_0 + R_2\left(1 + \dfrac{R_0}{R_1}\right)}{R_3}}{\dfrac{1}{R_1} + \dfrac{R_1 + R_2}{R_1 R_3}}$$

$$= \frac{R_0[R_1 + R_2 + R_3] + R_1[R_2 + R_3]}{R_1 + R_2 + R_3} \tag{42}$$

which is equal to Eq. (40) as expected. Let the values of the resistors in Fig. 2 be given as $R_0 = 1\,\Omega$, $R_1 = 3\,\Omega$, $R_2 = 5\,\Omega$ and $R_3 = 4\,\Omega$. Substituting these values into Eq. (42) gives the equivalent resistance as $R_E = 3.25\,\Omega$. Suppose that instead $R_E \equiv 3.25\,\Omega$ was given, as well as the resistor values except $R_2 =?$, then the problem requires finding the value of R_2 by using Eqs. (30)–(36) as discussed in the previous section. Since the network of Fig. 2 has already been solved in closed form, use of Eq. (42) can be made. Rearranging Eq. (42) for the unknown resistor R_2 gives:

$$R_2 = \frac{R_0(R_1 + R_3) + R_1 R_3 - R_E(R_1 + R_3)}{R_E - (R_0 + R_1)} = \frac{7 + 12 - 3.25 \times 7}{3.25 - 4} = 5\,\Omega \tag{43}$$

as expected. Typically for very large networks, closed form expressions become impractical so that numerical multiplications of the Q-matrices are easiest.

3.2. Analysis of Network 2

The second example consists of the slightly more complicated network as shown in Fig. 3. Let the resistors in Fig. 3 take the values: $R_0 = 1\,\Omega$, $R_1 = 12\,\Omega$, $R_2 = $ missing, $R_3 = 4\,\Omega$, $R_4 = 1\,\Omega$, $R_5 = 10\,\Omega$, $R_6 = 2\,\Omega$ and $R_7 = 8\,\Omega$. Recall that the missing resistor R_2 must be accounted for in the array A by setting its value to zero: $A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6, a_7] \equiv [1, 1/12, 0, 1/4, 1, 1/10, 2, 1/8]$. The
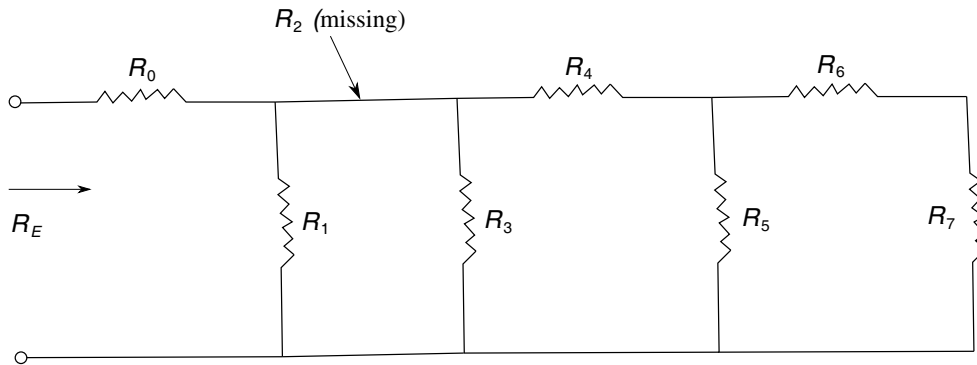

Figure 3. A resistor network consisting of $N = 8$ resistors in a series and parallel configuration. Even though R_2 is missing from the network, it is still counted as a component but its value is set to zero in the component array A.

equivalent resistance for the network is obtained via

$$\mathbf{R} = \prod_{n=0}^{7} \mathbf{Q}^{(a_n)}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{12} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{4} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{10} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{8} & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 9/4 & 82/15 \\ 3/4 & 34/15 \end{pmatrix} \tag{44}$$

The equivalent resistance for the network shown in Fig. 3 finally gives:

$$R_E = \frac{R_{11}}{R_{21}} = \frac{\frac{9}{4}}{\frac{3}{4}} = 3\,\Omega \tag{45}$$

The problem of determining the value of a component when the equivalent resistance is given is now revisited. Setting $R_E = 3\,\Omega$, what should the resistor R_7 be if all other components have values as given before? To determine this use is made once again of $R_{21}R_E - R_{11} = 0$. Multiplying all the Q-matrices with the last one containing R_7 gives:

$$\mathbf{R} = \begin{pmatrix} 47/30 & 82/15 \\ 7/15 & 34/15 \end{pmatrix} \begin{pmatrix} \frac{1}{R_7} & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 47/30 + (82/15)\dfrac{1}{R_7} & 82/15 \\ 7/15 + (34/15)\dfrac{1}{R_7} & 34/15 \end{pmatrix} \tag{46}$$

Then using $3R_{21} - R_{11} = 0$ and solving for R_7 gives the expected value $R_7 = 8\,\Omega$. On the other hand if there are multiple unknown components that need to be determined for a given R_E, the process is the same except that the equation for their solution takes the form $R_x + R_y + R_z + \dots = C$, where C represents all the other known resistor components. An equation such as this is known as a linear Diophantine equation [11] and solving for $R_x, R_y, R_z, \dots$ is well documented. For example, consider Fig. 2 and suppose that R_E and all other resistor values are given as before except for R_2 and R_3. This network has been solved above and is given by Eq. (42). Re-arranging for R_2 and R_3 gives:

$$R_2 + R_3 = \frac{(R_0 - R_E)R_1}{R_E - R_0 - R_1} \tag{47}$$

Substituting all the known values on the right side of Eq. (47) gives the Diophantine equation: $R_2 + R_3 = 9$ which can be solved using a Euclidean algorithm to obtain the solutions $(R_2, R_3) = [(1,8),(2,7),(3,6),(4,5),(5,4),(6,3),(7,2),(8,1)]$. These are the possible values that the resistor components R_2 and R_3 can take for a fixed $R_E = 3.25$. The set also includes the original values for them, namely $R_2 = 5$ and $R_3 = 4$. Some of these solutions can be obtained by inspecting the Diophantine equation; however for a larger number of unknown components, this is not so easy.

4. CONCLUSION

A simple matrix method has been proposed that allows computation for the equivalent resistance as well as the determination of unknown components in simple resistor networks. It can also be used to compute continued fractions. It is computationally efficient since it only requires the multiplication of a 2×2 matrix $(N-1)$ times where N is the total number of components in the network.

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