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Topological Circuit Theory: A Lie Group Perspective

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Abstract—We present a general theory of linear continuous circuits (microwave networks, waveguides, transmission lines, etc.) based on Lie theory. It is shown that the fundamental relationship between the low- and high-frequency circuits can be fully understood via the machinery of Lie groups. By identifying classes of distributed-parameter circuits with matrix (Lie) groups, we manage to derive the most general differential equation of the n-port network, in which its low-frequency (infinitesimal) circuit turns out to be the associated Lie algebra. This equation is based on identifying a circuit Hamiltonian derived by heavily exploiting the Lie-group-theoretic structure of continuous circuits. The solution of the equation yields the circuit propagator and is formally expressed in terms of ordered exponential operators similar to the quantum field theory's formula of perturbation theory (Dyson expansion). Moreover, the infinitesimal operators generating the per-unit-length lumped-element local circuit approximation appear to correspond to operators (such as observables) in quantum theory. This analogy between quantum theory and circuit theory through a shared Hamiltonian and propagator structure is expected to be beneficial for the two separate disciplines both conceptually and computationally. Several applications are presented in the field of microwave network analysis where we introduce and study the Lie algebras of important generic classes of circuits, such as lossless, reciprocal, and nonreciprocal networks. Applications to the problems of generalized matching and representation theorems in terms of uniform transmission lines are also outlined using topological methods derived from our Lie-theoretic formulation and exact theorems on continuous matching are obtained to illustrate the potential practical use of the theory.

1. INTRODUCTION

The cross-disciplinary subject of topology-physics interaction has recently experienced a revival through topics like topological insulators [1] and topological photonics [2]. Other such applications to physics include topological techniques in quantum field theory (QFT) [3], causal nets in cosmology [4], near-field analysis of electromagnetic radiation [5–8], and numerous others. While those fields aim at using topological methods to characterize interesting physical phenomena, another synergistic route goes through building *structural analogies* in which topological thinking could be considered a higher level technique or a more abstract approach for taking up in a unified manner what would otherwise initially appear as two very different domains of knowledge. The ultimate objective of such a line of research would be imposing some unity on disparate disciplines within the confines of a single investigatory framework. One method to accomplish this is exploiting the general and powerful apparatus of Lie groups [9]. Indeed, a vast amount of physics and engineering can be described by continuous groups of transformations [10, 11]. One of these applications, as will be proposed below, is the realization that signal or wave propagation in one-dimensional structures like waveguides or circuits is essentially a process of continual transformation of energy from electric type to magnetic type and

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vice verse [12]. Consequently, it is very often the case that this *propagation* process can be perfectly captured mathematically using linear matrices, leading to the transmission line (TL) model [13] or the matrix method of wave propagation [14, 15].

It is now fairly well established that the theory of Lie groups, i.e., the study of continuous groups of transformations [9], is fundamental to modern physics [11]. Moreover, Lie groups continue to play a major role in the calculations of fundamental interactions in particle physics, especially in connection with collision experiments within the framework of the standard model of particle physics [16], and in condensed-matter physics, chemistry, and physical chemistry [17]. However, the impact of this theory on other fields of physics and engineering has not received the same level of sustained attention as compared with gauge field theory. For example, while Lie groups have been applied in mechanical engineering and robotics [18] and information theory [19], the theory is rarely invoked at a fundamental level in optics or applied electromagnetics, not to mention other disciplines like molecular biology, biophysics, physical chemistry, and chemical engineering. The purpose of this paper is to forge a new cross-disciplinary bridge linking quantum theory and circuit theory, where the common structure at the foundation of this interrelation turns out to be precisely Lie theory. We focus in what follows on the so-called "old Lie theory," which dates from the publication of Sophus Lie's major book [9] to the appearance of the definitive text on the "modern theory" by Chavalley [20]. Afterwards, most researchers on the subject working within pure mathematics have concentrated on building the global topological properties of various important groups [21]. This modern theory also coincides in spirit and content with Weyl's influential books on quantum theory published in 1928 [10] and the equally important book on classical groups [22]. While we appreciate the structural depth of the modern theory with its explicit interest in characterising the global invariants of the group and the spaces linked to it, we believe that the simplicity and direct intuitive appeal of the "old Lie theory," especially in the way it starts with local representation, is closer to physics and engineering problems as will be demonstrated in this work. In fact, in recent decades, the "old Lie theory" was applied to rebuild the foundations of classical dynamics in a strikingly modern way [23]. The move from the local to the global is particularly clear in the old framework originally envisioned by Sophus Lie and his collaborators in the nineteenth century because it builds the extension by explicit construction of partial differential equations that when solved can generate proper flows on the group manifold. This flow often translates into a physical process. In this paper we show that such flow may be taken to correspond to the propagation of electromagnetic signals (voltages and currents) along transmission lines. For these reasons, we advocate in the theory briefly outlined below a cross-disciplinary approach to circuit theory linking the subject with quantum theory through their mutually shared deep Lie-theoretic structural substrate. The fundamental relevance of the general theory to be exposed below is by no means restricted to the microwave regime. In fact, transmission line theory is a theory of "continuous systems" underlying a vastly larger horizon of natural phenomena, including optical transmission in nonuniform media, electric power systems, and scattering processes of microscopic particles. However, for concreteness, in the applications sections of this paper we more often invoke the microwave circuit model as a transmission line.

A microwave *circuit* is essentially a transmission line (TL), which in turn is a *distributed*-element circuit [24]. In this view, a TL is understood as a high-frequency circuit that behaves "globally" in a very different manner compared to the low-frequency counterpart, which represents the *local* structure of the system. In particular, only an infinitesimal (electrically small) section of a TL looks like a lumped-element circuit. This immediately suggests to us the analogy with differentiable (topological) manifolds, which by definition look locally like Euclidian spaces. Indeed, the universe of lumped-element circuit theory is taken as the point of departure for any circuit theory. We know very well how to do things in the low-frequency regime. Subsequently, the behavior of the global system, i.e., the full *high*-frequency circuit, is constructed in the following way. We start with an initial small section of a TL. Cascade another small section. The resulting system is larger than the initial one, but sill lying in its "neighborhood" (a topological concept). The process is iterated by inserting new infinitesimal sections till the global (full) picture of the complete high-frequency system is obtained. In this case, any microwave circuit can be described in terms of repeated application of infinitesimal transformations, acting in an abstract differentiable (and hence topological) space. The most natural mathematical device to handle this problem is the theory of Lie groups [25], which is well developed for classical (matrix) groups. It is a happy coincidence that microwave networks are described by (invertible) square matrices, which motivates direct application of topology and group theory to the traditional area of microwave engineering. This we endeavor to achieve in the present paper.

The paper is organized as follows. In Section 2 we give the core of the Lie-theoretic approach proposed here and derive the general master equation of a TL using only Lie group arguments. The equation is shown to be equivalent to Schrodinger equation and we formally solve it using the Dyson series of quantum field theory (QFT). In Section 3 we develop the effective method needed for performing calculations through the introduction of the Lie algebra of distributed-parameter circuits. We give several examples and show how conventional TL theory can be recovered very efficiently from our Lie-theoretic reformulation of the problem. New insights into TLs are also obtained and reported. In Section 4, we outline some possible more advanced future applications of the theory in the field of microwave networks. A particular application, the problems of generalized matching and the representation of a generic network by a uniform TL are taken up in Section 5. A set of carefully written mathematical appendices are inserted at the end of the paper to provide a deeper look into some of the arguments intuitively sketched out throughout the main text. Finally, we end with conclusion.

2. MAIN FORMULATION OF THE LIE THEORY OF CONTINUOUS CIRCUIT

2.1. Derivation of the Master Differential Equation

Let G be a Lie group of dimension M, which is defined as an M-dimensional differential (smooth) manifold whose points are equipped with the algebraic structure of a group multiplication operation such that the product of two elements and the inverse operations are smooth. A *matrix* group is a Lie group.† The key idea behind the topological circuit theory proposed in this paper is that the chain matrix representation of TLs (see Figure 1) leads to a natural representation of signal propagation in terms of invertible linear matrices, the chain matrix itself. In the case of n input/output ports circuits, the largest possible G is the general linear group $\mathrm{GL}(n;\mathbb{C})$. This is the group of square $n \times n$ invertible matrices over the field $\mathbb{C}$ of complex numbers (we assume time-harmonic excitation throughout the paper.) Interesting physically realizable circuits are matrix *subgroups* of $\mathrm{GL}(n;\mathbb{C})$ (and hence Lie groups by a well-known

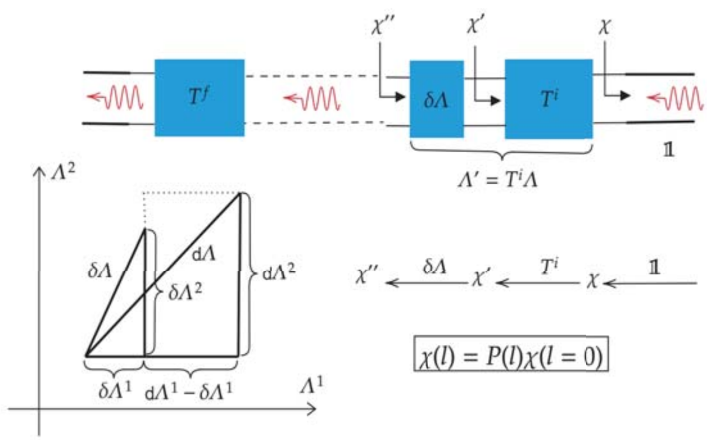


Figure 1. Derivation of the master equation in circuit theory. A transmission line is treated as an infinite number of infinitesimal circuits cascaded in series connection in order to enact propagation of electromagnetic signals (optical waves, voltage waves, etc.) Each signal space is represented by the vector χ of M parameters forming coordinates in the local manifold that is the chain matrix itself. The propagation of signals is now modeled by the propagator $P(l)$ defined by (13), which is an $M \times M$ matrix obtained by solving a Schrodinger-like Equation (17) with effective circuit Hamiltonian (24).

† A full review of the mathematical definition of general Lie group is provided in the Appendix. For an elementary introduction to Lie groups emphasizing matrix groups, see [26–28]. For more advanced treatment of classical (matrix) continuous groups, see [22, 29].

general theorem, see [28]) with examples given in Section 3. In general, each point $g \in \mathrm{GL}(n;\mathbb{C})$ in the differential manifold of the group will correspond to an n-port circuit. Concrete examples will be given in Sections 3 and 5. For the remaining parts of this section, the theory is developed at the most general level possible.

Let the M-parameter coordinate vectors for two elements of $g_1, g_2 \in G$ be Υ^m and χ^m (see Appendix B). When we compose these two elements we obtain a new group element $g_3 := g_1 g_2$, $g_3 \in G$, with parameter array χ'^m. The composition operations of Lie group turns out to be M analytic maps in the form [9, 30]

$$\chi'^m = \phi^m\left(\Upsilon^1, \ldots, \Upsilon^M, \chi^1, \ldots, \chi^M\right), \tag{1}$$

where $m = 1, 2, \ldots, M$. Here, one may think of the arrays of numbers χ'^m, χ^m, and Υ^m as "circuit coordinates" chosen from a coordinate patch containing the point (group element) g in the manifold G. It is to be understood everywhere that the representation of this group element is the usual chain matrix and the group operation of Eq. (1) captures — in the Lie group analytic manifold coordinate chart's language — the usual operation of matrix multiplication.

Now, consider Figure 1. We start with a reference line denoted by the identity operation $\mathbb{1}$ at the right. The line is followed by an arbitrary network described by the chain matrix T^i. At the other terminal there is the final network with a chain matrix T^f at the left. The infinitesimal circuits connecting T^i and T^f are described by the matrices whose Lie manifold coordinates are $\delta\Lambda$, where each such infinitesimal section is defined along the line going from right to left. Here, we denote by χ the total attained chain matrix while progressing with the infinitesimally continued steps $\delta\Lambda$. Again, χ really refers to the *coordinates* (parameters) of the Lie group representation of the chain matrix as per the discussion immediately following Equation (1) above.‡

We begin with the chain matrix χ seen after the reference (identity) line. The initial network T^i will transform this value into χ'. The continuation process is started by inserting a small section with a chain matrix $\delta\Lambda$, producing in turn χ''. We calculate the differential increment

$$\mathrm{d}\chi'^m = \chi''^m - \chi^m \tag{2}$$

with the help of the group operation functions in Eq. (1), and this leads to

$$\chi''^m = \chi'^m + \mathrm{d}\chi'^m = \phi^m\left(\delta\Lambda, \chi'\right). \tag{3}$$

Expanding the functions ϕ^m in Taylor series with respect to the first argument, we find that to the first-order approximation the following relation holds:

$$\mathrm{d}\chi'^m = \sum_{n=1}^{M} \delta\Lambda^n \, \frac{\partial}{\partial \Upsilon^n} \phi^m\left(\Upsilon, \chi'\right)\Big|_{\Upsilon=0}. \tag{4}$$

On the other hand, *we can compute the same quantity by treating the inserted section* $\delta\Lambda$ *as the germ of new analytical continuation of* χ. This is possible because both the new cascaded section's chain matrix and the law upon which we calculate the new value of χ are the same, i.e, matrix multiplication. To see this, notice that

$$\Lambda'^m = \Lambda^m + \mathrm{d}\Lambda^m = \phi^m\left(\delta\Lambda, \Lambda\right), \tag{5}$$

where Λ' can be interpreted as the new chain matrix obtained by cascading T^i and then $\delta\Lambda$ (see Figure 1); that is,

$$\Lambda' = \delta\Lambda T^i. \tag{6}$$

By again expanding Eq. (5) in Taylor series, we obtain to the first order

$$\mathrm{d}\Lambda^m = \sum_{n=1}^{M} \delta\Lambda^n \, \frac{\partial}{\partial \Upsilon^n} \phi^m\left(\Upsilon, \Lambda\right)\Big|_{\Upsilon=0}. \tag{7}$$

‡ It is important to acknowledge some abuse of notation we admit throughout this paper. Strictly speaking, we must distinguish between the actual chain matrix itself, which is an $n \times n$ complex matrix, and its "coordinates" or parameters, which are M in number. However, since the general linear group is a Lie group, it has a manifold structure as mentioned above and hence there is a local one-one correspondence between the matrices themselves and their coordinates. For that reason, we refer to the "chain matrix" $\delta\Lambda$ here by its *coordinates*, not the matrix itself. In general, when we say "chain matrix" we either refer to the matrix itself or to its array of M parameters. This loosening of the distinction between the coordinates and the abstract point is followed throughout.

Now, define the following $M \times M$ matrix

$$U_n^{-1,m}(\Lambda) := \frac{\partial}{\partial \Upsilon^n} \phi^m(\Upsilon, \Lambda)\bigg|_{\Upsilon=0}. \tag{8}$$

This matrix is invertible as can be established from the Lie theoretic properties of composition functions, see [9] for example. With the notation in Eq. (8), Equation (7) can be rewritten as

$$\delta\Lambda^m = \sum\nolimits_{n=1}^{M} U_n^m \mathrm{d}\Lambda^n, \tag{9}$$

where both sides were inverted after substitution. Inserting Eq. (9) into Eq. (4), we obtain

$$\mathrm{d}\chi'^m = \sum\nolimits_{n=1}^{M} \sum\nolimits_{n'=1}^{M} \mathrm{d}\Lambda^{n'} U_{n'}^n(\Lambda) V_n^m(\chi'), \tag{10}$$

where the matrix V is defined as

$$V_n^m(\chi') := \frac{\partial}{\partial \Upsilon^n} \phi^m(\Upsilon, \chi')\bigg|_{\Upsilon=0}. \tag{11}$$

Differentiating Eq. (10) and carefully labeling the indices, it easily follows that

$$\frac{\partial \chi'^m}{\partial \Lambda^n} = \sum\nolimits_{r=1}^{M} U_n^r(\Lambda) V_r^m(\chi'). \tag{12}$$

The previous derivation of Eq. (12) can be further appreciated when we remember that, generally speaking, $\delta\Lambda$ and $\mathrm{d}\Lambda$ are *essentially different* quantities: We can *freely* vary $\delta\Lambda$, which represents the inserted infinitesimal section of the tapered line. However, once we do so, we can *not* freely change $\mathrm{d}\Lambda$ (or equivalently $\mathrm{d}\chi$) since the latter are governed by the particular Lie group structure encapsulated in the functions in Eq. (1). Physically, the system of Equation (12) represents a coupled set of partial differential equations, most often nonlinear, and is very general in scope. It describes the "multi-dimensional dynamics" of wave propagation, i.e., not necessary along a preferred direction like time or the longitudinal extension of the TL, but instead with respect to all or subset of the total independent M parameters that describe the dynamics of continuous change (here wave propagation enacted via infinitesimal cascade modification of the chain matrices.) Next, we develop the special but important case of one-parameter subgroup of dynamical transformations.

Consider the one-parameter $l \in \mathbb{R}$, which is physically interpreted as the longitudinal coordinate (spatial index) of the TL. In this language, the germ state becomes the parameters χ^n, $n = 1, \ldots, M$, which are the coordinates of the initial chain matrix (initial state). We may express the response of the network at the location l as

$$\chi^m(l) = \sum\nolimits_{n=1}^{M} P_n^m(l)\, \chi^n(0), \tag{13}$$

where we define P as the *one-parameter propagator* of the network, clearly an $M \times M$ matrix parametrized by only one parameter l. Using the chain rule, the total derivative of the response χ can be computed, leading to

$$\frac{\mathrm{d}}{\mathrm{d}l}\chi^m(l) = \sum\nolimits_{n=1}^{M} \frac{\partial \chi^m(l)}{\partial \Lambda^n} \frac{\mathrm{d}\Lambda^n(l)}{\mathrm{d}l}. \tag{14}$$

Substituting Eq. (12) into Eq. (14), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}l}\chi^m(l) = \sum_{n=1}^{M} \sum_{r=1}^{M} U_n^r(\Lambda) V_r^m(\chi) \frac{\mathrm{d}\Lambda^n(l)}{\mathrm{d}l}. \tag{15}$$

We now introduce the crucial concept of the *infinitesimal generators* of the Lie group, which are defined by [27]

$$X_n(\chi) := -\sum\nolimits_{m=1}^{M} V_n^m(\chi) \frac{\partial}{\partial \chi^m}. \tag{16}$$

The infinitesimal generator can be interpreted in various ways. Technically, it is a differential operator on manifolds, the latter coinciding in our case with the group manifold itself [21]. More interestingly, it will be seen in Section 3 that the infinitesimal operator in fact directly captures the *low-frequency* or "infinitesimal section" of the TL circuit, and possibly can be interpreted as the circuit analog to observables in quantum theory.

With the help of Eq. (16), Equation (15) can be put in the convenient form

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}l} P_n^m(l) = -\sum_{v=1}^{M}\sum_{r=1}^{M} \frac{\mathrm{d}\Lambda^v(l)}{\mathrm{d}l} U_v^r[\Lambda(l)] X_r \left[\sum_{r=1}^{M} P_r^1(l)\,\chi^r(0), \ldots, \sum_{r=1}^{M} P_r^M(l)\,\chi^r(0)\right] P_n^m(l).} \tag{17}$$

To prove this, just substitute Eq. (16) to Eq. (15) and use the relation $\delta_n^m = \partial(x^m)/\partial x^n$ and the definition of the circuit propagator P in Eq. (13).

Equation (17) presents a set of first-order ordinary differential equations with the initial condition

$$P_n^m(l = 0) = \delta_n^m. \tag{18}$$

The system in Eqs. (17) and (18) constitutes the dynamical equations governing the propagation of electromagnetic signals along the n-port TL, and appears to be derived here for the first time at such very general level. It can be shown that Eq. (17) is reduced to the well known particular TL model obtained via Kirchhoff voltage and current laws (KCL/KVL) analysis of RLC infinitesimal sections as found in standard textbooks like [24]. The details of this reduction are given in Section 3.

2.2. Solution of the Master Equation Through Lie Algebra and Dyson Formula

Note that Eq. (17) was derived in a systematic manner for *arbitrary* n-port network without any special assumption about the physical realization of the low-frequency (infinitesimal) circuit. The physical content of the theory is injected into Eq. (17) through the *per-unit length circuit parameters* $\mathrm{d}\Lambda^v(l)/\mathrm{d}l$. However, the essential content of the relationship between the low- and high-frequency circuits is encoded in the Lie algebra (see Section 3), or the algebraic structure of the infinitesimal generators X_i. Our Equation (17) clearly relates these generators to the formal solution, although in a rather complicated way. In particular, the coupling between *multiple* ports (e.g., chain matrices in $\mathrm{GL}(n;\mathbb{C})$, $n > 1$) at the low-frequency level will be reflected into the rich mathematical structure of the Lie algebra, which encodes the appropriate form of the *internal* configuration of continuous circuits such as microwave networks.

The Lie algebra is the vector space spanned by the infinitesimal generators X_i closed under the commutation operation [25, 27, 29]

$$[X_i, X_k] := X_i X_k - X_k X_i \tag{19}$$

and satisfying the Jacobi identity [29]. Intuitively, the Lie algebra can be understood as a linearization of the Lie group, or a local viewpoint of what is otherwise a complicated global object (the group itself defined on its entire manifold [21].) Surprisingly, it turns out that one can learn much about the global behavior of the group from the structure of the Lie algebra [25, 29].

In order to motivate the concept of infinitesimal generator, it will be easier to work directly with a matrix group Λ than the original (more general) definition in Eq. (16). Expanding in Taylor series, we find

$$\Lambda = \mathbb{1} + \sum\nolimits_{i=1}^{M} \delta\Lambda^i X_i + O\left(\delta\Lambda^2\right), \tag{20}$$

where the infinitesimal matrix generators X_i are defined by [27]

$$X_i := \frac{\partial}{\partial \Lambda^i} \Lambda\left(\Lambda^1, \ldots, \Lambda^M\right). \tag{21}$$

We notice here that the parameters Λ^m are chosen such that $\Lambda(0) = \mathbb{1}$ and the derivatives are calculated at $\Lambda^i = 0$ for all i. To the first order, we approximate an infinitesimal section of a TL by the matrix

$$\delta\Lambda \approx \mathbb{1} + \sum\nolimits_{i=1}^{M} \delta\Lambda^i X_i. \tag{22}$$

Each infinitesimal network is the low-frequency (lumped-element) circuit with per-unit length parameters $\delta\Lambda$. The *effective* chain matrix of the TL is the multiplication of all these small sections. Therefore, by carrying out the formal limit, one can use this method to systematically compute the chain matrix of a *uniform* TL using the *matrix exponential* [27]

$$\Lambda = \exp\left(\sum\nolimits_{i=1}^{M} \xi^i X_i\right), \tag{23}$$

for real parameters ξ^i. Concrete calculations with familiar uniform TLs using Eq. (23) will be given in Sections 3 and 5. The generalization to *non*uniform TLs is the subject of the next paragraph.

We apply the above theory to derive a formal solution of generalized nonuniform transmission lines. Start by defining

$$-\frac{i}{\hbar}\mathcal{H}(l) := -\sum_{v=1}^{M}\sum_{r=1}^{M} \frac{\mathrm{d}\Lambda^v(l)}{\mathrm{d}l} U_v^r(\Lambda) X_r(\chi), \tag{24}$$

where $\mathcal{H}$ plays the role of the *Hamiltonian operator* in quantum field theory and $\hbar$ the Planck constant. The master equations assumes then the following compact form

$$\frac{\mathrm{d}}{\mathrm{d}l} P_n^m(l) = -\frac{i}{\hbar}\mathcal{H}(l) P_n^m(l), \tag{25}$$

which is clearly identical to the Schrodinger equation in the propagator formalism [31]. Integrating Eq. (25), we obtain the integral equation

$$P_n^m(l) = P_n^m(0) - \int_0^l \mathrm{d}l' \frac{i}{\hbar} H(l') P_n^m(l'), \tag{26}$$

which can be viewed as the first-order iteration. Reiterating, we arrive at the recursive integral relation

$$P_n^m(l) = P_n^m(0) - P_n^m(0)\frac{i}{\hbar}\int_0^l \mathrm{d}l' H(l') P_n^m(l') + \left(\frac{i}{\hbar}\right)^2 \int_0^l \mathrm{d}l' H(l') \int_0^{l'} \mathrm{d}l'' H(l'') P_n^m(l'') + \ldots, \tag{27}$$

which is the well-known perturbation series (Dyson expansion) of the propagator in quantum field theory [31].§ Its emergence here discloses the deep structural analogy between continuous-parameter circuits (TLs) and quantum processes: the shared structure of a propagator equation. For example, one may create Feynman diagrams for the circuit problem in analogy with the situation in QFT, etc. It will be interesting to explore in future work what the corresponding circuit processes have to offer QFT in terms of physical interpretation and insight. Conversely, circuit theory, a subject well developed computationally, may offer new algorithmic techniques to help facilitate solving challenging problems in QFT.

The final propagator expression can be written compactly as

$$\boxed{P_n^m(l) = \mathcal{T}\exp\left[-\frac{i}{\hbar}\int_0^l \mathrm{d}l' \mathcal{H}(l')\right],} \tag{28}$$

where $\mathcal{T}$ is the ordered exponential symbol [31]. The latter is simply defined as a recipe in which anything that comes after the symbol $\mathcal{T}$ is arranged according to its time argument in chronological order (increasing time) from right to left. Notice that this exponential is still difficult to compute in general. However, by analyzing the structure of special important circuit Lie groups, like the reciprocal/nonreciprocal and/or lossless groups in Sections 3 and 5, it is possible to considerably enhance our understanding of the general solution of general nonuniform TL. To our knowledge, the TL's propagator in Eq. (28) is derived here for the first time.

Figure 2 illustrates the space of the Lie algebra of a generalized TL problem. Two *non*uniform TLs are represented by paths I and II. The particular shape of the path reflects how the low-frequency circuit is changing while progressing along the TL. Path III models a *uniform* TL and can be computed by the exponential map in Eq. (23) as will be shown in Section 3.1. From Lie theory, a neighborhood of the identity 0 in the Lie algebra will be mapped injectively into the *connected* identity competent of a Lie group having the same Lie algebra [21, 25]. As it turns out, the exponential map is *not* always

§ Perturbation theory was also recently applied to antenna-antenna mutual coupling analysis in [32, 33].

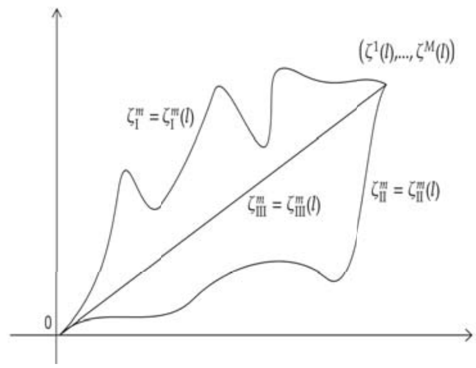


Figure 2. Various trajectories in the Lie algebra of continuous circuits. The straight line (Path III) is mapped to the chain matrix (Lie group) via the exponential map (23). This particular special case corresponds to *uniform* TL-type propagation. On the other hand, alternative, more complex trajectories like Paths I and II correspond to various instantiations of non-uniformity in continuous circuit signal propagation.

onto [29], which provides a significant insight into the relation between high- and low-frequency circuits. In general, our strategy will be to start from a given Lie algebra, i.e., a low-frequency description, and then study the structure of the microwave network, i.e., the high-frequency circuit, generated by the exponential relation.

3. THE LIE ALGEBRA OF DISTRIBUTED-PARAMETER CIRCUITS

The Lie algebra we introduced in Section 2.2 was defined as the vector space spanned by the infinitesimal generators X_i of Eq. (21) which are also closed under the commutation operation $[X_i, X_k] := X_i X_k - X_k X_i$ while satisfying the Jacobi identity. We will not rehearse all properties of Lie algebras in details since this subject tends to be extremely well covered in literature, e.g., see [17, 22, 25, 28, 31, 34]. The dimension of the Lie algebra is the number of linearly independent generators X_i. Lie algebra represents a linearization of the Lie group, or a local viewpoint. Surprisingly, it turns out that one can learn much about the global behavior of the group from the structure of the Lie algebra [17, 25, 31].

3.1. Basic Examples and Recovery of Conventional Transmission Line Theory

Let us calculate the Lie algebra of some simple low-frequency circuit topologies. Take the configuration shown in Figure 3. Applying Kirchhoff Current Law (KCL) and Kirchhoff Voltage Law (KVL), we arrive to the following chain ($ABCD$) matrix

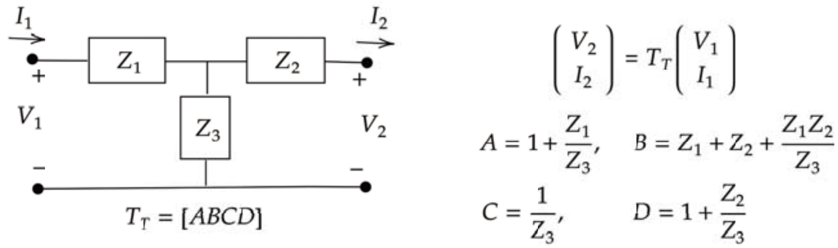
$$T_{\rm T} = \begin{pmatrix} 1 + \dfrac{Z_1}{Z_3} & Z_1 + Z_2 + \dfrac{Z_1 Z_2}{Z_3} \\ \dfrac{1}{Z_3} & 1 + \dfrac{Z_2}{Z_3} \end{pmatrix}. \tag{29}$$

Let us try to apply this method to calculate the chain matrix of a piece of transmission line with length l. Consider the Γ-network obtained by setting $Z_2 = 0$ in Figure 3. The resulting chain matrix is

$$T_\Gamma = \begin{pmatrix} 1 + ZY & Z \\ Y & 1 \end{pmatrix}, \quad Y := \frac{1}{Z_3} = jY_{\rm cap}, \quad Z := Z_1 = jZ_{\rm ind}, \tag{30}$$

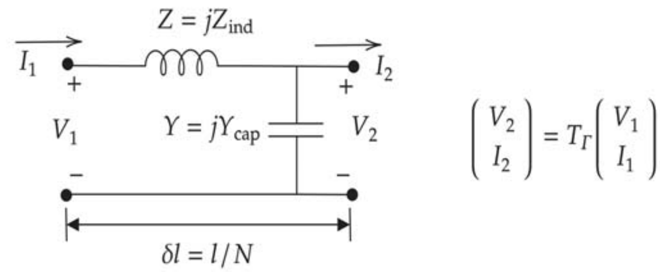
with $Z_{\rm ind}$, $Z_{\rm cap} \in \mathbb{R}^+$, representing the magnitudes of the inductive and capacitive impedance, respectively. The per-unit-length infinitesimal circuit itself is shown in Figure 4. The infinitesimal (low-frequency) circuit is an Γ-section with length approximated as

$$\delta l = \frac{l}{N}, \tag{31}$$



(a) (b)

Figure 3. (a) T-section impedance transformer circuit with $ABCD$ matrix T_{T}. (b) The $ABCD$ parameters.



(a) (b)

Figure 4. (a) Γ-section per-unit-length (infinitesimal or low-frequency) circuit of a conventional transmission line. (b) The $ABCD$ matrix relation of the infinitesimal section (b).

for N some large integer. The corresponding chain matrix is given by Eq. (30). There, the impedance and admittance can be written as

$$Z = j\rho_Z \delta l, \quad Y = j\rho_Y \delta l, \tag{32}$$

with the impedance and admittance *densities* (per-unit-length parameters) denoted by ρ_Z and ρ_Y, respectively. The infinitesimal matrix generators can be calculated from Eq. (21), and the results are

$$Z_{\mathrm{ind}} X_1 = \frac{Z_{\mathrm{ind}}\partial}{\partial Z_{\mathrm{ind}}} \begin{pmatrix} 1+ZY & jZ_{\mathrm{ind}} \\ Y & 1 \end{pmatrix}\Bigg|_{Y_{\mathrm{cap}}, Z_{\mathrm{ind}}=0} = Z_{\mathrm{ind}} \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix} = \rho_Z \delta l \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix} = \frac{\rho_Z l}{N} \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}, \tag{33}$$

$$Y_{\mathrm{cap}} X_2 = \frac{Y_{\mathrm{cap}}\partial}{\partial Y_{\mathrm{cap}}} \begin{pmatrix} 1+ZY & Z \\ jY_{\mathrm{cap}} & 1 \end{pmatrix}\Bigg|_{Y_{\mathrm{cap}}, Z_{\mathrm{ind}}=0} = Y_{\mathrm{cap}} \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix} = \rho_Y \delta l \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix} = \frac{\rho_Y l}{N} \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}, \tag{34}$$

where Eqs. (31) and (32) were used. Therefore, it may be concluded from the matrix exponential law in Eq. (23) that

$$\Lambda = \lim_{N\to\infty} \delta\Lambda^N = \lim_{N\to\infty} \left(\mathbb{1} + \frac{\rho_Z l}{N} X_1 + \frac{\rho_Y l}{N} X_2 \right)^N = \exp\left[(\rho_Z l)\, X_1 + (\rho_Y l)\, X_2 \right]. \tag{35}$$

The matrix exponential in Eq. (35) can be computed directly using the standard methods of matrix analysis [28]. We provide the final result and omit the details for brevity. It emerges that the TL full chain matrix is given by the expression:

$$\Lambda = \begin{pmatrix} \cos & jZ_c \sin\beta l \\ jY_c \sin\beta l & \cos\beta l \end{pmatrix}, \quad \beta = \sqrt{\rho_Z \rho_Y}, \quad Z_c = \frac{1}{Y_c} = \sqrt{\frac{\rho_Z}{\rho_Y}}, \tag{36}$$

where β and z_c are the propagation constant and characteristic impedance, respectively, whose per-unit-length circuit is the one in Figure 4. The matrix in Eq. (36) is nothing but the chain matrix of a uniform TL with length and per-unit inductance and capacitance given by ρ_Z and ρ_Y [13]. We have successfully recovered the correct traditional transmission line chain matrix from our proposed Lie group theory of continuous circuits.

It should be remembered that a relatively straightforward calculation of the matrix exponential function yielding Eq. (36) was possible only because the densities ρ_Z and ρ_Y are constant along the line. We have already shown in Section 2 how to deal with the fully general situation in which the TL is nonuniform. In that case, the more laborious treatment adopted there resulted in a general solution that can be obtained by careful integration of the master Equation (17) along a trajectory as in Figure 2, resulting in the Dyson formula (28). Thus, each nonuniform line will correspond to a path in the Lie algebra. The direct case of uniform line will correspond to integration through a line (Taylor Theorem for Lie Groups.)

3.2. The Matrix Exponential and Transmission Line Theory: Additional Insights into the Structure of the Problem of Signal Propagation

One may use the methods outlined above in order to systematically calculate the chain matrix of a network with arbitrary length and arbitrary infinitesimal (per unit length) parameters using the matrix exponential formula (23) restated as follows for the special case of the TL infinitesimal (per-unit) model in Figure 4:

$$\Lambda\left(\xi^1,\xi^2\right)=\exp\left(\xi^1 X_1+\xi^2 X_2\right), \tag{37}$$

where the explicit dependence on the real per-unit-length parameters ξ^i is emphasized to highlight the connection with the physical problem at hand (propagation of signals along TLs). Again, this expression is valid only for uniform lines in which the low-frequency circuits, parameterized by ξ^i, are linearly varying with a universal index like the length along the line. For the nonuniform case, the more complicated Dyson formula (28) must be used. In any case, it should be noted that $X_1X_2 - X_2X_1$ cannot be written as a linear combination of X_1 and X_2, which means that the subspace of matrices spanned by X_1 and X_2 is not closed under the commutator operation and hence is not a Lie subalgebra. This also implies that the class of chain matrices obtained by repeated multiplication of infinitesimal operators in the form $\mathbb{1}+\sum_{i=1}^{2} a^i X_i$ does not satisfy the group closure property. The resolution of this problem is given in Section 3.4 where we show that the proper Lie algebra of TL problems (reciprocal and lossless class) contains *three* infinitesimal generators, where a third one X_3 will be added to the two generators X_1 and X_2 we have already found.

Now for the most general situation, a mechanism for generating Lie group elements can be implemented in the following way:

(i) The generation of a finite transformation (signal propagation over a finite length) proceeds first by finding a suitable parametrization of the continuous group. The choice of the parametrization is not unique but must be made to avoid singularities around the identity element.

(ii) In the case when the low-frequency circuit is known, we can start by building a matrix representation of an arbitrary element of the associated Lie algebra. This happens because the low-frequency circuit contains at least the first-order information of the high-frequency circuit (more on this in Section 3.3).

(iii) By performing the partial derivatives in the definition of the infinitesimal generators in Eq. (21), one may isolate those pure first-order operation corresponding to each parameter appearing in the description of the low-frequency circuit.

(iv) Moreover, it should be clear from this description that the chain matrix of the low-frequency circuit reduces to the identity when the values of the parameters in the circuit description are zero. If this is not the case, a new parametrization must be attempted in order to achieve the requirements mentioned before.

(v) These matrices (infinitesimal generators) are taken to be the infinitesimal matrix generators of the group.

(vi) The finite group elements connected with the unity are then obtained by a suitable exponentiation operation in Eq. (23).

(vii) Transformations associated with disjoint elements of the Lie group can be obtained by introducing a discrete group [27–29].

In the author's opinion, the greatest, advantage of this connection between the low-frequency circuits and Lie algebra is that it is easy to study the structure of a microwave network or a general TL by simply starting from the low-frequency model. In other words, when Maxwell's equations places global restrictions on the chain matrices of important subgroups (reciprocal, lossless, etc., see Section 3.4), this *defines* our Lie group, which is inherently a global (and nonlinear structure). In contrast, the low-frequency circuit models can provide a *linearization* (hence a way for realization) of the original more complicated Lie group. We can then relate our study of the parent group to our knowledge of the performance (behavior) of the low-frequency model using the extremely well-developed and sophisticated methods of Lie algebras [21].

3.3. Generalized Per-Unit-Length (Low-Frequency) Circuits and the Connection with the Lie Infinitesimal Generators

We would like now to show that the compact analysis of the problem of signal propagation provided in Section 3.1 is far from being a special case. Lie theory provides an extremely powerful tool to generalize the concept of the infinitesimal (per-unit-length) circuit that goes beyond conventional TLs models. Since the recent subject of metamaterials [35–37] is interested in building novel structures by arranging new unit cells at the subwavelength scale, this might be a good place to investigate the subject of signal propagation in distributed-parameter systems at greater depth.

Consider the infinitesimal (low-frequency) circuit model shown in Figure 5. We wish to investigate how to proceed from the general circuit model shown there to the fundamental equations of a transmission line obtained by repeated cascading of such infinitesimal sections. The matrix T describes the transfer matrix of the low-frequency approximation of the transmission line. The length of the small section is given by δl. The relation between the input and output signals is given by

$$V_2 = AV_1 + BI_1, \quad I_2 = CV_1 + DI_1, \tag{38}$$

where

$$V_1 = V\left(l\right), \quad I_1 = I\left(l\right), \quad V_2 = V\left(l + \delta l\right), \quad I_2 = I\left(l + \delta l\right). \tag{39}$$

Notice that these relations are completely general. They apply to any low-frequency circuit, regardless of its topology, and in particular, they are not exclusively tied up with the conventional TL per-unit-length model shown in Figure 4.

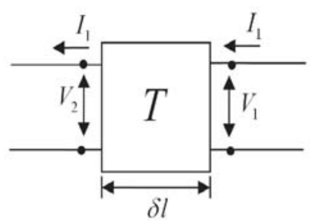


Figure 5. Generalized Infinitesimal (low-frequency) circuit model. The model shown in Figure 4 should be considered a special case of this circuit.

Let us expand the parameters of the chain matrix T in powers of the length δl as follows:

$$\begin{aligned} &A\left(l + \delta l\right) = 1 + A_1\left(l\right)\delta l + A_2\left(l\right)\left(\delta l\right)^2 + \ldots, \quad B\left(l + \delta l\right) = B_1\left(l\right)\delta l + B_2\left(l\right)\left(\delta l\right)^2 + \ldots, \\ &C\left(l + \delta l\right) = C_1\left(l\right)\delta l + C_2\left(l\right)\left(\delta l\right)^2 + \ldots, \quad D\left(l + \delta l\right) = 1 + D_1\left(l\right)\delta l + D_2\left(l\right)\left(\delta l\right)^2 + \ldots, \end{aligned} \tag{40}$$

where the choice of the constant terms in the Taylor series expansion above ($A_0 = D_0 = 1,\ B_0 = C_0 = 0$) was made based on the constraint that for zero length the network T must be reduced to the unit matrix $\mathbb{1}$, i.e., we must have the condition

$$T\left(l + \delta l\right)|_{\delta l \to 0} = \mathbb{1}. \tag{41}$$

Now consider the dimensions of the various entries of the chain matrix. The quantities A and D are dimensionless, while B has the unit of impedance and C has the unit of admittance. Since the circuit is linear, it can be described in the spectral domain by a set of linear algebraic equations. Therefore, the functional form of all the elements of the chain matrix contains only rational polynomials in impedances and admittances. Therefore, the only possible combination able to produce a dimensionless quantity is that when both A and D are expressed as multiplication of impedances and admittances. However, according to the choices we made in writing Equation (40), this multiplication can have only second-order terms proportional to $(\delta l)^2$ and higher. We conclude then that

$$A_1(l) = 0, \quad D_1(l) = 0. \tag{42}$$

Substituting Equations (40) and (42) into Equation (38) and keeping only the first-order terms, i.e., ignoring all terms with power $(\delta l)^2$ and higher, we find

$$V_2 = V_1 + (B_1 \delta l) I_1, \quad I_2 = (C_1 \delta l) V_1 + I_2. \tag{43}$$

By dividing through δl, we obtain the familiar TL (telegrapher) equations [13]

$$\frac{d}{dl} V(l) = \lim_{\delta l \to 0} \frac{V(l+\delta l) - V(l)}{\delta l} = B_1(l) I(l), \quad \frac{d}{dl} I(l) = \lim_{\delta l \to 0} \frac{I(l+\delta l) - I(l)}{\delta l} = C_1(l) V(l), \tag{44}$$

Therefore, it can be seen that in terms of the original transfer matrix of the infinitesimal (low-frequency) circuit, only the following first-order chain matrix

$$T_1(l, \delta l) = \begin{pmatrix} 1 & B_1(l)\delta l \\ C_1(l)\delta l & 1 \end{pmatrix} = \begin{pmatrix} 1 & \delta l \left.\dfrac{dB}{dl'}\right|_{l'=l} \\ \delta l \left.\dfrac{dC}{dl'}\right|_{l'=l} & 1 \end{pmatrix} \tag{45}$$

is relevant to the system of differential equations governing the propagation of signals along the transmission line.

As can be seen from this derivation, all higher-order information contained in the chain matrix T eventually drops out when writing down the system of differential Equation (44), while the remaining essential first-order information appears in the off-diagonal terms in Eq. (45) in a form identical to the infinitesimal generators of one-parameter matrix groups in Eq. (21). We have discovered then a natural and direct motivation for applying Lie groups to circuit theory or reformulating circuit theory in terms of Lie groups. Indeed, Lie's original fundamental insight amounting to the fact that the local nonlinear group structure can be recovered from the much simpler behaviour of the linear structure of first-order infinitesimal operators is inherent in the very fabric of the problem of signal propagation through transmission line, leading to the one-parameter group as the underlying mathematical model of wave propagation. Since — as we have already seen in Section 2 — the one-parameter group is closely related to the Lie algebra of the original (chain) matrix group, we conclude that the Lie-algebraic theory developed in this paper provides a systematization of this basic procedure in terms of matrix Lie group generators.

3.4. On the Lie Algebra of Lossless and Reciprocal 2-Port Microwave Networks

As a concrete example, we will identify in this section the Lie algebra of the group of lossless and reciprocal microwave networks. The infinitesimal generators of the Lie algebra will first be computed, then utilized to find the most important data about any Lie algebra: its structure constants, the regular representation, and the associated Cartan-Killing form.

One can use Maxwell's equations to show that any microwave network corresponding to a lossless system has pure imaginary impedance matrix element [13]. Furthermore, as will be established in Section 5, if the system is reciprocal, it follows that the determinant of the chain matrix is unity.[‖] By employing the well-known transformation equations between impedance and chain matrices [13], we may write the most general expression of the chain matrix in the following form

$$T = \begin{pmatrix} a & jb \\ jc & d \end{pmatrix}, \tag{46}$$

‖ See in particular the discussion around Eq. (57) there.

where a, b, c and d are real. For unit determinant, we obtain a special form given by

$$T\left(\Lambda^1, \Lambda^2, \Lambda^3\right) = \begin{pmatrix} 1+\Lambda^1 & j\Lambda^2 \\ j\Lambda^3 & \dfrac{1+\Lambda^1\Lambda^3}{1+\Lambda^1} \end{pmatrix}, \tag{47}$$

where this parametrization was chosen such that $T(0,0,0) = \mathbb{1}$. It is a straightforward calculation to show that this matrix form constitutes a matrix group satisfying all the group conditions listed in Appendix C. We can then directly compute the infinitesimal generators of this group using the defining relation in Eq. (21). The results are:

$$X_1 = \left.\frac{\partial T}{\partial \Lambda^1}\right|_{\Lambda=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad X_2 = \left.\frac{\partial T}{\partial \Lambda^2}\right|_{\Lambda=0} = \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix}, \quad X_3 = \left.\frac{\partial T}{\partial \Lambda^3}\right|_{\Lambda=0} = \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix}. \tag{48}$$

Note that the infinitesimal matrix generators are traceless as expected. The general element in this Lie algebra takes the form

$$X = \Lambda^1 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \Lambda^2 \begin{pmatrix} 0 & j \\ 0 & 0 \end{pmatrix} + \Lambda^3 \begin{pmatrix} 0 & 0 \\ j & 0 \end{pmatrix} = \begin{pmatrix} \Lambda^1 & j\Lambda^2 \\ j\Lambda^3 & -\Lambda^1 \end{pmatrix}. \tag{49}$$

We can easily compute the *structure constants*¶ of the fundamental Lie algebra of lossless and reciprocal circuits. First, the commutation relations are:

$$[X_1, X_2] = 2X_2, \quad [X_1, X_3] = -2X_3, \quad [X_2, X_3] = -X_1. \tag{50}$$

Therefore, the structure constants of this canonical case are given by

$$C^1_{23} = -C^1_{32} = -1, \quad C^2_{12} = -C^2_{21} = 2, \quad C^3_{13} = -C^3_{31} = -2. \tag{51}$$

On the other hand, the *regular representation* of the Lie algebra, which is the fundamental tool in studying the structure of this type of algebras [27–29, 38], is given by

$$R\left(X_n\right)^r_m = C^r_{nm}. \tag{52}$$

In light of Eq. (49), the general element of this matrix representation will then be given as

$$\begin{aligned} R(X) &= R\left(\Lambda^1 X_1 + \Lambda^2 X_2 + \Lambda^3 X_3\right) = \Lambda^1 R(X_1) + \Lambda^2 R(X_2) + \Lambda^3 R(X_3) \\ &= \Lambda^1 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{pmatrix} + \Lambda^2 \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + \Lambda^3 \begin{pmatrix} 0 & 0 & 2 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2\Lambda^2 & 2\Lambda^3 \\ \Lambda^3 & 2\Lambda^1 & 0 \\ -\Lambda^2 & 0 & -2\Lambda^1 \end{pmatrix}. \end{aligned} \tag{53}$$

We may view Eq. (53) as supplying the general expression of any element belonging to the Lie algebra of lossless reciprocal microwave networks.

To proceed in the analysis of the structure of this particular Lie algebra, let us evaluate the Cartan-Killing form [27]

$$\left(X_n, X_m\right) := \mathrm{Tr}\left\{R\left(X_n\right) R\left(X_m\right)\right\} = \sum_r \sum_s C^s_{nr} C^r_{ms}, \tag{54}$$

where Tr is the matrix trace operation. Calculating Eq. (54) by means of Eq. (53), we find

$$(X, X) = \mathrm{Tr} \begin{pmatrix} 0 & -2\Lambda^2 & 2\Lambda^3 \\ \Lambda^3 & 2\Lambda^1 & 0 \\ -\Lambda^2 & 0 & -2\Lambda^1 \end{pmatrix}^2 = \begin{pmatrix} \Lambda^1 & \Lambda^2 & \Lambda^3 \end{pmatrix} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} \Lambda^1 \\ \Lambda^2 \\ \Lambda^3 \end{pmatrix}. \tag{55}$$

The 3×3 numerical matrix appearing in the quadratic form at far left of Eq. (54) plays a fundamental role in the theory of Lie algebra. In fact, its eigenvalues can give immediate information about the *global* or *topological* structure of the associated Lie group [27]. We will not further pursue this line of thought here but leave a more detailed examination of the Cartan-Killing form to a future work.

Finally, we can provide a physical interpretation for the three infinitesimal matrix generators obtained previously. In Figure 6, we show three circuits corresponding to the three infinitesimal matrix generators. It turns out that the algebraic structure of lossless reciprocal microwave circuits is completely reducible to the three 1-dimensional vector subspaces spanned by

¶ The structure constants of a Lie algebra X are defined by the relation $[X_n, X_m] = \sum_l C^l_{nm} X_l$, see [9, 23, 25] for more information on the fundamental role they play in Lie theory.

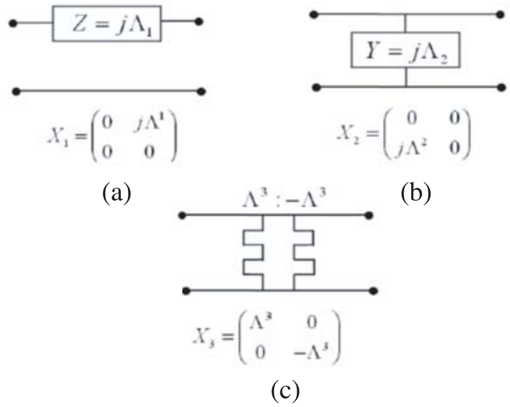


Figure 6. Physical interpretation for the three infinitesimal matrix generators of the lossless/reciprocal microwave network group. The three proposed circuits provide the physical content of the Lie algebra of this class of circuits.

(i) Pure series reactive impedance: Figure 6(a).
(ii) Pure shunt reactive admittance: Figure 6(b).
(iii) A circuit we call the infinitesimal transformer: Figure 6(c).

This last circuit can be interpreted as the infinitesimally small section of an ideal transformer. To see this, carry out the exponential of the matrix X_3, and you will immediately obtain the correct transfer matrix of the ideal transformer.

4. INTERLUDE: POTENTIAL FURTHER DEVELOPMENT AND APPLICATIONS OF THE THEORY

Let us assume that we can study the complete structure of a given Lie algebra. For example, we may obtain the decomposition of a semisimple Lie algebra into direct sums of ideals [29, 39]. This means that the TL circuit corresponding to the semisimple Lie algebra can be expressed in terms of simpler circuits, each corresponding to an ideal of the Lie algebra. This can then be taken as an application of the structure theory of Lie algebra to microwave circuits. This direction of application will not be further pursued in the present paper since it requires very extensive exposure to the theory of Lie algebra, which is outside the scope of our work here.

In a slightly different approach, we may proceed by studying the various subspaces associated with a given low-frequency circuit. In particular, we can explore possible subspaces formed by classic circuit topologies such as the T- pi-, X-, and T-bridge topologies. We would then try to construct the circuits that corresponds to the key subspaces associated with a given Lie algebra. For instance, some of the promising interesting questions to ask are: What are the circuits corresponding to nilpotent (V_0), compact (V_-), and noncompact (V_+)?+ Notice that this procedure can give us information about the topology of the induced Lie group. For example, compactness is a topological property. By pure algebraic calculations, we can deduce something about the global topological properties of the microwave circuit. This will be illustrated later in Section 5.

Another line of possible development, however, is more directly connected with some of the possible application of our theory to microwave networks [13, 24] and transformation optics [35]. First, note that both reciprocal and lossless networks form matrix (Lie) subgroups of the general linear group (Section 3.4). By studying the group's *topology* of these and other classes, we may obtain general theorems about *path-connectedness*, a key topological property [40]. This property can be translated into the possibility of performing *continuous* matching (taper design [13].) This train of thought will be further persued for some key special examples in Section 5.1.

+ For definitions of these important technical terminologies within the general subject of Lie algebra, see [27].

Finally, we add one more possible direction for expanding the topological theory of continuous circuits. By applying techniques from linear algebra to analyze the structure of the corresponding Lie algebras, it will become possible to decide which microwave circuits can be represented by uniform TLs, a cascade connections of two uniform TLs, etc.. Also, new light will be shed on the analysis and design of nonuniform TLs by exploiting the logic of the interrelation between Lie subalgebras and Lie subgroups, a topic that is extremely well-attested in the mathematical literature on Lie theory, e.g., see [21, 28, 29]. This particular application will also be briefly touched upon in Section 5.2.

5. TOPOLOGICAL APPLICATIONS IN MICROWAVE CIRCUIT THEORY

We sketch out some possible further development of the general theory outlined above. It has already be en shown in Section 3 that both reciprocal and lossless networks form matrix (Lie) subgroups of the general linear group. By studying the group's topology of these and other classes, we obtain general theorems about path-connectedness, which is translated into the possibility of performing *continuous* matching (taper design [24].) On the other hand, by applying techniques from linear algebra to analyze the structure of the corresponding Lie algebras, it will become possible to decide which microwave circuits can be represented by uniform TLs, a cascade connections of two uniform TLs, etc. Also, new light will be shed on the analysis and design of nonuniform TLs by exploiting the logic of the interrelation between Lie subalgebras and Lie subgroups. These further developments, however, are quite lengthy and require more refined mathematical tools borrowed from the extensively developed apparatus of Lie algebras so in what follows our discussion is brief.

5.1. Generalized Matching Problem: Topological Perspective

The chain matrix of a general microwave network will be taken to be invertible, i.e., we assume (as done throughout this paper) that $T \in \mathrm{GL}(n, \mathbb{C})$. However, in microwave theory one can talk about subclasses of $\mathrm{GL}(n, \mathbb{C})$ that define classes of particular interest. For example, let us relate the chain matrix to the impedance matrix Z_{nm}. For 2-port networks, we know that [13]

$$A = \frac{Z_{11}}{Z_{21}}, \quad B = \frac{Z_{11}Z_{22} - Z_{12}Z_{21}}{Z_{21}}, \quad C = \frac{1}{Z_{21}}, \quad D = \frac{Z_{22}}{Z_{21}}. \tag{56}$$

Therefore, the determinant of the chain matrix is given by

$$\det T = \frac{Z_{12}}{Z_{21}}. \tag{57}$$

It follows then that for reciprocal networks, in which the impedance matrix has to be symmetric, i.e., $Z_{21} = Z_{12}$, the determinant of the chain matrix is unity. Thus, the chain matrix of a reciprocal network is a member of the special linear classical group $\mathrm{SL}(n, \mathbb{C})$.

For lossless networks, it is well established that the elements of the impedance matrix are all pure imaginary [13]. It follows then that for nonreciprocal but lossless networks, the chain matrix will have a determinant in the form $\det T = a/b$, where $a = -jZ_{12}$ and $b = -jZ_{21}$. The real numbers a and b can be positive or negative, depending on whether the coupling between the ports is capacitive or inductive. Therefore, for arbitrary nonreciprocal lossless network, the determinant can be either positive or negative: positive for the case when the coupling between the ports both ways is either inductive or capacitive; negative when the coupling between the two ports is inductive in one direction and capacitive in the other direction.

Let us turn now into some direct application from topology. The reader should be familiar with the topological concepts of connectedness, path-connectedness, and homotopy (a review is provided in Appendix F.) From the information we just stated about the determinant of reciprocal lossless networks and those of nonreciprocal lossless networks, the following two theorems follow immediately. The proof technique is similar to the standard procedure used to investigate connectivity properties of classical (matrix) groups and the reader is referred to [28] for details.

Theorem 5.1 *Two reciprocal lossless networks can always be joined together.*

That is, it is possible to continuously deform a given reciprocal network to reach any other.

Theorem 5.2 *Two nonreciprocal lossless networks cannot always be joined together. In particular, a nonreciprocal network with* $z_{12}/z_{21} < 0$ *cannot be continuously deformed till it becomes another nonreciprocal network with* $z_{12}/z_{21} > 0$*. Also, a lossless reciprocal network and a lossless nonreciprocal network with* $z_{12}/z_{21} < 0$ *cannot be joined together.*

The reader should consult Theorems D.3 and E.2 for more details about the proof.♯ In physical terms, if for nonreciprocal networks the coupling between the two ports is capacitive one way and inductive the other way, then this network cannot be matched with other networks which are either (1) reciprocal or (2) nonreciprocal with coupling between the two ports is either capacitive or indicative in the two directions.

The important lesson we have learned so far is that each transmission line can be viewed as a path in its Lie algebra. A uniform transmission line maps to a straight line passing through the origin, while a general nonuniform line is mapped to a particular trajectory (see Figure 2). The *generalized matching problem* poses the following question: When can we be able to match a given microwave network to another one using a continuous taper? It follows immediately from the topology of Lie groups that⋆

Theorem 5.3 *Two microwave networks can be matched to each other by a continuous taper if and only if they both belong to the same connected component of their corresponding Lie group.*

Note that in circuit design through optimization, we would like to generate all possible continuous tapers by building uninterrupted circuit sections that can match two given microwave networks without introducing scattering discontinuity. In order to evaluate desired performance measures, e.g., return loss, transmission characteristics, etc., we need then to know how to deform a given initial microwave network into another. Rephrasing this question using the terminology of the theory developed in this paper, we see that the generation of one continuous matching circuit from another can be viewed as whether two paths in the topological space of the corresponding Lie group are deformable to each other or not. In the jargon of topology, we ask whether they are homotopic or not. A homotopy class with respect to a given path is the set of all paths that are homotopic to the given path. That is we have proved the following theorem.

Theorem 5.4 *The set of all continuous matching microwave networks that can be continuously generated from a given network* T_0 *is the homotopy class of* T_0*.*

For a review of homotopy, see Appendix F and the related appendices before. Advanced techniques from homotopy theory [3, 41, 42] can be directly applied now to investigate all possible connectivity properties of microwave networks at a very general level. As can be noted from our formulation, the entire subject is developed *topologically* right from the Lie-theoretic structure of continuous circuits. There is no need to solve Maxwell's equations since topology is a covering (larger) theory with respect to Maxwell's theory.

5.2. Decidability of the Problem of the Existence of a Transmission Line Model Realization of Arbitrary Microwave Network

The topological theory developed in this paper can be utilized in an important way to pose a well-formulated practical problem in the field of microwave circuit analysis and to provide effective general means for its solution. Our problem is:

Problem Description (*Representation of a Generic Microwave Network by a TL*): *Given a microwave network represented by its transfer matrix, is it possible to realize it by a transmission line circuit?*

Without Lie theory, this problem is ill-formulated in the following way. It is not clear what are the specifications of the transmission line that are relevant to the microwave network in the realization under question. However, Lie algebras represent the tangent plane of the matrix group at the identity element. Since the structure constants of the Lie algebra are indeed constants, it follows that the behavior of the Lie group around the identity is enough to get a very general view about the global behavior of the matrix group. Indeed, this is the core idea behind the concept of a *homogeneous topological space* and

♯ These two theorems are also used for proving some of the other results below.

⋆ See the Appendix in general, and Theorem D.1 in particular together with the discussion surrounding it there.

it is shown in the Appendix that all topological groups *are* homogeneous spaces. It appears then very natural to focus the search for TL realizations on mainly those elements belonging to the Lie algebra associated with the given matrix since it is precisely those realizations that behave as the limit of infinitesimal transformations "close" the given microwave network.[&] We then reformulate our problem in the following manner:

Alternative Problem Description: *Given a microwave network represented by its transfer matrix subgroup* $G \leq \mathrm{GL}(n, \mathbb{C})$, *is it possible to realize this network by a transmission line circuit belonging to the Lie algebra* $\mathfrak{g} \leq \mathfrak{gl}(n, \mathbb{C})$*?*

Here, $\mathfrak{g}$ and $\mathfrak{gl}$ stand for the Lie algebras of the Lie groups G and GL, respectively. The notation $\mathfrak{g} \leq \mathfrak{gl}$ means that $\mathfrak{g}$ is a Lie subalgebra of $\mathfrak{gl}$. The answer to this question depends crucially on purely topological considerations. For example, a connected compact Lie group can always by represented as an exponential of an element belonging to the corresponding Lie algebra [21]. However, it is known that when the Lie group fails to be topologically compact, it is not possible to answer the decidability problem in general [27].

Let us give a concrete example. Consider our Lie group of reciprocal microwave networks $\mathrm{SL}(n, \mathbb{C})$. This group is connected but not compact [28]. Indeed, we will show now that

Theorem 5.5 *There exists a class of lossless reciprocal microwave networks that cannot be realized by transmission line models. In particular, any lossless and reciprocal microwave network with transfer matrix* T *such that* $\mathrm{Tr}\, T < -2$ *cannot be represented by any (lossless) uniform transmission line whatsoever.*

For the proof, notice that the well-known theorem [17, 28]

$$\det \exp(A) = e^{\mathrm{Tr}\, A} \tag{58}$$

immediately forces us to look for TLs belonging only to the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$. Otherwise, for non-traceless TL models, the determinant of the microwave network transfer matrix will cease to be unity. Now, the general element of the Lie group of *lossless* reciprocal microwave networks that can be given by exponentiating an element of the Lie algebra in Eq. (49) can be shown by brute force calculation to be

$$\exp(X) = \begin{pmatrix} \cosh\Theta + \left(\Lambda^1/\Theta\right)\sin\Theta & \left(j\Lambda^2/\Theta\right)\sin\Theta \\ \left(j\Lambda^3/\Theta\right)\sin\Theta & \cosh\Theta - \left(\Lambda^1/\Theta\right)\sin\Theta \end{pmatrix}, \quad \Theta = \sqrt{-\det X} = \sqrt{a^2 - bc}. \tag{59}$$

Since a, b, and c are real, Θ is either purely real or purely imaginary. Therefore, we have

$$\mathrm{Tr}\ \exp(X) = 2\cosh\sqrt{\det X} \geq -2. \tag{60}$$

It follows then that the microwave network corresponding to a transfer matrix with trace less than -2 cannot be realized by a uniform lossless reciprocal TLs.

We have enough evidence now to claim that the expression (49) represents a closed-form relation between a general microwave lossless and reciprocal network and its TL model, which is parameterized by the weights of the infinitesimal generators of the TL, here Λ^i, $i = 1, 2, 3$. We can generate the transfer matrix closed-form expression corresponding to various TL topologies. To do this, we compute the infinitesimal matrix generators for each topology, and then calculate the matrix exponential. This will generate standard parametrization of the microwave network transfer matrix. We can carry out this procedure systematically in order to populate a table of standard parametrization for classical TL unit cell topologies.

6. CONCLUSION

The theory developed above is expected to help conducting new investigations into fundamental aspects related to how low- and high-frequency phenomena in electromagnetic wave propagation, examined here within the specialized context of circuit theory, are structurally interrelated to each other. We anticipate that the theory will provide a rigorous conceptual framework for current and future microwave circuit

[&] However, a complete study of all available Lie groups and Lie algebras is possible in principle, and the question of the decidability whether a given matrix can be represented in terms of TL model can be attacked within the framework of this paper.

research. From the fundamental physics viewpoint, the analogy with quantum theory is striking. It appears that there is a correspondence between the concepts of propagator, Hamiltonian, and observable in QFT and analogous quantities in the circuit universe. For example, observables in quantum physics correspond to the infinitesimal operators responsible of generating the per-unit length lumped-element circuit that in turn represents the local approximation of the original continuous circuit. The most interesting finding is how topology enters the picture through the well attested progressive determination of a group germ extended analytically from one local domain to another. This process seems to underlie how electromagnetic signals propagate in a continuous circuit (one-dimensional medium) and may open the doors for new duality relations or numerical methods in quantum physics coming from circuit analysis or vice versa.

APPENDIX A. TOPOLOGICAL SPACES

A *topological space* is defined as the ordered pair $(A, \mathfrak{S})$. Here, A is a set of points endowed with a topological structure $\mathfrak{S}$, which consists of a collection of subsets $\mathfrak{S} = \{S,\, S \subseteq A\}$. Each element of $\mathfrak{S}$ is called an *open sets* in the topology $(A, \mathfrak{S})$, and it satisfies the following usual axioms

$$\emptyset, A \in \mathfrak{S}, \quad \bigcap_{n=1}^{\text{finite}} S_n \in \mathfrak{S}, \quad \bigcup_{n=1}^{\text{finite/infinite}} S_n \in \mathfrak{S}. \tag{A1}$$

The topological space described by the previous axioms too big for the applications is presented in this paper. We therefore add the following extra axiom, which is called the axiom of separability [43]

$$\forall p, q \in A, \quad p \neq q, \quad \exists S_p, S_q \in F | \, p \in S_p, \quad q \in S_q, \quad \text{and} \quad S_p \cap S_q = \emptyset. \tag{A2}$$

Topological spaces that satisfy this axiom are called Hausdroff spaces. An open set S_p containing p is called a *neighborhood* of p. A sequence x_n *converges* to x if, for every neighborhood U of x, there exists an integer N such that $x_n \in U$ for $n > N$. The axiom of separation ensures that some "pathological" situations are avoided, e.g., the same sequence converges to two different limits. A topological space is *compact* if every infinite sequence of points $\{a_n\}_{n=1}^{\infty}$, $a_n \in A$, contains a subsequence that converges to a point in the space itself. A set is *closed* if it contains all its limit points. A set S together with all its limit points is called the closure of S and is denoted by $\bar{S}$.

We next define maps and continuous maps. Let φ be a mapping that links a topological space $T_1 = (A_1, \mathfrak{S}_1)$ into another topological space $T_2 = (A_1, \mathfrak{S}_2)$. That is, we have $\varphi : A_1 \to A_2$ and the *image* of $a_1 \in A_1$ is $a_2 = \phi(a_1) \in A_2$. The set of all points that map to a single point in a_2 is called the *inverse image* of that point. Now we come to the most important definition in topology. A map φ is said to be *continuous* if the inverse image of an open set is an open set. It can be easily shown that in the case of metric spaces, this topological definition is reduced to the familiar $\varepsilon - \delta$ definition in real analysis [44, 45]. A *homeomorphism* $\varphi : X \to Y$ is an injective map such that both φ and φ^{-1} are continuous. Two topological spaces that can be linked to each other through a homomorphism are called *homeomorphic*, or *topologically equivalent*. They carry the same topological structure.^

APPENDIX B. DIFFERENTIAL MANIFOLDS

A *differentiable manifold* is a Hausdroff topological space equipped with additional structure consisting of a collection of charts $\varphi_p : T_p \to \mathbb{R}^M,\ p \in T_p \subset T$, satisfying the following axiom

(i) Each φ_n is a homeomorphism that links an open set $T_p,\ p \in T_p$, into an open set in the M-dimensional Euclidian space $\mathbb{R}^M$.

(ii) The union of all open sets $T_p,\ p \in T_p$ covers the original space T. That is, $\bigcup T_p = T$.

(iii) Let p and q be any different points in T. Since T is a topological space, then, provided that $T_p \cap T_q \neq \emptyset$, the set $T_p \cap T_q$ is an open set in the topology of T. In general, the mappings φ_p and φ_q produce different sets $\varphi_p\,(T_p \cap T_q)$ and $\varphi_q\,(T_p \cap T_q)$. However, we require that the mapping $\varphi_p \circ \varphi_q^{-1} : \mathbb{R}^M \to \mathbb{R}^M$ is continuous and "sufficiently" differentiable.

^ The subject of topological spaces is covered in many excellent books. We recommend [40, 43].

The *dimension* of the chart $\varphi : U \subset T \to \mathbb{R}^M$ is M. The *coordinate functions* of the chart are defined as $x^m = u^m \circ \varphi$, where u^m is the standard coordinate function in $\mathbb{R}^M$ given by $u^m(x^1, x^2, \dots, x^m, \dots, x^M) = x^m$. As we can see from the definitions above, since a topological manifold is equipped with a chart, any neighborhood in T can be mapped into a neighborhood in the conventional Euclidean space $\mathbb{R}^M$. This means that a differentiable manifold looks *locally* like a Euclidean space. Moreover, the maps $\varphi_p \circ \varphi_q^{-1} : \mathbb{R}^M \to \mathbb{R}^M$ accomplish a transformation between coordinates (charts) that is itself differentiable. This allows us to bring the full machinery of calculus into work in the analysis of manifolds.[≀]

APPENDIX C. GROUPS

Groups are the most important algebraic structures. A binary operation $*$ acting on two elements a and b in the set G is a rule assigning to them a third element $c \in A$. We write $a * b = c := ab$. A *group* consists of a set G equipped with a binary operation satisfying the following axioms

(i) Closure: $a, b \in G \Rightarrow c = ab \in G$.
(ii) Associativity: $a, b, c \in G \Rightarrow (ab)\,c = a\,(bc)$.
(iii) Existence of identity element: $\exists e \in G | \forall a \in G,\ ae = ea = a$.
(iv) Existence of right and left inverses: $\forall a \in G,\ \exists a^{-1} \in G | aa^{-1} = a^{-1}a = e$.

Technically speaking, a group is defined as the ordered pair $(G, *)$, but we follow the convention in referring to the group by the set G. The identity element and inverses are unique in any group. It can be easily shown, starting from the group axioms above, that $(ab)^{-1} = b^{-1}a^{-1}$ for any $a, b \in G$. A *subgroup* $H \subseteq G$ is a subset that forms a group on its own and is closed under the usual multiplication operation inherited from G. Notice that it follows that H must contain the identity element of G. When H is a subgroup of G, we write $H \leq G$.

APPENDIX D. TOPOLOGICAL GROUPS

A *Topological group* (or sometimes called continuous group) is a set of points equipped with both algebraic and topological structures such that[▷]

(i) There exits a topological space $\mathcal{M}$.
(ii) (this condition is optional) There exists an M-dimensional differential manifold defined on the topological pace $\mathcal{M}$.
(iii) There exists an algebraic structure defined on the topological space $\mathcal{M}$ by the operation ϕ mapping each pair of points (χ, ξ) in the manifold into a third point γ in the same manifold.
(iv) Let the coordinate system of the points χ and ξ be $\chi^1, \chi^2, \dots, \chi^M$ and $\xi^1, \xi^2, \dots, \xi^M$, respectively. Then, in terms of this chart defined in the neighborhood of the two points, the coordinates of the new point γ are also in the same neighborhood, and are given by the functions

$$\gamma^m = \phi^m \left(\chi^1, \chi^2, \dots, \chi^M; \xi^1, \xi^2, \dots, \xi^M\right), \quad m = 1, 2, \dots, M. \tag{D1}$$

Formally, the requirement "the new point $\gamma = \phi(\chi, \xi)$ generated by the algebraic operation ϕ should lie in the neighborhood of the two points χ and ξ" is expressed by the conditions that the maps ϕ and υ below are both continuous

$$\phi : \chi \times \xi \to \gamma = \phi\,(\chi, \xi) := \chi\xi, \quad \upsilon : \xi \to \xi^{-1}. \tag{D2}$$

The following is a general procedure to prove that the multiplication operation $\phi : G \times G \to G$, which is defined as $\phi : (x, y) \to xy$, is continuous. Let N be a neighborhood of $uv \in G$. The objective will be to find in the topology of G two open sets G_x and G_y such that $x \in G_x$ and $y \in G_y$. It follows that the set $G_x \times G_y$ is open in the product topology $G \times G$. We form the set $\phi^{-1}(N) = \{(u, v) \in G \times G, uv \in N\}$.

≀ A great place to learn about differential manifolds in general is [46, 47].
▷ For readers interested in learning more, the best book on topological groups (and probably Lie theory in general) remains [48]. Other books that discuss the subject in depth include [21, 49, 50].

If we can show that the product set $G_xG_y = \{xy, x \in G_x, y \in G_y\} \subseteq N$, then $G_x \times G_y \subseteq \phi^{-1}(N)$. From basic set theory, this last condition means that we can write $\phi^{-1}(N) = \cup\{G_x \times G_y, xy \in N\}$. Since each subset $G_x \times G_y$ is open in the product topology $G \times G$, and $\phi^{-1}(N)$ is the union of such open sets, it follows that $\phi^{-1}(N)$ is open also in $G \times G$, and therefore, the map ϕ is continuous. QED.

Moreover, the group multiplication operation must satisfy the following additional axioms

(i) Closure: $\chi, \xi \in \mathcal{M} \Rightarrow \gamma = \phi(\chi, \xi) \in \mathcal{M}$.
(ii) Associativity: $\phi(\gamma, \phi(\chi, \xi)) = \phi(\phi(\gamma, \chi), \xi)$.
(iii) Identity: $\forall \xi \in \mathcal{M},\ \exists\, e \in \mathcal{M} \mid \phi(\xi, e) = \phi(e, \xi) = \xi$.
(iv) Inverse: $\forall \xi \in \mathcal{M}, \exists \xi^{-1} \in \mathcal{M} \mid \phi\left(\xi, \xi^{-1}\right) = \phi\left(\xi^{-1}, \xi\right) = e$.

In other words, the continuous operation functions $\phi^m,\ m = 1, \ldots, M$, defined on the topological space $\mathcal{M}$ must be compatible with the regular axioms of the group. It is evident from the definition that in a topological group, both the group operation and the inverse map send points close enough to each other into points also lying in the neighborhood of each other.

In this paper, the most important topological property of a given group will be connectivity. We say that a space is *connected* if any two points in the space can be joined with each other by a line, and all the points on this line are contained in the original space. Every topological space may be partitioned into connected components. The *connected component* of a point p (also called a *sheet*) is the set of all points that can be joined with p by a line as described above. The number of connected components in a given space is a topological invariant, i.e., is preserved precisely between homeomorphic spaces.

A connected component of a topological group must contain the identity element in order to be able to become a group itself. We have the following important theorem

Theorem D.1 *The component of a topological group G that is connected to the identity forms a group G_0.*

A connected space is *simply connected* if any line joining two points in this space can be continuously deformed into every other curve connecting the same two points.$^\diamond$ We note the following assuring theorem which states that topological groups give rise to regular (nonpathological) topological spaces:

Theorem D.2 *Each topological group is Hausdroff.*

For classical groups, we have the following important result:

Theorem D.3 *The general linear group* $\mathrm{GL}(n, \mathbb{C})$ *is neither compact nor connected. Two connected components constitute the topological space of the group, one with negative determinant and the other with positive determinant. The group* $\mathrm{SL}(n, \mathbb{C})$ *is simply connected.*

This theorem was used extensively in Section 5 on the topological applications to generalized matching in microwave networks.[††]

APPENDIX E. TOPOLOGICAL HOMOGENEOUS SPACES

The concept of homogeneous space is fundamental in this paper. As we will see, topological groups possess a very unique structure called homogeneity (to be defined below.) It amounts to the observation that the group looks topologically the same when being viewed from any particular location inside its own space. This will prove to be of fundamental importance in the systematic analysis of Lie groups since one can focus only on one point, say the identity element, where performing calculations is considerably simplified by the existence of Lie algebras. Moreover, the conclusions obtained from such a study, say the topological behavior around this identity, can be straightforwardly transported to any other location.

Let G be a topological group. Fix any element $g \in G$. We say that g defines a *left-translation* function $L_g : G \to G$ such that for any $h \in G$, we have $L_g(h) = gh$. That is, the left-translation of h by g is nothing but their multiplication gh. Similarly, *right-translations* can be defined as functions

$^\diamond$ For a rigorous definition of the technical term *continuous deformation*, see [3, 31, 40].
†† For a good general background on group theory, see [10, 48].

$R_g : G \to G$ with $R_g(h) = hg$ for any $h \in G$. The conclusions to be discussed below about left-translations applies exactly to the right-translation just defined. Now, let the group multiplication operation be given by $m : G \times G \to G$, and let $\mu_g : G \to G \times G$ be the function defined by $\mu_g(h) = (g, h)$. It follows that $L_g = m \circ \mu_g$. Now we show that μ_g is continuous. Take any base of $G \times G$, i.e., a "rectangle" composed of direct product $U = G_1 \times G_2$ of two open sets G_1 and G_2 in G. Now U is by definition open in the product space $G \times G$. The inverse image of U under μ_g is just the set $\mu_g^{-1}[G_1 \times G_2] = G_2$, which is open in G. This shows that μ_g is continuous because the inverse image of open sets is open. QED. Next, by the axioms of topological groups, the function m is also continuous. Since the composition of two continuous functions is continuous, it follows that L_g is continuous for every $g \in G$.[‡‡] But the same argument can be developed for the case of L_g^{-1}, the left-translation by g^{-1}. Moreover, we can compute $L_g L_{g^{-1}}(h) = L_g(L_{g^{-1}}(h)) = L_g(g^{-1}h) = gg^{-1}h = h$, and hence $L_g L_{g^{-1}} = 1_G$. Similarly, we can find $L_{g^{-1}} L_g = 1_G$. Therefore, $L_g^{-1} = L_{g^{-1}}$. To summarize the results we have just obtained, the function L_g is a homeomorphism from G to G.

A topological space T is called *homogenous* iff it has the following property: For any two points p_1 and p_2 in T, there exists a homeomorphism f from T to itself such that f takes p_1 into p_2, i.e., $p_2 = f(p_1)$. It follows immediately that every topological group is also a homogenous space; indeed, the construction $L_{p_2} L_{p_1^{-1}}$ is a homeomorphism doing exactly this. Such spaces look topologically the same when being viewed from any point. To understand this, assume for example that we let a topological group G be locally homeomorphic to the Euclidean space, i.e., there exists a neighborhood N to the identity element $e \in G$ homeomorphic to $\mathbb{R}^n$. But consider the left translation of N to $L_g[N]$. It is obvious that $g = ge \in L_g[N]$. Since L_g is a homeomorphism, every open set containing e is translated into an open set containing g (the converse is also true.) Thus, $L_g[N]$ is a neighborhood of g. It is implied then that this neighborhood is also homeomorphic to $\mathbb{R}^n$ because N is so.

Therefore, given a topological group, the fact that the underlying topological space is homogenous implies that if we know the local topological behavior around a given point (say we know the *local* topological base at the identity element), then we have a complete knowledge of the local topological behavior around *any* point in G whatsoever (the local base there is nothing but the proper translation of the given local base.) This fact is of paramount importance in the theory of Lie groups and the way we employ this theory to study the topological structure of microwave circuit theory. Therefore, it is worthy spending some time contemplating it.

We define a *nucleus* as a neighborhood of identity. This terminology reflects the fact that such neighborhoods are special. Indeed, since algebraic manipulations around the identity takes a simpler form, it is tempting to use knowledge obtained by studying the local behavior around this element of the topological group to obtain knowledge about the neighborhood surrounding other elements. Consider the situation depicted in Figure E1 where we show a nucleus of the identity element e in a given topological group. It is obvious that the left translation L_g will take e into g, where $g \in G$ is not contained in N. Moreover, since this is a homeomorphism it is also open, and therefore, the image $L_g(N)$ is also a neighborhood of g. Conversely, if $L_g(N)$ is a neighborhood of g, then it contains an open set containing g. The inverse image of this set under the map L_g^{-1} is open since this function is continuous. Now this argument can be repeated for every point contained in N. Indeed, take x to be in N and define the map $L_y L_x^{-1}$, where y is a point in the neighborhood of g (not necessary in N). This map will take x into y. Moreover, since it is homeomorphism, any open set containing x will be mapped into an open set containing y. We phrase this result in the following way: If A is a family of nuclei of the topological group G, and $g \in G$, then the family of all neighborhoods of g is the translations of members of A, that is, the set $\{L_g(N), N \,\text{is a nucleus of}\, G\}$. Now one can use this relation to do topology at any location in the topological group starting from the knowledge of the behavior around the identity. For example, suppose that we are given a class of sets containing g and that we want to know whether they are open, closed, etc., in the standard topology of G. We know that a given subset S is open iff for each point $s \in S$, and there exists a nucleus N such that $L_g(N) \subseteq S$, which amounts of course to saying that S is a neighborhood to each of its points. Therefore, *any statement on topology surrounding an element in the topological group can be translated into the corresponding statement about the topology around the*

[‡‡] Notice that the projections maps from $p_i : G \times G \to G, i = 1, 2$ are open, and hence one can rephrase the previous result in the following manner: *The subspace $g \times G$ is homeomorphic to G.*

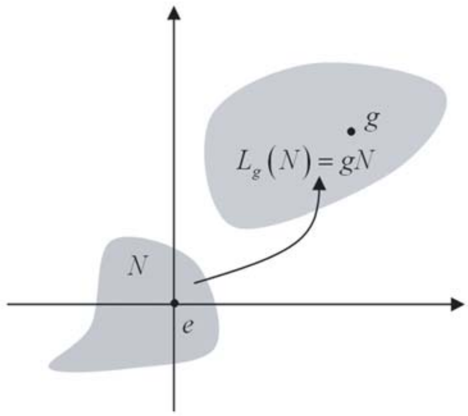


Figure E1. Translation of a nucleus into a new neighborhood around g.

identity element e. This latter statement can be checked or analyzed by means of the theory developed for the nucleus. We have established the following fact about topological groups:

Theorem E.1 *Homogeneity is a topological invariant.*

As a further illustration of the previous facts, let us introduce the following definition. A topological space is called *locally connected* if every neighborhood of any point in this space contains a connected open neighborhood. It follows immediately then that a topological group is locally connected if it is locally connected at the identity. Suppose that G is locally connected at e. Then, there exists an open connected set V containing e. Now, take any other element $g \in G$. The left translated set $L_g(V)$ containing g is also open and connected (since L_g is a homeomorphism.) Then G is locally connected at g. QED.

Let U be any subset $U \subseteq G$ of the topological group G. We call the smallest subgroup containing U the *subgroup generated by* U.

Theorem E.2 *In any locally connected group G, the component of the identify G_0 is generated by any connected neighborhood of the identity e.*

Consider the topological group $\mathrm{GL}(n, \mathbb{C})$. Since it is a metric space topology, it is locally connected. It follows that given any microwave circuit m, there exists a connected open set U_m containing m. In metric spaces, connected sets are path connected. Then, we can continuously match m with any circuit within U_m. However, this set may be "small" in the sense that practically we cannot detect those circuits that can be matched to m. We need then to introduce the measure of *distance* in the topological (Lie) group to quantify how "close" two microwave circuits are to each other. The usual metric defined on $\mathrm{GL}(n, \mathbb{C})$ should do the job.[ll]

APPENDIX F. HOMOTOPY

Homotopy is of fundamental importance in topology since it provides a natural methodology for the investigation of the connectivity properties of various topological spaces. In this paper, homotopic path is shown to be the essential deciding factor in whether two given microwave networks can be *continuously* deformed to match each other. A *curve* is a continuous map from the real interval $I = [0, 1]$ to a given topological space $\mathfrak{S}$. A path *joining two points* p and q is defined as a continuous map $t : I \longrightarrow \mathfrak{S}$ such that $t(0) = p$ and $t(1) = q$. Consider two curves $t_1 = t_1(s)$ and $t_2 = t_2(s)$ that are continuous functions on the parameter $s \in [0, 1]$ with a common end points, i.e., we have $t_1(0) = t_2(0)$ and $t_1(1) = t_2(1)$. The two curves are said to be *homotopic* to each other if there exists a continuous function $t(r, s)$ defined on the intervals $0 \leq r,\ s \leq 1$, such that $t(0, s) = t_1(s)$ and $t(1, s) = t_2(s)$. It follows then that if two curves

[ll] The subject of topological homogeneous space (sometimes called uniform spaces) is covered in [21, 28, 48].

are homotopic to each other, then one is obtained from the other by a contentious deformation in the parameter r. The above definition aims then to capture in rigorous manner the intuitive conception of how two paths can be matched to each other in a continuous fashion. The set of all curves homotopic to a given curve t is called the *homotopy class* of t and is designated $[t]$.[§§]

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[§§] For an intuitive and physics-oriented overview on homotopy, see [31]. For a more extensive treatment, see [3, 40].

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