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Compressed Sensing DOA Estimation in the Presence of Unknown Noise

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Abstract—A new compressive sensing-based direction of arrival (DOA) estimation technique for source signal detection in the presence of unknown noise, based on the generalized correlation decomposition (GCD) algorithm, is presented. The proposed algorithm does not depend on the singular value decomposition nor on the orthogonality of the signal and the noise subspaces. Hence, the DOA estimation can be done without an a priori knowledge of the number of sources. The proposed algorithm can estimate more sources than the number of physical sensors used without any constraints or assumptions about the nature of the signal sources. It can estimate coherent source signals as well as closely-spaced sources using a small number of snapshots.

1. INTRODUCTION

Array signal processing has been of great research interest over the past decades [1]. Array signal processing applications include radar [2], sonar [3], seismic event prediction [4], microphone sensors [5], and wireless communication systems [6]. Direction of arrival (DOA) estimation involves estimating the direction from which the signal sources impinge on a sensor array; which is a set of sensors arranged in a specific configuration, and able to measure the values of the impinging source signals. Common DOA estimation methods include conventional beamforming techniques [7], subspace-based techniques like MUSIC [8] and ESPRIT [9], and maximum likelihood (ML) methods [10, 11].

Most of the DOA estimation techniques are developed using uniform arrays, and the number of source signals to be estimated is upper bounded by the number of sensors in the array. For example, a uniform linear array (ULA) containing M sensors will be able to detect up to $(M-1)$ source signals. In order to increase the available degrees of freedom and consequently be able to estimate more sources than the available sensors, various techniques based on sparse sensor arrays, such as the minimum redundancy arrays (MRA) along with augmented covariance matrices (ACM) [12–14], Khatri-Rao (KR) product [15, 16], co-prime arrays [17–19], and nested arrays [20], have been presented. Among all these techniques, nested arrays have shown superior performance compared to that of the others [20].

Many of the DOA techniques that have been proposed assume spatially white noise [21–26]. Hence, the array noise covariance matrix is related to the noise power through an identity matrix. However, the assumption of spatially white noise is not realistic in many practical applications [27–35], where the noise fields are spatially colored. The colored noise significantly degrades the performance of the DOA estimator. Furthermore, estimating the number of signal sources becomes a problem. In addition, some of the peaks due to the non-white noise background may be identified as source signals.

To overcome this degradation, certain constraints are imposed on the signal or on the colored noise. In [23], the signal is assumed to be partially known as a linear combination of a set of basis functions, while in [36] the noise is modeled as an autoregressive process. However, these assumptions are still not

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realistic, and furthermore, if they are not satisfied, then the DOA performance will be highly degraded. In [16], an underdetermined KR based technique using a ULA was proposed for DOA estimation in unknown spatial noise covariance. However, the source signals are assumed to be quasi-stationary. Iterative methods using ULAs for DOA estimation in nonuniform noise were proposed in [37]. These methods are based on estimating the signal subspace and noise covariance matrices simultaneously. Yet, the number of sources to be estimated is assumed to be known in advance and the methods are computationally intensive.

Sparse arrays are used to avoid the above unrealistic assumptions for DOA estimation in the presence of spatially colored noise [28, 38]. In [38], the separation between the sub-arrays is chosen such that the noise is uncorrelated between the sub-arrays. In this situation, the noise covariance matrix has a block-diagonal structure, which allows the DOA estimation to be done accurately. In [28], the DOA estimation was explored using two separated sub-arrays and based on the generalized correlation (GC) analysis, a new method for DOA estimation in unknown noise (UN) fields known as, UN-MUSIC, is proposed for DOA estimation. However, two separate ULAs are used for the DOA estimation and in order to be able to decompose the received signal into its unique subspaces a long procedure is required. In [29], an ML technique on a sparse sensor array is proposed for DOA estimation in the presence of spatially colored noise. However, the technique requires a large number of snapshots. Furthermore, the algorithm is based on the ML technique, which is computationally the most intensive amongst the DOA estimation methods [39] and further, the number of sources to be estimated is assumed to be known a priori [40].

Since the source signals being received by the sensor arrays can be considered as sparse signals, compressive sensing (CS) techniques have received recent attention in array signal processing. Compressive sensing was introduced for the DOA estimation problem in [41], where a new recursive weighted minimum-norm algorithm known as the focal underdetermined system solver (FOCUSS) was applied to the DOA estimation problem. In the work developed by Fuchs [42, 43], it was assumed that the source signals are uncorrelated and that a large number of snapshots are available. Malioutov et al. [44] proposed a new ℓ_1-norm based on the singular value decomposition (SVD), namely, ℓ_1-SVD. However, in low signal to noise ratio (SNR), the signal subspace of the SVD will be dominated by the noise effect and the ℓ_1-SVD performance will be degraded. Salama et al. [45] developed a new adaptable least absolute shrinkage and selection operator (A-LASSO) for the DOA estimation problem. It was shown that A-LASSO-based DOA estimation performance is superior to that of both the classical and subspace-based DOA estimation techniques. Furthermore, the A-LASSO technique is able to detect the sources even in very low SNR scenarios. It should be pointed out that the above-mentioned techniques use ULA, except for [45]. Furthermore, the noise covariance structure is assumed to be known.

In this paper, using a single sparse linear array, we propose a new CS-based DOA estimation technique, called the generalized correlation decomposition (GCD) A-LASSO technique, that is capable of performing DOA estimation for source signals in the presence of unknown noise fields. In Section 2, we briefly describe the the co-array principle. In Section 3, GCD A-LASSO DOA estimation in unknown noise fields frameworks is presented. In Section 4, the performance of the proposed technique is studied using simulations and finally, conclusions are drawn in Section 5.

Notations

Superscript ^{H} denotes the conjugate transpose; superscript * denotes the conjugation without transpose; and ^{T} denotes the transpose operation. The symbol $\odot$ denotes the KR product [46] between two matrices of appropriate sizes.

2. DIFFERENCE CO-ARRAY

Consider L narrowband far-field source signals $s_1, \ldots, s_l, \ldots, s_L$, impinging on a linear array (LA), uniform or non-uniform, and consisting of M sensors, with angles of arrival (AOA) $\theta_1, \ldots, \theta_l, \ldots, \theta_L$. Let $x_m(t)$ denote the resulting signal related to the mth sensor with time index t, where $m = 1, \ldots, M$.

The output of the sensor array at the tth sample can be written as

$$\begin{aligned}\mathbf{x}(t) &= [\mathbf{a}(\theta_1)\ \mathbf{a}(\theta_2)\ \dots\ \mathbf{a}(\theta_L)] \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_L \end{bmatrix} + \mathbf{n}(t) \\ &= \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \end{aligned} \tag{1}$$

where $\mathbf{x} \in \mathbb{C}^{M\times 1}$ is the output of the sensor array; $\mathbf{A} \in \mathbb{C}^{M\times L} = [\mathbf{a}(\theta_1)\ \dots\ \mathbf{a}(\theta_l)\ \dots\ \mathbf{a}(\theta_L)]$ is the array manifold matrix, with $\mathbf{a}(\theta_l)$ being the steering vector corresponding to AOA (θ_l), whose ith element is $e^{-jk_o d_m \cos(\theta_l)}$; $\mathbf{s}(t) \in \mathbb{C}^{L\times 1} = [s_1 s_2 \dots s_L]^T$ is the vector which represents the source signals with $k_o = 2\pi/\lambda$ being the wavenumber, d_m the mth sensor position in the array, and λ the wavelength of the propagating waves; and $\mathbf{n}(t) \in \mathbb{C}^{M\times 1}$ is an additive white Gaussian noise (AWGN) that is uncorrelated with the source signals.

Given the sensor array output, $\mathbf{x}$, the covariance matrix of the received signals, $\mathbf{R}_{xx} \in \mathbb{C}^{M\times M}$, can be obtained as [7]

$$\begin{aligned}\mathbf{R}_{xx} &= \mathrm{E}\left[\mathbf{x}\mathbf{x}^H\right] \\ &= \mathbf{A}\mathbf{R}_{ss}\mathbf{A}^H + \sigma_n^2 \mathbf{I}\end{aligned} \tag{2}$$

where $\mathbf{R}_{ss} \in \mathbb{C}^{L\times L} = E[\mathbf{s}\mathbf{s}^H]$ is a block diagonal matrix containing the received source signal powers $\sigma_l^2,\ l = 1,\dots,L$; $\mathbf{I}$ is the identity matrix of size $(M \times M)$; and σ_n^2 is the noise power. One can now vectorize $\mathbf{R}_{xx}$ [15, 16] as

$$\mathbf{V} = \mathrm{vec}\left(\mathbf{R}_{xx}\right) = (\mathbf{A}^* \odot \mathbf{A})\mathbf{p} + \sigma_n^2 \mathbf{1} \tag{3}$$

where $\mathbf{p} \in \mathbb{C}^{L\times 1} = [\sigma_1^2\ \sigma_2^2\ \dots\ \sigma_L^2]^T$ and $\mathbf{1} \in \mathbb{C}^{M\times M} = [\mathbf{e}_1^T\ \dots\ \mathbf{e}_m^T\ \dots\ \mathbf{e}_M^T]^T$ with $\mathbf{e}_m \in \mathbb{C}^{M\times 1}$ being a column vector of zeros except for a 1 at the mth position. Comparing Eq. (3) with Eq. (1), we can see that $\mathbf{V} \in \mathbb{C}^{M^2\times 1}$ in Eq. (3) have the same structure as that of the output of a sensor array. However, in this case, the array manifold is given by $(\mathbf{A}^* \odot \mathbf{A})$, $\mathbf{p}$ representing the equivalent source signals and the noise given by $\sigma_n^2\mathbf{1}$ being deterministic. Further, it is noted that the dimension of $\mathbf{V}$ is M^2 which is greater than the physical array dimension M. Thus, underdetermined DOA estimation can be performed. The distinct rows of $(\mathbf{A}^* \odot \mathbf{A})$ form a linear virtual array, and the locations of whose distinct sensors are given by the set

$$D = \{d_i - d_j\}, \quad \forall i, j = 1, 2, \dots, M \tag{4}$$

where d_i is the position vector of the ith sensor in the original array. This array is known as the *difference co-array* [20].

We now assume that the original array is a two-level nested array [20]. Detailed information about the number of sensors in each level, the distinct sensors in the *difference co-array*, $\bar{M}$, and the maximum number of source signals that can be estimated for odd and even M sensors, are as shown in Table 1. In each case, the virtual array is a ULA consisting of $\bar{M}$ sensors which are located from $-(\bar{M}-1)d/2$ to $(\bar{M}-1)d/2$ [20, 45].

Table 1. Sensors distribution for a two-level nested array.

M	1st level	2nd level	$\bar{M}$	Max L
Odd	$(M-1)/2$	$(M+1)/2$	$(M^2-1)/2+M$	$((M^2-1)/2+M-1)/2=(\bar{M}-1)/2$
Even	$M/2$	$M/2$	$M^2/2+M-1$	$(M^2/2+M-2)/2=(\bar{M}-1)/2$

It should be noted that the equivalent source signal vector $\mathbf{p}$ (for the difference co-array) contains the sources powers σ_l^2, $l = 1,\dots,L$. Therefore, they act like fully-correlated sources. A spatial smoothing technique was suggested by Pal and Vaidyanathan [20] to overcome this problem of correlated sources. However, in [45], it was shown that using CS-based DOA estimation technique, spatial smoothing is no longer needed, and source signals could be estimated directly without any preprocessing scheme.

3. GCD A-LASSO FOR DOA ESTIMATION IN UNKNOWN NOISE FIELDS

Taking into account that the source signals are far-field sources, they can be considered as point sources and consequently become sparse in space. Hence, the output of the sensor array, $\mathbf{y} \in \mathbb{C}^{\bar{M}\times 1}$, can be expressed as

$$\mathbf{y}(t) = \Phi\bar{\mathbf{s}}(t) + \bar{\mathbf{n}}(t) \tag{5}$$

where $\Phi \in \mathbb{C}^{\bar{M}\times N}$ is the overcomplete steering matrix and is given by

$$\begin{aligned}\Phi &= [\mathbf{a}'(\bar{\theta}_1)\ \mathbf{a}'(\bar{\theta}_2)\ \dots\ \mathbf{a}'(\bar{\theta}_n)\ \dots\ \mathbf{a}'(\bar{\theta}_N)] \\ &= \begin{bmatrix} e^{jk_o d(-(\bar{M}-1)/2)\cos\bar{\theta}_1} & e^{jk_o d(-(\bar{M}-1)/2)\cos\bar{\theta}_2} & \dots & e^{jk_o d(-(\bar{M}-1)/2)\cos\bar{\theta}_N} \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \ddots & 1 \\ \vdots & \vdots & & \vdots \\ e^{jk_o d((\bar{M}-1)/2)\cos\bar{\theta}_1} & e^{jk_o d((\bar{M}-1)/2)\cos\bar{\theta}_2} & \dots & e^{jk_o d((\bar{M}-1)/2)\cos\bar{\theta}_N} \end{bmatrix}\end{aligned} \tag{6}$$

and $\bar{\mathbf{n}} \in \mathbb{C}^{\bar{M}\times 1}$ is an AWGN. Denoting $\mathbf{a}'(\bar{\theta}_n) \in \mathbb{C}^{\bar{M}\times 1}$ as the steering vector of the virtual array corresponding to AOA of $(\bar{\theta}_n)$, where $\{\bar{\theta}_n\}_{n=1}^{N}$ denotes a grid that covers the set of all possible locations, Ω and $N \gg L$. In this case, the source signal vector $\bar{\mathbf{s}} \in \mathbb{C}^{N\times 1}$ is given by

$$\bar{\mathbf{s}}(t) = [\bar{\sigma}_1\ \bar{\sigma}_2\ \dots\ \bar{\sigma}_n\ \dots\ \bar{\sigma}_N]^T \tag{7}$$

where the nth element of $\bar{\mathbf{s}}(t)$, $\bar{s}_n(t)$, is nonzero only if $(\bar{\theta}_n = \theta_l)$ and, in that case, $\bar{\sigma}_n = \sigma_l$. The compressing sensing (CS) technique is to estimate the signal energy as a function of the source signal locations given the sensor array output, $\mathbf{y}$. In a noise free scenario, a direct way to investigate the sparsity on $\bar{\mathbf{s}}$ is by minimizing the ℓ_0-norm, which counts the number of nonzero elements in the vector $\bar{\mathbf{s}}$, as follows

$$\min_{\bar{\mathbf{s}}} \|\bar{\mathbf{s}}\|_0 \text{ subject to } \mathbf{y} = \Phi\bar{\mathbf{s}} \tag{8}$$

However, this minimization is an NP-hard problem [47], which becomes, even for a moderate dimensional problem, computationally intractable. For that reason, different alternative approaches to approximate the solution of ℓ_0-norm problems were presented in [47–50]. It has been proven that, for sufficiently sparse signals and sensing matrices with sufficiently incoherent columns [51, 52], the ℓ_0-norm problem is equivalent to the ℓ_1-norm one [53–55], where ℓ_1 minimization is given by

$$\min_{\bar{\mathbf{s}}} \|\bar{\mathbf{s}}\|_1 \text{ subject to } \mathbf{y} = \Phi\bar{\mathbf{s}} \tag{9}$$

Furthermore, ℓ_2-norm could be used as an alternative approach to solve ℓ_0-norm problem by relaxing ℓ_0-norm into ℓ_2-norm as follows

$$\min_{\bar{\mathbf{s}}} \|\bar{\mathbf{s}}\|_2 \text{ subject to } \mathbf{y} = \Phi\bar{\mathbf{s}} \tag{10}$$

which is a convex problem and has an analytic solution given by

$$\hat{\bar{\mathbf{s}}} = \Phi^H \left(\Phi\Phi^H\right)^{-1} \mathbf{y} \tag{11}$$

However, ℓ_1-norm problem favors sparse signals than the ℓ_2-norm. Furthermore, ℓ_1-norm relaxation is the closest convex optimization to that of the ℓ_0-norm and it converges to the global minimum [56]. In practice, CS can be extended to noisy measurement scenarios. The ℓ_1-norm problem for a noisy measurement can written as

$$\min_{\bar{\mathbf{s}}} \|\bar{\mathbf{s}}\|_1 \text{ subject to } \|\Phi\bar{\mathbf{s}} - \mathbf{y}\|_2 \le \beta \tag{12}$$

where β is an error tolerance parameter ($\beta > 0$). The ℓ_2-norm used for evaluating the error $\Phi\bar{\mathbf{s}} - \mathbf{y}$ can be replaced by any other norm, such as ℓ_∞ or ℓ_p, $0 < p < 1$. Proper choice of β is an important issue for the success of minimization in Eq. (12) [57, 58]. An ℓ_1-norm constrained form of Eq. (12) is known as LASSO [59]. The LASSO minimization problem can be written as

$$\min_{\bar{\mathbf{s}}} \|\mathbf{y} - \Phi\bar{\mathbf{s}}\|_2^2 + \tau \|\bar{\mathbf{s}}\|_1 \tag{13}$$

where τ is a nonnegative regularization parameter. The ℓ_1 penalization approach is also known as the *basis pursuit* [60]. Two iterative versions of LASSO, namely, the ordinary least squares (OLS) A-LASSO and the minimum variance distortionless response (MVDR) A-LASSO, were introduced in [45]. It was shown that the performance of these A-LASSO techniques is superior to that of the classical DOA estimation techniques and LASSO-based DOA estimation. The A-LASSO is given by [45]

$$\hat{\bar{\mathbf{s}}}^{(k)} = \min_{\bar{\mathbf{s}}} \|\mathbf{y} - \Phi\bar{\mathbf{s}}\|_2^2 + \tau_k \sum_{n=1}^{N} \hat{w}_n |\bar{s}_n| \tag{14}$$

where k is the iteration number, and $\hat{w}_n$ is the n-th element of the weight vector, $\hat{\mathbf{w}}$ which is given by OLS or MVDR in the first iteration, $k = 1$.

It should be pointed out that in most of the CS-based DOA estimation techniques, the noise covariance structure is known in advance; it is assumed to be AWGN (see [1] and the references therein). However, in practice, this assumption does not hold and the noise covariance structure is probably unknown. Thus, the DOA estimator performance is highly degraded when the noise covariance is not known. Further, in such scenarios, more false source signal peaks could appear due to the background noise.

In order to overcome the above mentioned problem, we adopt the following technique for signal source DOA estimation in unknown correlated noise fields [28]. Consider two ULAs whose output vectors can be written as

$$\begin{aligned} \mathbf{x}_1(t) &= \mathbf{A}_1\mathbf{s}_1(t) + \mathbf{n}_1(t) \\ \mathbf{x}_2(t) &= \mathbf{A}_2\mathbf{s}_2(t) + \mathbf{n}_2(t) \end{aligned} \tag{15}$$

where $\mathbf{x}_1(t)$ and $\mathbf{x}_2(t)$ are the data vectors of dimensions M_1 and M_2, respectively; $\mathbf{A}_1 \in \mathbb{C}^{M_1 \times L}$ and $\mathbf{A}_2 \in \mathbb{C}^{M_2 \times L}$ are the steering matrices of the arrays; and $\mathbf{s}_1(t)$ and $\mathbf{s}_2(t)$ are the signal vectors. The outputs of the two sub-arrays can be considered to be the same, but one is a delayed version of the other. The noise vectors $\mathbf{n}_1(t)$ and $\mathbf{n}_2(t)$ are assumed to be stationary, zero-mean, Gaussian with the joint covariance, $\mathbf{J}$, given by

$$\mathbf{J} = \left\{ \begin{bmatrix} \mathbf{n}_1 \\ \mathbf{n}_2 \end{bmatrix} \begin{bmatrix} \mathbf{n}_1^H & \mathbf{n}_2^H \end{bmatrix} \right\} = \begin{bmatrix} \mathbf{R}_{nn_1} & 0 \\ 0 & \mathbf{R}_{nn_2} \end{bmatrix} \tag{16}$$

$\mathbf{R}_{nn_1}$ and $\mathbf{R}_{nn_2}$ are unknown covariance matrices of the noise of the two sub-arrays. The joint covariance matrix of the received data from the two sub-arrays, $\Sigma \in \mathbb{C}^{2(M_1+M_2)\times 2(M_1+M_2)}$, can be written as [28]

$$\Sigma = \left\{ \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^H & \mathbf{x}_2^H \end{bmatrix} \right\} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix} \tag{17}$$

where

$$\begin{aligned} \mathbf{R}_{ii} &= \mathbf{A}_i\mathbf{R}_{ss_i}\mathbf{A}_i^H + \sigma_n^2\mathbf{R}_{nn_i}, \quad i = 1, 2 \\ \mathbf{R}_{12} &= \mathbf{R}_{21}^H = \mathbf{A}_1\mathbf{R}_{ss_{12}}\mathbf{A}_2^H \end{aligned} \tag{18}$$

where $\mathbf{R}_{ss_i}$ is the auto-correlation, and $\mathbf{R}_{ss_{12}}$ is the cross-correlation of the signals such that

$$\begin{aligned} \mathbf{R}_{ss_i} &= \mathrm{E}\left\{\mathbf{s}_i\mathbf{s}_i^H\right\}, \quad i = 1, 2 \\ \mathbf{R}_{ss_{12}} &= \mathrm{E}\left\{\mathbf{s}_1\mathbf{s}_2^H\right\} \end{aligned} \tag{19}$$

and both are assumed to be of full rank. In practice, we do not know the true value of Σ, and therefore, we use the average of the outer products of the output data as an estimate of Σ such that

$$\widehat{\Sigma} = \frac{1}{T}\sum_{n=1}^{N} \begin{bmatrix} \mathbf{x}_1(n) \\ \mathbf{x}_2(n) \end{bmatrix} \begin{bmatrix} \mathbf{x}_1^H(n) & \mathbf{x}_2^H(n) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{R}}_{11} & \hat{\mathbf{R}}_{12} \\ \hat{\mathbf{R}}_{21} & \hat{\mathbf{R}}_{22} \end{bmatrix} \tag{20}$$

where T is the snapshot number. It should be noted that $\mathbf{R}_{12}$ and $\mathbf{R}_{21}$ contain noiseless DOA information. So, we can proceed using any technique such as MUSIC [8] to estimate the DOA. However, the signal subspace estimation from $\hat{\mathbf{R}}_{12}$ is not unique. To uniquely estimate the signal subspace from $\hat{\mathbf{R}}_{12}$, the GCD is used to develop the UN-MUSIC algorithm in [28].

Consider now a two-level nested array containing an odd number of sensors, M, as presented in Section 2; the resulting virtual array will contain $\bar{M}$ virtual sensors, as given in Table 1. Assume that the first sub-array contains the virtual sensors from the first virtual sensor to the $(\bar{M}-Q)$-th sensor and the second sub-array contains the sensors from $(Q+1)$-th to the last virtual sensor, so that the total number of sensors in each of the sub-arrays is $\bar{M}-Q$. It should be pointed out that the maximum number of sources to be estimated will be affected by Q and is given by $(\bar{M}-Q-1)/2$. Due to the overlapping of the two sub-arrays and because the source signals are assumed to be located in the far-field, the steering matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of the two sub-arrays could be assumed to be the same, that is $\mathbf{A}_1 = \mathbf{A}_2 = \bar{\mathbf{A}}$, where $\bar{\mathbf{A}}$ is the steering matrix of the sensor array for which the total number of sensors is $\bar{M}-Q$ and is given by

$$\bar{\mathbf{A}} = \left[\bar{\mathbf{a}}(\bar{\theta}_1)\ \bar{\mathbf{a}}(\bar{\theta}_2)\ \ldots\ \bar{\mathbf{a}}(\bar{\theta}_N)\right] \tag{21}$$

where $\bar{\mathbf{a}}(\bar{\theta}_n) \in \mathbb{C}^{(\bar{M}-Q)\times 1}$ as the steering vector of the sensor array whose $\bar{m}$th element can be written as

$$\bar{a}_{\bar{m}}(\bar{\theta}_n) = e^{jk_o d\bar{m}\cos\bar{\theta}_n}, \quad \bar{m} = \begin{cases} -\frac{\bar{M}-Q-1}{2}, \ldots, \frac{\bar{M}-Q-1}{2} & \text{if } Q \text{ is even} \\ -\frac{\bar{M}-Q}{2}+1, \ldots, \frac{\bar{M}-Q}{2} & \text{if } Q \text{ is odd} \end{cases} \tag{22}$$

Consider extracting $\hat{\mathbf{R}}_{12} \in \mathbb{C}^{\bar{M}-Q\times\bar{M}-Q}$ from Eq. (20) which can be written as

$$\hat{\mathbf{R}}_{12} = \mathbf{A}_1\mathbf{R}_{ss_{12}}\mathbf{A}_2^H = \bar{\mathbf{A}}\mathbf{R}_{ss_{12}}\bar{\mathbf{A}}^H \tag{23}$$

Following linear algebra theory, each column (vector) of $\hat{\mathbf{R}}_{12}$ can be linearly represented by any complete basis in the $(\bar{M}-Q)$-dimensional complex vector space [61]. The qth column of $\hat{\mathbf{R}}_{12}$ can be written as

$$\hat{\mathbf{r}}_q = \Phi^*\mathbf{b}_q, \quad q = 1, \ldots, \bar{M}-Q \tag{24}$$

where $\mathbf{b}_q$ is the representation coefficient vector in terms of the overcomplete steering matrix, and $\Phi^* \in \mathbb{C}^{(\bar{M}-Q)\times N}$ is the overcomplete steering matrix for the sensor array for which the total number of sensors is $\bar{M}-Q$. In matrix form, Eq. (24) can be written as

$$\hat{\mathbf{R}}_{12} = \Phi\mathbf{B} \tag{25}$$

where $\mathbf{B} = [\mathbf{b}_1, \ldots, \mathbf{b}_q, \ldots, \mathbf{b}_{\bar{M}-Q}]$. It should be noted that $\{\mathbf{b}_q\}_{q=1}^{\bar{M}-Q}$ have the same sparsity structure, i.e., the non-zero elements of each vector of $\mathbf{B}$ appear in the same index [61]. Based on Eq. (24), DOA estimation is the same as seeking the sparsity of $\mathbf{b}_q$, which has the same structure as that of the signal to be estimated. Using Eq. (24), the DOA estimation problem can be reformulated using A-LASSO [45] as follows:

$$\mathbf{b}_q^{(k)} = \min_{\mathbf{b}_q} \|\hat{\mathbf{r}}_q - \Phi\mathbf{b}_q\|_2^2 + \tau_k \sum_{n=1}^{N} \hat{w}_n |b_{q_n}| \tag{26}$$

We denote Eq. (26) as GCD A-LASSO. Algorithm 1 illustrates single iteration of the GCD A-LASSO technique. Following [45], two initial weights are considered for the first iteration ($k = 1$) of the GCD A-LASSO algorithm, these initial weights are given by OLS or MVDR. Depending on whether OLS or MVDR weights are used as initial weights, the algorithm will be known as GCD OLS A-LASSO or GCD MVDR A-LASSO, respectively.

4. EXPERIMENTAL RESULTS

Consider a sparse linear two-level nested array, for which M is odd, consisting of three elements in the first level and with four elements in the second level. Thus, the total number of sensors is $M = 7$, as shown in Fig. 1. Investigating the array output by applying Eqs. (1)–(3), and extracting the equivalent distinct virtual elements from the virtual array manifold $(\mathbf{A}^* \odot \mathbf{A})$, one can see that the virtual array is a uniform linear array containing $\bar{M} = 31$ elements. The number of sensors in each of the two sub-arrays of the virtual array is chosen to be 29, that is, $Q = 2$. As a sequence, we see from Eq. (22) that $\bar{m} = (-14, \ldots, 14)$. We shall call such an array of antennas as array #1. The sampling grid

Algorithm 1 GCD A-LASSO

1: Collect T snapshots of the received signals, $\mathbf{x}(t)$.

2: Calculate the covariance matrix, $\mathbf{R}_{xx}$.

3: Vectorize $\mathbf{R}_{xx}$ and construct the virtual sensor array output as given in Section 2.

4: Divide the virtual array into two equal uniform linear sub-array with $\bar{M}-Q$ virtual sensor in each sub-array.

5: Calculate the joint covariance matrix, Σ, from Eq. (20) and extract $\hat{\mathbf{R}}_{12}$ from the result.

6: Select q-th column of $\hat{\mathbf{R}}_{12}$ where $q = 1, \ldots, \bar{M}-Q$.

7: Compute the initial estimate for the signal, $\bar{s}$, using OLS or MVDR as initial weights.

8: Find $\hat{\mathbf{w}}$, where the n-th element of $\hat{\mathbf{w}}$, $\hat{w}_n$, is given by $\hat{w}_n = 1/|\hat{s}_n|^{\gamma}, \ n = 1, \ldots, N$.

9: Define $\Phi' \in \mathbb{C}^{(\bar{M}-Q)\times N}$ matrix, such that its (q, n)-th element is given by $\phi_{qn}/\hat{w}_n$, where $q = 1, \ldots, \bar{M}-Q$ and $n = 1, \ldots, N$.

10: **for** $k = 1, 2, \ldots, K$ iterations **do**

Solve the LASSO problem as:

$$\mathbf{b}_q^* = \min_{\mathbf{b}_q} \left\|\hat{\mathbf{r}}_q - \Phi'\mathbf{b}_q\right\|_2^2 + \tau_k \left\|\mathbf{b}_q\right\|_1$$

Calculate $b^{(k)} = b_n^*/\hat{w}_n$, $n = 1, 2, \ldots, N$.

11: **end for**

12: Find the final DOA estimation.

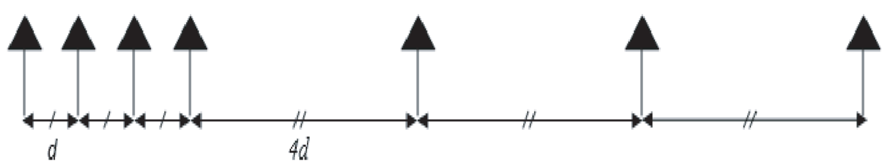


Figure 1. Two level nested array with 7 elements.

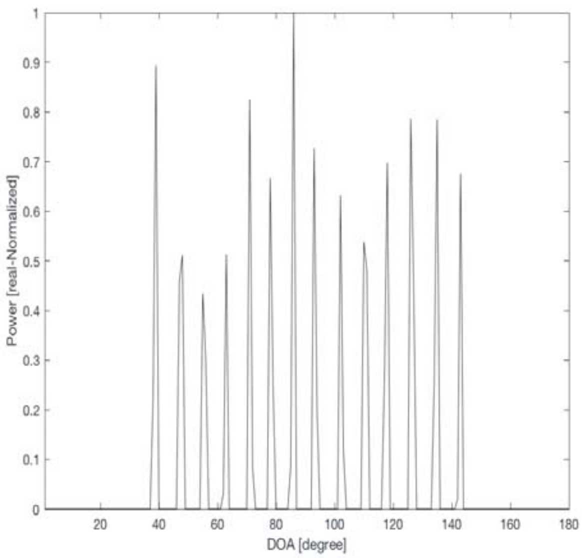
$\bar{\theta}_n \in [1° : 180°]$ that covers Ω is chosen to be of 1° step. The received signals are assumed to be contaminated by a mixture of correlated noise and AWGN in all the simulations.

The signal sources are modeled as $e^{j2\pi f_d t}$ where f_d is the Doppler frequency, and 10 snapshots are assumed for the simulations, except in the first simulation. A single iteration A-LASSO [45] is considered for all the simulations. All the simulated source signals are assumed to be equi-power and uncorrelated with one another or with the noise except in the third simulation, where the sources are assumed to be correlated with each other. The total number of trials, N_{sim}, is set to 100 for each observation point. For each experiment, the regularization parameter, τ, is selected based on the idea of the L-Curve [62, 63] and following the same procedure as given in [45].

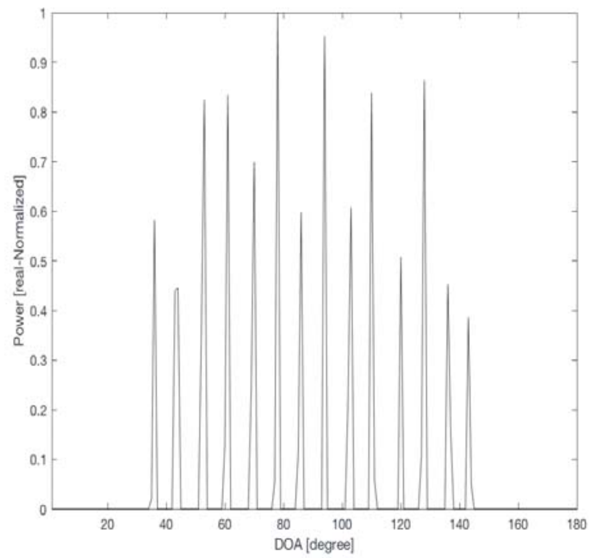
The CVX toolbox [64, 65] for convex optimization that is available within the MATLAB environment is used for examining the performance of the proposed algorithms.

We investigate, in the first experiment, the capabilities of the proposed algorithm in detecting the sources even when the number of sources exceeds the number of physical array elements in the presence of unknown noise fields. In other words, the proposed algorithm is for an underdetermined DOA scenario. For that purpose, let 14 fixed source signals impinge the array from uniformly distributed DOAs over $\theta = [38°, 142°]$. The number of snapshots is set to 100, SNR is set to 0 dB and the noise is a mixture of AWGN and pink noise.

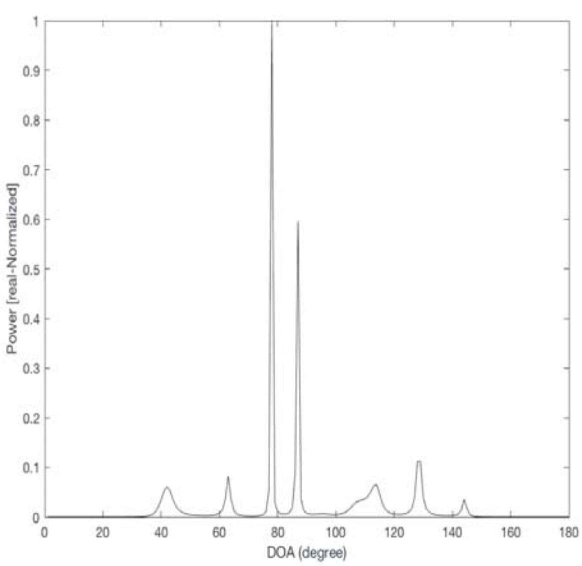
For UN-MUSIC simulations, two different scenarios are assumed. In the first scenario, two separate ULAs are considered for the simulation and the number of sensors in each one of them is chosen to be 7, that is, $M_1 = M_2 = 7$ (the same number of sensors as that of the two-level nested array). Such an array will be denoted as array #2. However, using this scenario, we cannot identify the 14 source signals because the maximum number of sources that can be estimated using UN-MUSIC is upper limited by the number of the used sensors, that is, $L_{\max} < \{M_1, M_2\}$ [28]. Hence, one can detect only up to 6 source signals using UN-MUSIC. Furthermore, the number of sources to be estimated is assumed to be known in advance in the UN-MUSIC technique.



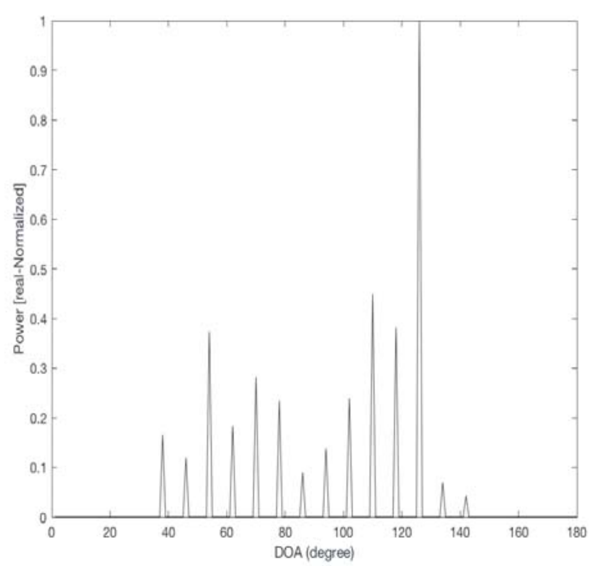
(a) GCD MVDR A-LASSO, array #1, 10 snapshots



(b) GCD OLS A-LASSO, array #1, 10 snapshots



(c) UN-MUSIC, array #3, 10 snapshots



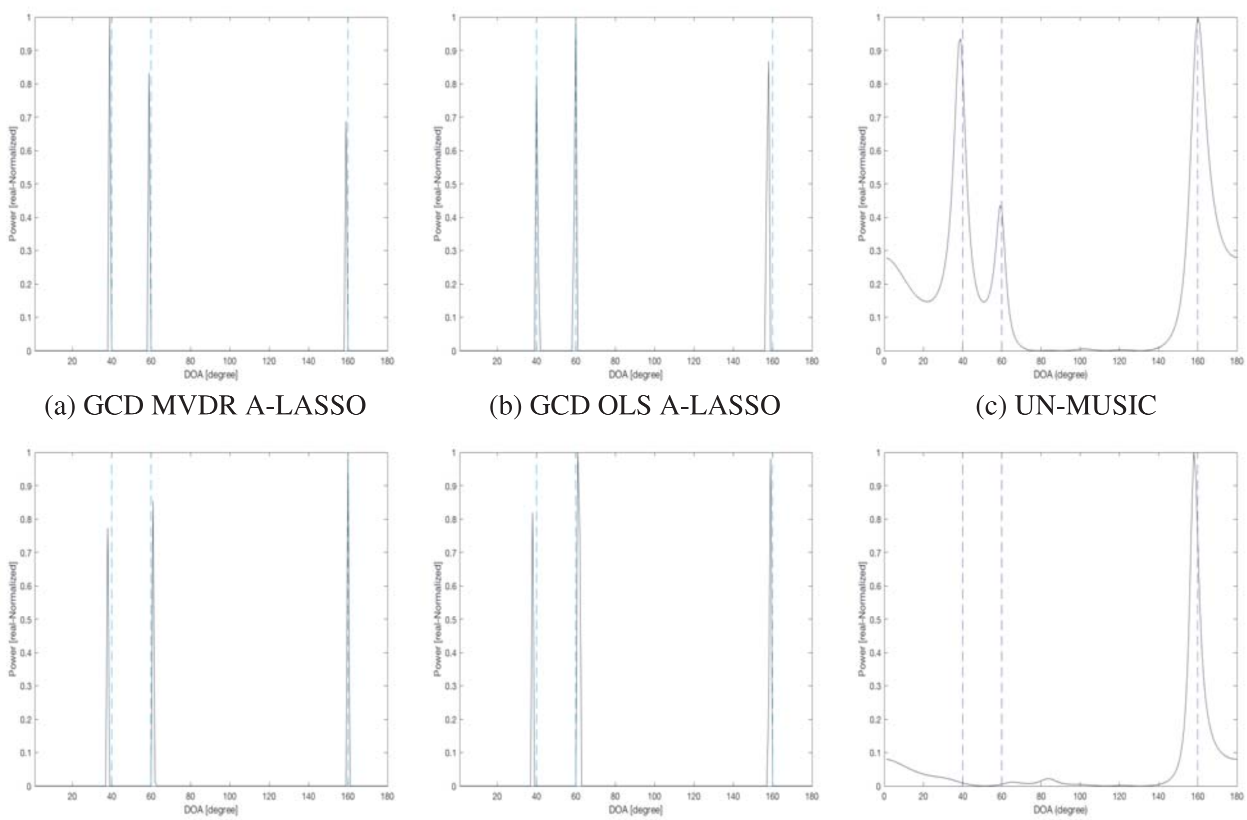
(d) UN-MUSIC, array #3, 105 snapshots

Figure 2. DOA estimation when the number of sources is more than the number of sensors, 200 snapshots, SNR $= 0\,\text{dB}$, using (a), (b) array #1 and 10 snapshots, (c) array #3 and 10 snapshots, and (d) array #3 and 10^5 snapshots.

We therefore assume, in the second scenario, for UN-MUSIC technique, two separate ULAs each containing 15 elements, that is, $M_1 = M_2 = 15$. Therefore, the maximum number of sources that can be estimated using this array, which we shall call array #3, is 14 sources [28].

Simulations are carried out on array #1 using both of the proposed techniques, namely, GCD OLS A-LASSO and GCD MVDR A-LASSO, as well as array #3 using the UN-MUSIC technique to identify the 14 sources. The results are as shown in Fig. 2. Figs. 2(a) and 2(b), show that all of the 14 source signals are identified correctly by both the GCD OLS A-LASSO and GCD MVDR A-LASSO, even in the presence of unknown noise, whereas even when we use 15 sensors and theoretically the maximum number of sources that can be identified is 14, UN-MUSIC has identified only 6 source signals, see Fig. 2(c). However, increasing the number of snapshots to be 10^5 snapshots, UN-MUSIC is able to identify the 14 sources as shown in Fig. 2(d). This clearly shows the capability of the proposed techniques in being able to identify all the sources $((\bar{M} - Q - 1)/2)$, which is exactly the maximum number of sources that our method is supposed to be able to identify. In view of this result, for the succeeding experiments we assume the number of sources to be three, except in the last one where we assume only 2 sources.

In the second experiment, we consider two cases: (a) three uncorrelated signals impinging on array #1 and 2 at $40°, 60°$ and $160°$, and (b) three signals impinging at the same angles, but with the first two signals being fully correlated (coherent). The received signal is assumed to be contaminated by pink noise and AWGN with SNR set to 0 dB and only 10 snapshots are employed. Figs. 3(a), (b), and (c) show that all the three techniques (namely, the two proposed and the UN-MUSIC) are able to identify the three signals when they are uncorrelated. However, when two of the sources are correlated, Figs. 3(d),



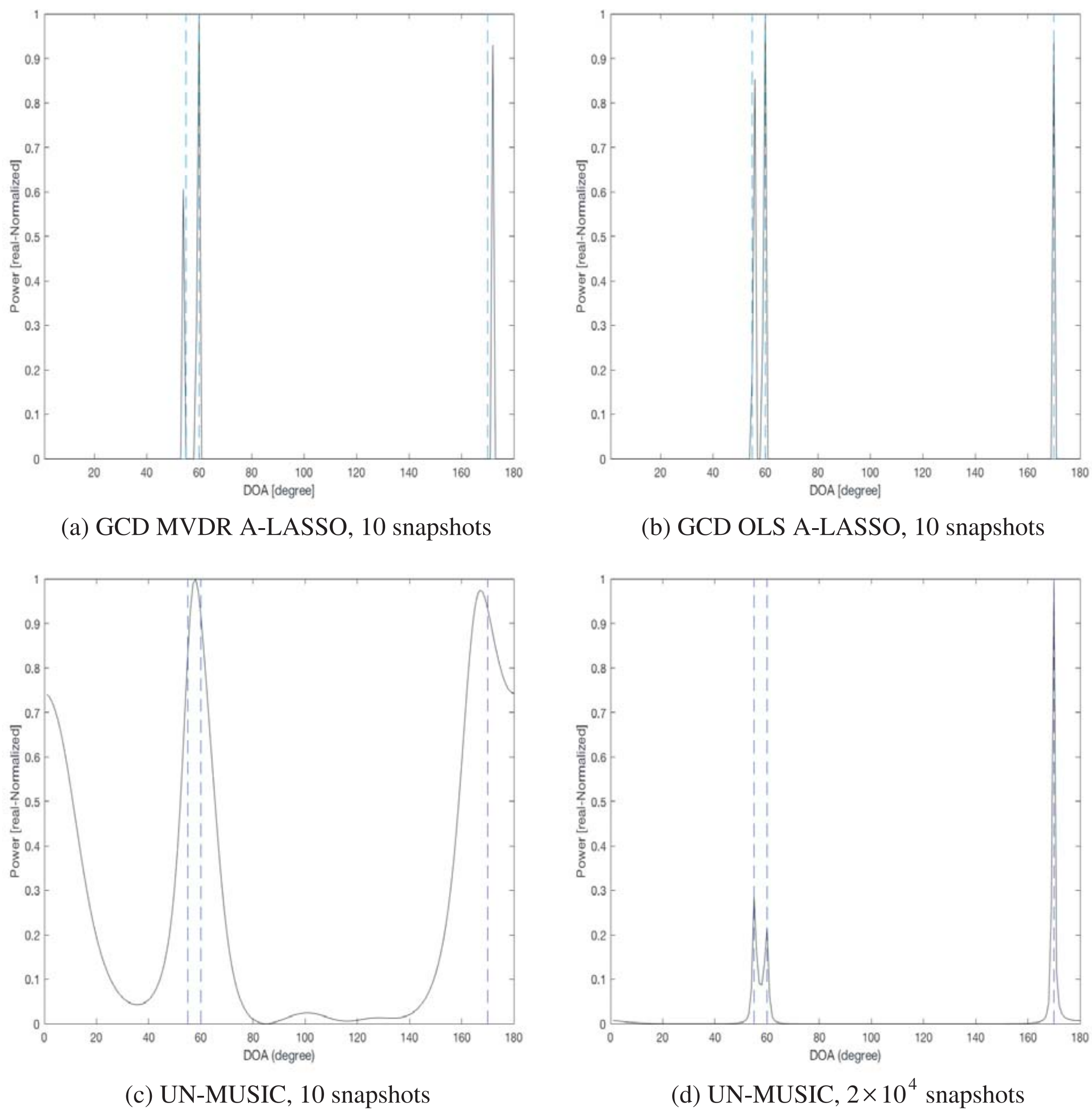
(a) GCD MVDR A-LASSO (b) GCD OLS A-LASSO (c) UN-MUSIC

(d) GCD MVDR A-LASSO (e) GCD OLS A-LASSO (f) UN-MUSIC

Figure 3. DOA estimation for 3 source signals with DOAs 40°, 60° and 160°, where the first 2 source signals are fully correlated, 10 snapshots, and pink noise and AWGN with SNR = 0 dB, using (a) and (d) GCD MVDR A-LASSO, (b) and (e) GCD MVDR A-LASSO, and (c) and (f) UN-MUSIC, (a)–(c) uncorrelated sources and (d)–(f) correlated sources.

(e), and (f) show that all the source signals are correctly identified by the two proposed techniques, whereas UN-MUSIC is not able to do so. In fact, UN-MUSIC is unable to identify correlated signals.

In the third experiment, we examine the capability of the proposed techniques in identifying closely-spaced sources. For this purpose, let three sources impinge arrays #1 and 2 form DOAs of 55°, 60°, and 170°. The received signal sources are assumed to be contaminated with pink noise with SNR is set to 0 dB, and 10 snapshots of the received data are used. The simulations results are shown in Fig. 4. From this figure, three peaks can easily be identified using GCD MVDR A-LASSO and GCD MVDR A-LASSO, thus identifying the three sources. However, UN-MUSIC is not able to identify the sources properly, and the two closely-spaced sources are identified as a single source. However, increasing the



(a) GCD MVDR A-LASSO, 10 snapshots

(b) GCD OLS A-LASSO, 10 snapshots

(c) UN-MUSIC, 10 snapshots

(d) UN-MUSIC, 2×10^4 snapshots

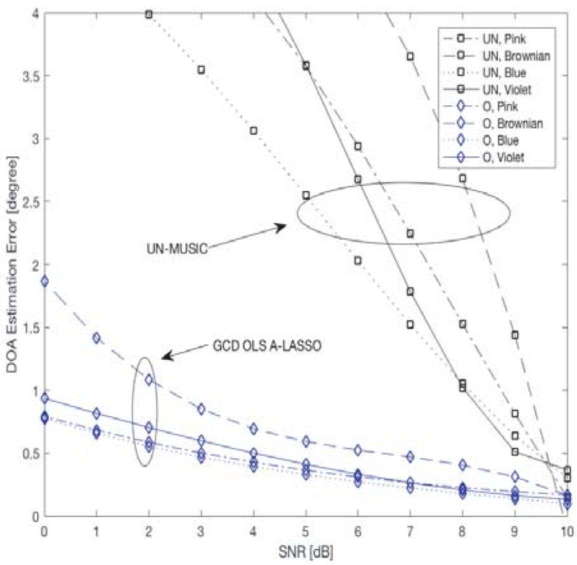
Figure 4. DOAA estimation for spatially closed two source signals, pink Noise with $SNR = 0\,\text{dB}$, two source signals at DOAs 60° and 64°, 10 snapshots, using (a) GCD MVDR A-LASSO, (b) GCD MVDR A-LASSO, and (c) and (d) UN-MUSIC.

number of snapshots to be 2×10^4 snapshots, UN-MUSIC is able to discriminate the sources.

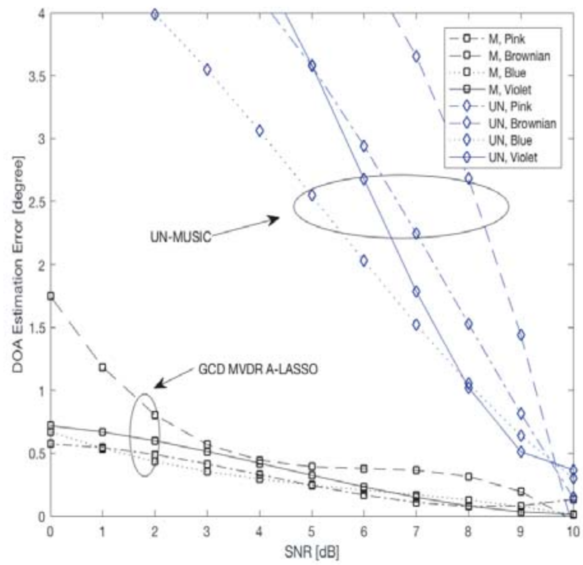
In the next two experiment, the root mean square error (RMSE) is used as the performance measure, which is given by

$$\mathrm{RMSE} = \frac{1}{L}\sum_{l=1}^{L}\sqrt{\frac{1}{N_{sim}}\sum_{n=1}^{N_{sim}}\left(\widehat{\theta}_{l,n} - \theta_l\right)^2} \tag{27}$$

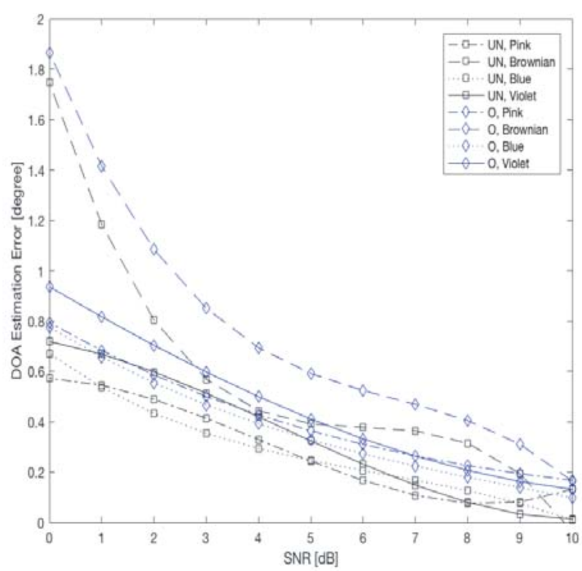
where $\widehat{\theta}_{l,n}$ is the estimate of the DOA angle θ_l of the n-th Monte Carlo trial.



(a) UN-MUSIC VS. GCD OLS A-LASSO



(b) UN-MUSIC VS. GCD MVDR A-LASSO



(c) GCD OLS A-LASSO vs. GCD MVDR A-LASSO

Figure 5. Performance comparison of the different algorithms as SNR is varied using UN-MUSIC, GCD MVDR A-LASSO (single iteration), and GCD OLS A-LASSO (single iteration). (a) GCD MVDR A-LASSO vs. GCD MVDR A-LASSO, (b) UN-MUSIC vs. GCD MVDR A-LASSO, and (c) UN-MUSIC vs. GCD MVDR A-LASSO.

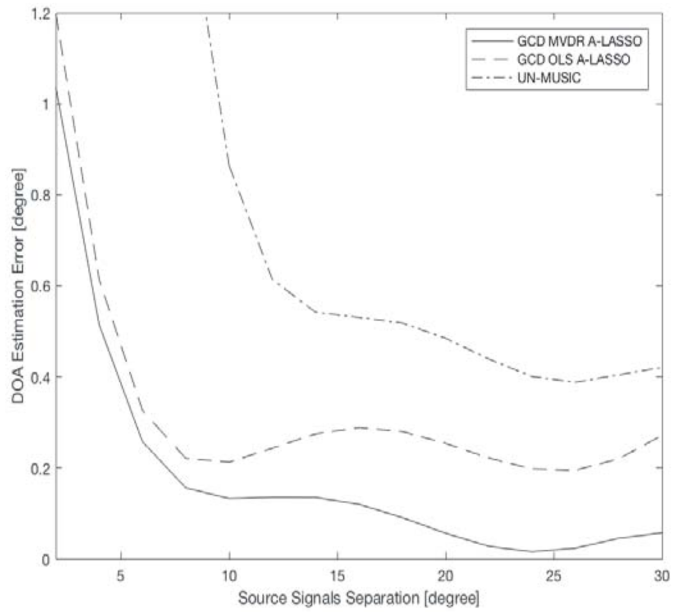


Figure 6. DOA estimation error for two sources as a function of separation between the two sources, SNR = 5 dB, 10 snapshots.

In the fourth experiment, we investigate the performance of the GCD MVDR A-LASSO, GCD MVDR A-LASSO, and UN-MUSICC algorithms as we vary SNR. For this purpose, let three source signals impinge on the arrays from DOA of 60°, 70° and 120°. For UN-MUSIC, as before two separated ULAs are used wherein each ULA contains 7 sensors ($M_1 = M_2 = 7$). The performance of the various algorithms as SNR is varied is shown in Fig. 5. It is observed from the figure that both GCD OLS A-LASSO and GCD MVDR A-LASSO outperform UN-MUSIC algorithm for the four assumed different noise mixtures. Furthermore, GCD MVDR A-LASSO performs better than GCD OLS A-LASSO for all the different noise mixtures.

The final experiment involves the investigation of the effect of varying the angular separation between the source signals. Consider two source signals, the first being held fixed at DOA of 60°, while for the second one the DOA ranges from 62° to 90° in steps of 2°. The SNR is set to be 5 dB, 10 snapshots are considered for the simulation, 100 trials for each observation point, and a sampling grid varying from 1° to 180° with 1° steps. In UN-MUSIC, for source signals with separation $\leq 10°$, the DOA estimation error is high. Hence, the simulations for the UN-MUSIC are conducted starting form a source signal separation $> 10°$. Fig. 6 illustrates the DOA estimation error as a function of the angular separation between the two source signals using the proposed DOA estimation techniques. It can be seen from this figure that, the performance of GCD OLS A-LASSO and GCD MVDR A-LASSO are superior to that of the UN-MUSIC technique. Moreover, The DOA estimation error for of the GCD MVDR A-LASSO technique is always less than that of the GCD OLS A-LASSO; in fact the DOA of GCD MVDR A-LASSO estimation error is $< 0.2°$ for an angular separation of $\geq 8°$.

5. CONCLUSION

This paper has presented two novel techniques using the compressive sensing framework on a sparse linear array for DOA estimation in the presence of unknown noise; based on the generalized correlation decomposition (GCD); these have referred to as GCD OLS A-LASSO and GCD MVDR A-LASSO, depending on whether ordinary least squares or minimum variance distortionless response is used as the initial weights. The performance of the proposed techniques is studied and compared with that of the UN-MUSIC technique. Neither of the proposed techniques require a priori knowledge about the number of source signals. The proposed algorithms are able to perform the DOA estimation using a small number of snapshots and are able to estimate correlated source signals and closely-spaced source signals in the presence of unknown noise. The proposed algorithms can identify source signals of order $O(M^2)$ using an array of order $O(M)$ sensor, with high resolution. For UN-MUSIC, even when the number of antennas is more than the number of sources, it is not able to distinguish source signals that are close to one another nor able to identify coherent sources. Even when the sources are not correlated, the UN-MUSIC technique requires more number of snapshots than that required by the proposed techniques in order to identify the sources but not necessarily all of them. It has been shown that the DOA estimation performance using the proposed techniques is superior to that of the UN-MUSIC; further, the performance of GCD MVDR A-LASSO is better than that of GCD MVDR A-LASSO.

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