

# A Computationally Efficient Modified MUSIC Spectrum for Resolving DOAs of Multiple Closely Spaced Non-Gaussian Sources

Chandrasekaran Ashok<sup>1, \*</sup> and Narasimhan Venkateswaran<sup>2</sup>

**Abstract**—The objective of this work is to estimate the Direction of Arrival (DOA) of signals from multiple closely spaced non-Gaussian sources corrupted by additive Gaussian noise. Generally, this is achieved by using higher order statistics (HoS) based MUSIC spectrum. In HoS, the fourth-order cumulant is utilized because of its property of insensitivity to Gaussian process. But in the case of resolving closely spaced sources, a large number of sensor elements are required; otherwise, the resolution gets deteriorated. The large number of sensor elements leads to high computational burden. We propose a computationally efficient modified spectrum that combines fourth-order cumulant based MUSIC spectrum and its second-order differential counterparts. The proposed spectrum for DOA estimation offers good statistical performance and better accuracy the existing methods even in the case of extremely closely spaced signal sources. The improvement in the aspects of resolution and accuracy is substantiated by means of various simulation results such as Monte Carlo simulations, spectral width, resolution with respect to angular separation, and comparison of RMSE with respect to number of array elements, number of snapshots, and SNR. The computational complexity analysis of the proposed method is also presented.

## 1. INTRODUCTION

In the field of array signal processing, DOA estimation is utilized in various applications like military, mine fields, and other areas where there is no ease of access. The method of DOA estimation can also be used in underwater applications in order to locate and track the sources of various signals. Nowadays, wireless technology is growing fast, and 5G technology will soon occupy a place of prime importance [1–3]. This can be achieved only with a new design of antennas such as Smart Antenna systems [4], in which DOA estimation plays an important role in localizing the required entities in order to direct the beam towards the source and a null in the other directions. Audio Zooming is one application wherein DOA estimations are used [5]. The source of the sound is located, and the particular sound can be amplified or zoomed in order to get a clearer and louder sound. So the algorithm used in DOA estimation should be highly precise in locating the signal sources and also insensitive to the noise.

The most widely used DOA estimation method in wide range of applications is MUSIC (Multiple Signal Classification) spectrum [6] over other methods [7–10]. This method involves the formulation of auto-correlation matrix from the noise corrupted received signal upon which eigen decomposition is carried out to bifurcate the obtained eigenvectors into the signal and noise subspaces. In the case of second order statistics auto-correlation matrices, there is a disadvantage of impossibility to completely eliminate the effect of additive Gaussian noise, if the noise varies from sensor to sensor in an unknown way. This can be battled by the use of higher order statistics — fourth-order cumulants to compute received signal correlation [11–13], whereas the cumulant based MUSIC spectrum makes it possible

---

*Received 6 February 2020, Accepted 27 May 2020, Scheduled 10 June 2020*

\* Corresponding author: Chandrasekaran Ashok (cashokengineer@gmail.com).

<sup>1</sup> Department of Electronics and Communication Engineering, St. Joseph's College of Engineering, Chennai, India. <sup>2</sup> Department of Electronics and Communication Engineering, SSN College of Engineering, Chennai, India.

to exploit the sensor-to-sensor independence of noise which results in better resolution for the case of non-Gaussian sources signal getting corrupted with additive Gaussian noise [14–16]. However, in many practical situations, the sources are found to be spaced very close to each other, which makes it more difficult to localize individual sources [18–20]. To resolve the DOAs of closely spaced sources, a large number of sensor elements and snapshots are required in the existing methods, which leads to high computational burden. This problem of estimating the DOAs of multiple closely spaced sources with low computational complexity needs to be solved, to make it applicable to hardware implementation to various practical applications.

In this work, a computationally efficient modified spectrum is proposed which combines fourth-order cumulant based MUSIC spectrum and its second-order differential counterparts. The proposed spectrum for DOA estimation offers good statistical performance and better accuracy the existing methods even in the case of extremely closely spaced signal sources.

The remainder of the paper is organized as follows. Section 2, furnishes the mathematical formulation of the signal model, cumulant based MUSIC spectrum, and the proposed spectrum. Section 3 presents several simulation results with relevant discussions. Section 4 gives the conclusion and future directions of the presented work.

## 2. MATHEMATICAL FORMULATION

### 2.1. Signal Model

In this work, it is assumed that there are  $R$  narrowband non-Gaussian signals impinging on an  $N$  element antenna array. The inter-element spacing of the array elements is taken to be half-a-wavelength. The non-Gaussian signals are considered to be QPSK signals. The narrow band approximation of the received signal is represented as given,

$$x_k[n] = \sum_{i=0}^{R-1} s_i[n] a_k(\theta_i) + v_k[n], \quad k = 0, 1, 2, \dots, N-1 \quad (1)$$

where  $a_k(\theta_i) = e^{-j\pi k \sin \theta_i}$  is the response of the  $k$ th array element with respect to the  $i$ th signal;  $\theta_i$  is the DOA of the  $i$ th signal  $s_i[n]$ ; and  $v_k[n]$  is the additive noise at the  $k$ th element. In the vector form, the above Equation (1) can be represented as

$$\mathbf{X}[n] = \mathbf{A}\mathbf{S}[n] + \mathbf{V}[n] \quad (2)$$

where  $\mathbf{A}$  is the array steering matrix of incoming signal.  $\mathbf{V}[n]$  is the vector having random variables which has Gaussian distribution, i.e., additive white Gaussian noise with zero mean and variance  $\sigma^2$ . The matrix form of Equation (1) can be written as given below.

$$\begin{bmatrix} x_0[n] \\ x_1[n] \\ \vdots \\ x_{N-1}[n] \end{bmatrix} = \begin{bmatrix} a_0(\theta_0) & a_0(\theta_1) & \cdots & a_0(\theta_{r-1}) \\ a_1(\theta_0) & \cdot & \cdot & \cdot \\ \vdots & \vdots & \vdots & \vdots \\ a_{N-1}(\theta_0) & \cdot & \cdot & a_{N-1}(\theta_{r-1}) \end{bmatrix} \begin{bmatrix} s_0[n] \\ s_1[n] \\ \vdots \\ s_{r-1}[n] \end{bmatrix} + \begin{bmatrix} v_0[n] \\ v_1[n] \\ \vdots \\ v_{N-1}[n] \end{bmatrix} \quad (3)$$

### 2.2. Fourth Order Cumulant Based MUSIC Spectrum

Due to inability to separate non-Gaussian signals from Gaussian additive noise posed by the second order statistics based method, the use of higher order statistical computations is preferred, in order to obtain a more accurate estimation of DOA of the signals from various sources.

The fourth-order cumulant of the received array element output can be expressed in matrix notation as given below,

$$C = E \left[ (x \otimes x^*) (x \otimes x^*)^H \right] - E [(x \otimes x^*)] E \left[ (x \otimes x^*)^H \right] - E [xx^H] \otimes E \left[ (xx^H)^* \right] \quad (4)$$

where  $\otimes$  denotes the Kronecker product, and the dimension of the cumulant matrix  $C$  is  $N^2 \times N^2$ . Since the fourth-order cumulant of Gaussian process is zero, Eq. (4) can be written as

$$\mathbf{C} = (\mathbf{A} \otimes \mathbf{A}^*) \mathbf{C}_s (\mathbf{A} \otimes \mathbf{A}^*)^H \quad (5)$$

where

$$\mathbf{C}_s \underline{\Delta} E \left[ (s \otimes s^*) (s \otimes s^*)^H \right] - E \left[ (s \otimes s^*) \right] E \left[ (s \otimes s^*)^H \right] - E \left[ s s^H \right] \otimes E \left[ (s s^H)^* \right] \quad (6)$$

The above Equation (6) represents the fourth-order cumulant of sources  $S$ . Since the sources are statistically independent, the fourth-order cumulant of the signal vector  $S$  is a diagonal matrix, so Eq. (5) can be written as follows,

$$\mathbf{C} = \sum_{i=0}^{R-1} (a(\theta_i) \otimes a^*(\theta_i)) \mu_i (a(\theta_i) \otimes a^*(\theta_i))^H \quad (7)$$

where  $\mu_i$  is the fourth-order cumulant of  $s_i$  for  $i = 0, 1, 2, \dots, R - 1$ , then Eq. (7) can be written as

$$\mathbf{C} = \mathbf{BDB}^H \quad (8)$$

where

$$\begin{aligned} \mathbf{D} &= \text{diag}\{\mu_0, \mu_2, \dots, \mu_{R-1}\} \\ \mathbf{B} &= [b(\theta_0), b(\theta_1), \dots, b(\theta_{R-1})] \end{aligned}$$

since  $\mathbf{B}$  is composed of  $b(\theta_i) = a(\theta_i) \otimes a^*(\theta_i)$ ,  $i = 0, 1, \dots, R - 1$ , which are linearly independent, it has full column rank, and since the sources are independent,  $\mathbf{D}$  is non-singular. The range of  $\mathbf{C}$ , which is a  $N^2$  dimensional space, can be decomposed into two orthogonal subspaces:

*i.* An  $R$  dimensional subspace, called signal subspace, which is spanned by the eigenvectors corresponding to the  $R$  largest eigenvalues.

*ii.* The complementary  $N^2 - R$  dimensional subspace called noise subspace. The signal subspace and then noise subspace can be found by eigen decomposition of the cumulant matrix  $\mathbf{C}$ .

Let  $\lambda_1 \lambda_2, \dots, \lambda_{N^2}$  denote the eigen values of  $\mathbf{C}$  and  $v_1, v_2, \dots, v_{N^2}$  the corresponding eigen vectors. Then, the eigen vectors satisfy the equations

$$\mathbf{C}v_k = \lambda_k v_k \text{ for } k = 1, 2, \dots, N^2 \text{ for } k = 1, 2, \dots, N^2 \quad (9)$$

From elementary linear algebra, it can be shown that  $N^2(-R)$  eigenvalues of  $\mathbf{C}$  will be equal to zero. Without loss of generality, the eigenvalues can be arranged in non-decreasing order as

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_R > \lambda_{R+1} = \lambda_{R+2} = \dots = \lambda_{N^2} = 0$$

Let signal subspace  $\mathbf{Q} \underline{\Delta} [v_1, v_2, \dots, v_R]$  be a  $N^2 \times R$  matrix and noise subspace  $\mathbf{V} \underline{\Delta} [v_{R+1} v_{R+2}, \dots, v_{N^2}]$  be a  $N^2 \times (N^2 - R)$  matrix, then

$$\mathbf{B}^H \mathbf{V} = \mathbf{0} \quad (10)$$

This means that the eigenvectors, associated with the eigenvalue zero of multiplicity  $(N^2 - R)$ , are orthogonal to the  $\mathbf{B}$  vector, which is given below

$$\begin{aligned} \mathbf{B} &= [b(\theta_0), b(\theta_1), \dots, b(\theta_{R-1})] \\ b(\theta_i) &= a(\theta_i) \otimes a^*(\theta_i), \quad i = 0, 1, \dots, R - 1 \end{aligned} \quad (11)$$

This implies that the vectors,  $b(\theta_i)$ ,  $i = 0, 1, 2, \dots, R - 1$ , are orthogonal to the eigenvectors corresponding to the zero eigenvalues, and the true DOA parameters  $\theta_i$ ,  $i = 0, 1, 2, \dots, R - 1$ , of the sources are the unique solutions of the equation

$$b^H(\theta) \mathbf{V} \mathbf{V}^H b(\theta) = 0 \quad (12)$$

Practically, the actual eigenvalues and eigenvectors of fourth-order cumulant are not known, and they ought to be estimated from the received signal snapshots, given by

$$\hat{\mathbf{C}} = \frac{1}{K} \sum_{t=1}^K y(t) y^H(t) - \frac{1}{K^2} \sum_{t=1}^K y(t) \sum_{t=1}^K y^H(t) - (\hat{R} \otimes \hat{R}^*) \quad (13)$$

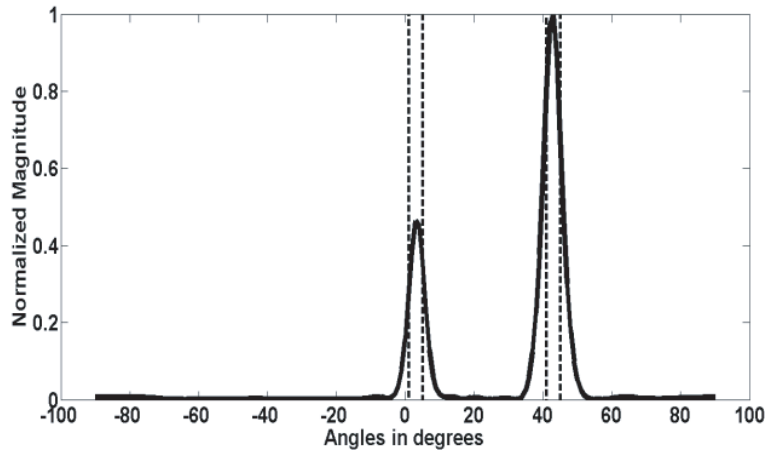
where  $y(t) \underline{\Delta} x(t) \otimes x(t)^*$ ,  $\hat{R} \underline{\Delta} \frac{1}{K} \sum_{t=1}^K x(t) x^H(t)$  and  $K$  refers to the number of snapshots utilized.

Thus, one can estimate the vectors  $b(\theta_i)$ ,  $i = 0, 1, 2, \dots, R - 1$  by finding the vectors which are most nearly orthogonal to the eigenvectors corresponding to the  $(N^2 - R)$  eigenvalues of  $\hat{C}$  which are approximately zero. Let  $\hat{V}$  denote the matrix defined similarly to  $\mathbf{V}$ , but made from the eigenvectors of  $\hat{C}$ . Then, the DOA of the multiple source signals can be estimated by locating the  $R$  largest peaks of the spatial spectrum given by

$$F(\theta) = \frac{1}{b^H(\theta) \mathbf{V} \mathbf{V}^H b(\theta)}, \quad \theta \in \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] \quad (14)$$

Even though fourth-order statistics based MUSIC algorithm provides improved estimation of signal parameters and has the ability to separate statistically independent sources, it is found to be inefficient in distinguishing signal sources which are very close to each other. This is provided with a simple illustration given below.

When the signals coming from multiple sources with an angle of  $\theta = 1^\circ, 5^\circ, 41^\circ$  and  $45^\circ$  are received by an  $N = 14$  element antenna array with  $K = 10$  snapshots and noise variance of 0.1, the resulting fourth-order cumulant based MUSIC spectrum is given in Figure 1.



**Figure 1.** Fourth-order cumulant based MUSIC spectrum.

From Figure 1, It is clearly evident that the fourth-order cumulant based MUSIC spectrum fails to resolve the sources. In this work, this problem is overcome by modifying the existing method.

### 2.3. Proposed Method

The main drawback of the existing method is that it is unable to resolve sources as individual distinct ones when they are very close to each other. This problem is overcome by modifying the existing method by a sequence of procedures.

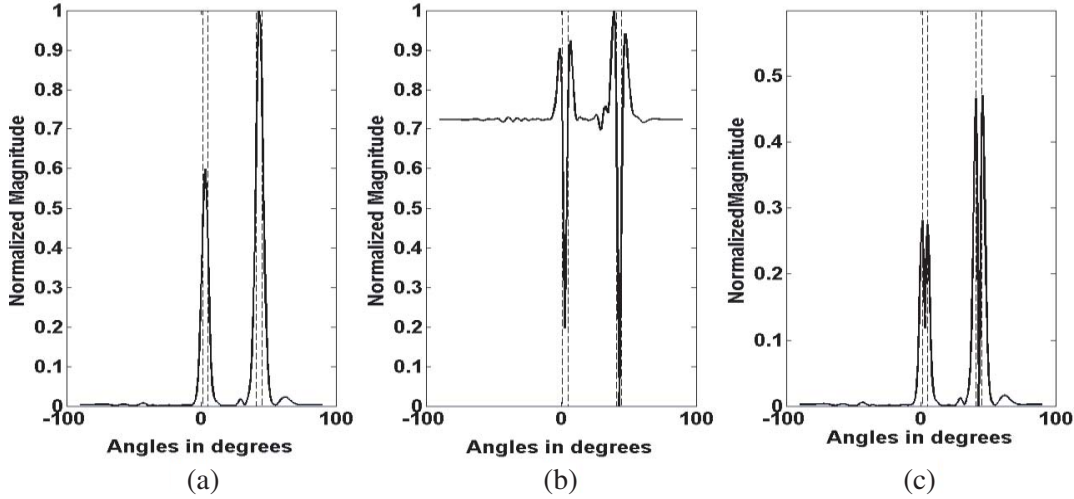
Firstly, the fourth-order cumulant based MUSIC spectrum is obtained, which is given by  $P_{C\text{-MUSIC}}(\theta)$

$$P_{C\text{-MUSIC}}(\theta) = \frac{1}{b^H(\theta) \mathbf{V} \mathbf{V}^H b(\theta)}, \quad \theta \in \left[ \frac{-\pi}{2}, \frac{\pi}{2} \right] \quad (15)$$

Then, the second-order differentiation is applied on the fourth-order cumulants based MUSIC spectrum. This is given by

$$P''(\theta) = \left( \frac{(-P_{C\text{-MUSIC}}(\theta - \Delta\theta) + 2P_{C\text{-MUSIC}}(\theta) - P_{C\text{-MUSIC}}(\theta + \Delta\theta))}{(\Delta\theta)^2} \right) \quad (16)$$

where  $\Delta\theta$  is discrete angle interval, and applying the values of  $P_{C\text{-MUSIC}}$  in Eq. (16), the resultant



**Figure 2.** (a)  $P_{C-MUSIC}(\theta)$ : Fourth order Cumulant based MUSIC spectrum; (b)  $P''(\theta)$ : Second order differentiation of  $P_{C-MUSIC}(\theta)$ ; (c)  $P_{MC-MUSIC}(\theta)$ : Modified spectrum.

equation is given by

$$P''(\theta) = \left( \frac{\left( -\frac{1}{b^H(\theta - \Delta\theta)VV^Hb(\theta - \Delta\theta)} + 2\frac{1}{b^H(\theta)VV^Hb(\theta)} - \frac{1}{b^H(\theta + \Delta\theta)VV^Hb(\theta + \Delta\theta)} \right)}{(\Delta\theta)^2} \right) \tag{17}$$

The second-order differentiation of fourth-order cumulant based MUSIC spectrum  $P''(\theta)$  results in peaks on the signal directions, which is not identified in the  $P_{C-MUSIC}(\theta)$  spectrum as shown in Figure 2(a). In addition, there are also lots of other spurious peaks, shown in Figure 2(b). In order to extract only the necessary signal peaks, the differentiated MUSIC spectrum  $P''(\theta)$  is multiplied with the resultant of undifferentiated fourth-order cumulant based MUSIC spectrum as shown in Figure 2(c) which is given by

$$P_{MC-MUSIC}(\theta) = P_{C-MUSIC}(\theta) \cdot P''(\theta) \tag{18}$$

Applying  $P_{C-MUSIC}(\theta)$  and  $P''(\theta)$  in the equation of  $P_{MC-MUSIC}(\theta)$  is given by

$$P_{MC-MUSIC}(\theta) = \left( \frac{1}{b^H(\theta)VV^Hb(\theta)} \right) \cdot \left( \frac{\left( -\frac{1}{b^H(\theta - \Delta\theta)VV^Hb(\theta - \Delta\theta)} + 2\frac{1}{b^H(\theta)VV^Hb(\theta)} - \frac{1}{b^H(\theta + \Delta\theta)VV^Hb(\theta + \Delta\theta)} \right)}{(\Delta\theta)^2} \right) \tag{19}$$

### 3. SIMULATION RESULTS AND DISCUSSION

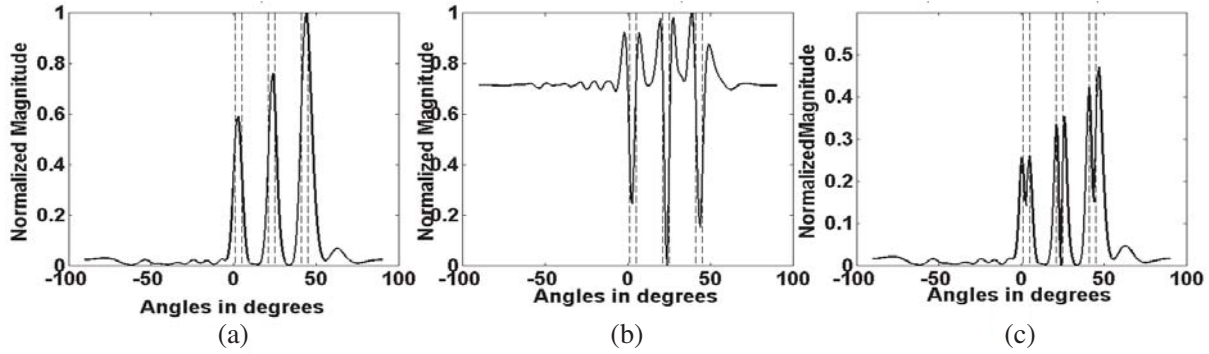
In this section, the potential of the proposed method is validated with several computer simulations. The simulation setup used for the validation is described as follows: A uniform linear array (ULA) with half wavelength spacing for each sensor is used for receiving multiple signals for DOA estimation. The received signals at the array are narrowband non-Gaussian digitally modulated quadrature phase shift keying (QPSK) which is considered. The incoming signals are corrupted with uncorrelated additive complex white Gaussian noise with variance  $\sigma^2$ . All the computer simulations are performed on a PC equipped with a 1.80 GHz processor and RAM of 8 GB. The implementation is performed in MATLAB.

The performance metric used to measure the accuracy of DOA estimation is given by root mean square error (RMSE) and is defined by

$$\text{RMSE} = \sqrt{\frac{1}{RL} \sum_{k=1}^R \sum_{l=1}^L (\hat{\theta}_{kl} - \theta_k)^2}$$

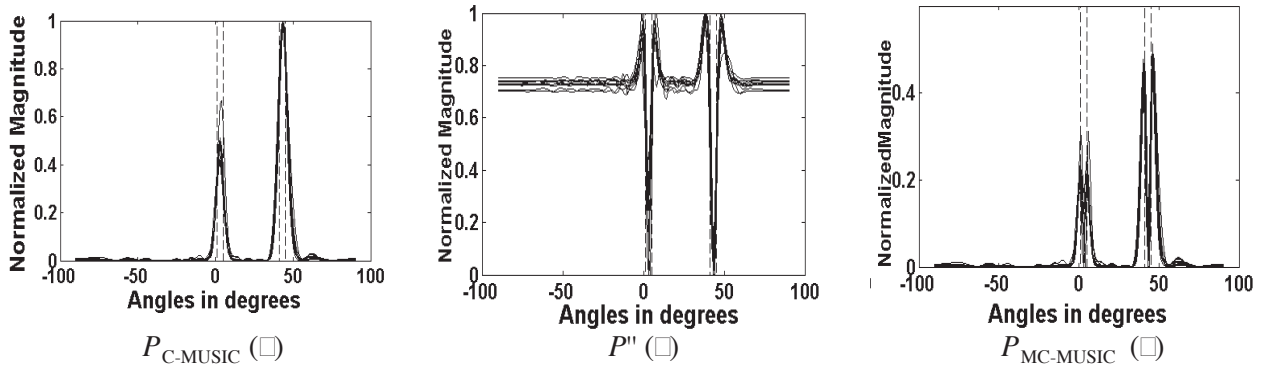
where  $L$  is the number of Monte Carlo runs,  $\hat{\theta}_{kl}$  the estimated DOA, and  $\theta_k$  the actual DOA of the  $k$ th signal in the  $l$ th experiment [7].

The proposed method is validated for the case of signal from six sources with equal power ranging from angles  $\theta = 1^\circ, 5^\circ, 21^\circ, 25^\circ, 41^\circ, 45^\circ$  which is received using a ULA with  $N = 14$  with  $K = 10$  snapshots and noise variance of  $\sigma^2 = 0.1$ , and the resulting spectrum is shown in Figure 3.



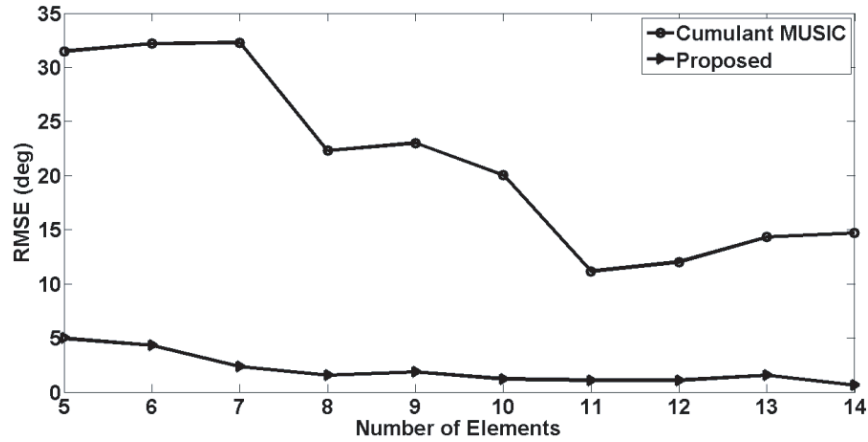
**Figure 3.** (a)  $P_{C\text{-MUSIC}}(\theta)$ : Fourth-order cumulant based MUSIC spectrum; (b)  $P''(\theta)$ : Second-order differentiation of  $P_{C\text{-MUSIC}}(\theta)$ ; (c)  $P_{MC\text{-MUSIC}}(\theta)$ : Modified spectrum.

To ensure that the performance of proposed method is satisfactory, Monte Carlo trials are carried out. The number of Monte Carlo trails is taken to be 10 for the case of signal from four sources with angles of  $\theta = 1^\circ, 5^\circ, 41^\circ,$  and  $45^\circ$  which is received by an  $N = 14$  element antenna array with  $K = 10$  snapshots and noise variance of 0.1. The resulting spatial spectra of both the Cumulant MUSIC and proposed method are presented in Figure 4.



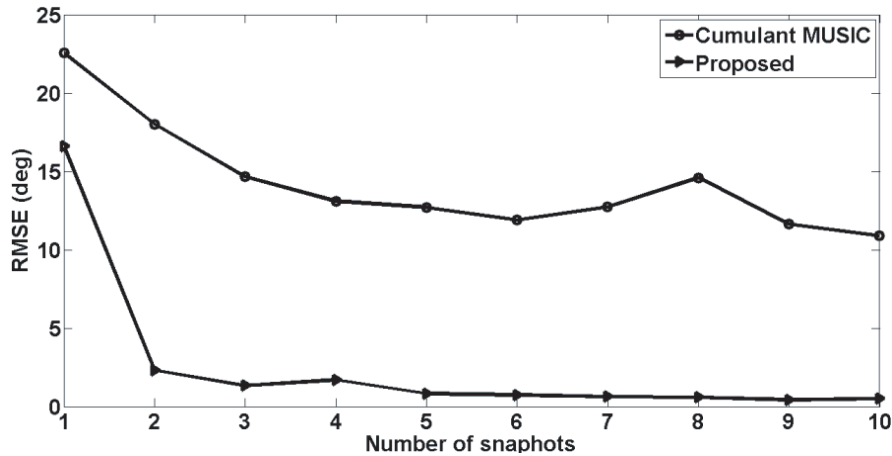
**Figure 4.** Comparison of 10 Monte Carlo trials of  $P_{C\text{-MUSIC}}(\theta)$  and  $P_{MC\text{-MUSIC}}(\theta)$ .

For the same simulation setup, the root means square error (RMSE) performance of the cumulant based MUSIC spectrum and proposed spectrum for DOA estimation of with respect to number of elements is obtained. In that, the number of sensor elements is varied from 5 to 14. From Figure 5 it is evident that the proposed method results in good estimation accuracy as the number of elements increases over the Cumulant MUSIC method.



**Figure 5.** Comparison of RMSE performance of cumulant based MUSIC spectrum and proposed spectrum with respect to number of elements.

In Figure 6, the performance of the proposed spectrum is also evaluated by considering RMSE as a function of number of snapshots. The snapshots are varied from  $K = 1$  to 10. It is observed that the proposed spectrum gives a robust performance of estimation accuracy even in the case of low snapshot considerations.

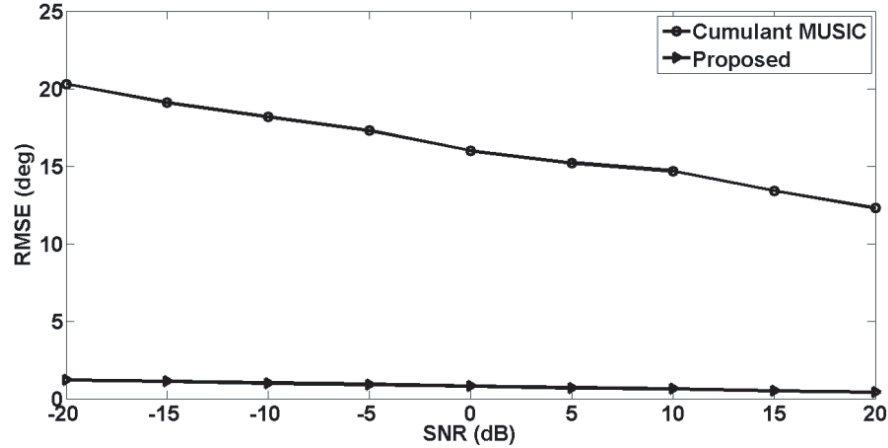


**Figure 6.** Comparison of RMSE of cumulant based MUSIC spectrum and proposed spectrum with respect to number of snapshots.

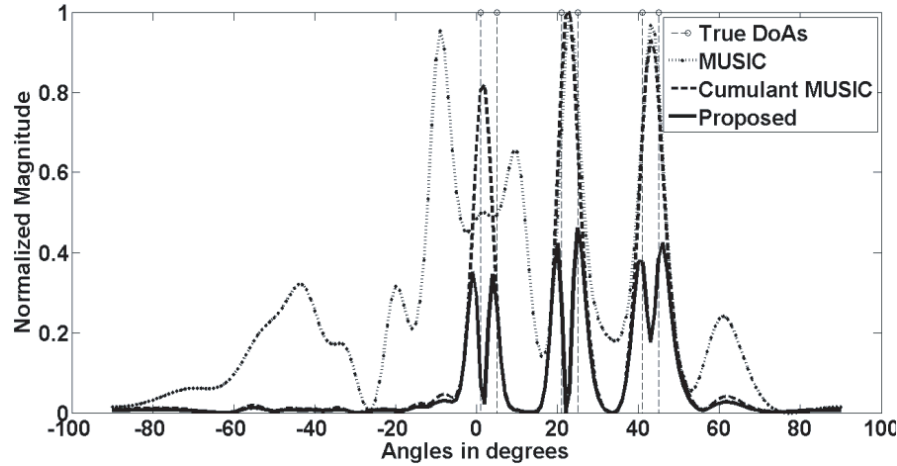
In Figure 7, the performance of the proposed spectrum is also evaluated by considering RMSE as a function of SNR. The SNR is varied from  $-20$  dB to  $20$  dB, it is observed that the proposed spectrum gives a robust performance of estimation accuracy even at low SNR regime.

Also, the proposed spectrum is compared with the existing spectra such as MUSIC spectrum, Cumulant MUSIC spectrum for DOA estimation for the case of signal from six sources with equal power ranging from angles  $\theta = 1^\circ, 5^\circ, 21^\circ, 25^\circ, 41^\circ, 45^\circ$  which is received using a ULA with  $N = 14$  with  $K = 10$  snapshots and noise variance of  $\sigma^2 = 0.1$ . It is inferred that the proposed spectrum offers a better performance than the existing spectrum-based methods, shown in Figure 8.

The resolution of the proposed spectrum is evaluated with respect to the angular separation for the case of signal from two sources with equal power which is received using a ULA with  $N = 14$  with  $K = 10$  snapshots and noise variance of  $\sigma^2 = 0.1$ . The resolution capability of the proposed spectrum for resolving two sources is enumerated with the detection probability. So here, the angular separation considered ranges from  $2^\circ, 3^\circ, 4^\circ, 5^\circ, 6^\circ, 7^\circ, 8^\circ$  to  $9^\circ$ . From Figure 9, it is very evident that



**Figure 7.** Comparison of RMSE of cumulant based MUSIC spectrum and proposed spectrum with respect to SNR.



**Figure 8.** Comparison of proposed spectrum with MUSIC and cumulant based MUSIC spectrum.

the proposed spectrum resolves the two sources with two peaks even at the angular separation of  $2^\circ$  whereas the Propagator [16] resolves them at the angular separation of  $8^\circ$ , and cumulant based MUSIC spectrum [17] and Propagator-Cumulant [21] resolve them at the angular separation of  $6^\circ$ , respectively. Also, the resolution performance of the proposed spectrum is evaluated by computing the half-power spectral width, i.e., 3 dB around the spectral peak of the source, which is tabulated in Table 1.

**Table 1.** Resolution of the proposed spectrum.

DOAs	$1^\circ$	$5^\circ$	$41^\circ$	$45^\circ$
3-dB Spectral width	$3.3^\circ$	$3.1^\circ$	$2.9^\circ$	$3.4^\circ$

The computational complexity for constructing the cumulant matrix is given by  $O((N^2)^2K)$ . So, the number of complex multiplications required by the proposed spectrum with respect to angular separation over the existing method is presented in Figure 10. It is very evident that the proposed spectrum has the potential to resolve very closely spaced sources with minimum number of complex multiplications over the Cumulant MUSIC spectrum.



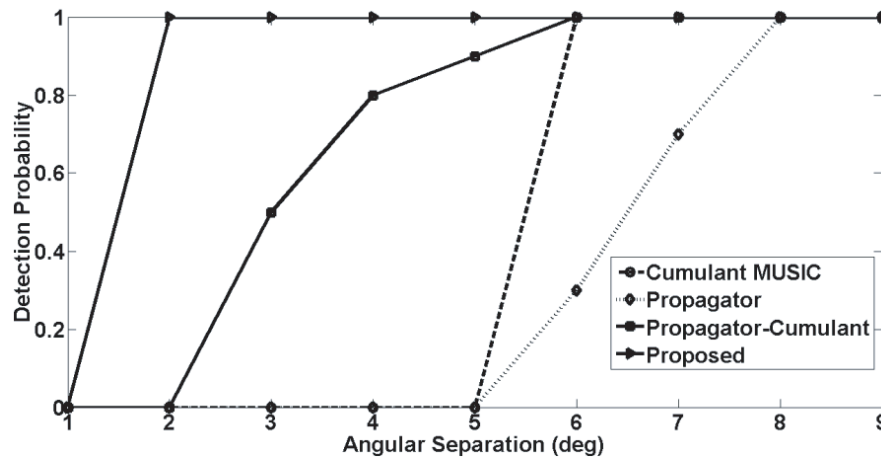


Figure 9. Comparison of detection probability of proposed spectrum, Cumulant MUSIC, propagator, propagator-Cumulant with respect to angular separation.

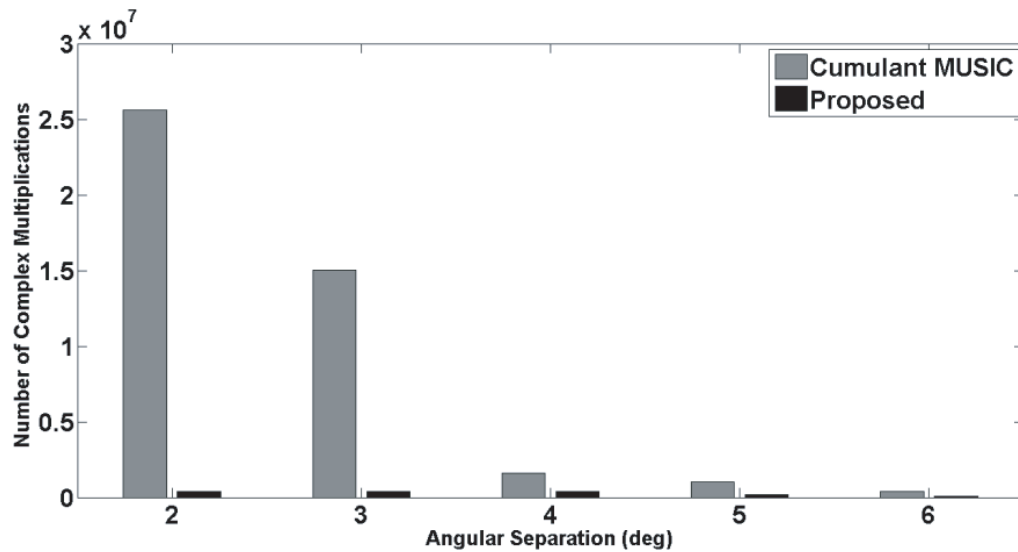


Figure 10. Computational complexity.

#### 4. CONCLUSION

In this work, the problem of resolving multiple closely spaced non-Gaussian sources corrupted with Gaussian noise in DOA estimation is addressed. It is well known that higher order statistics (HoS) based MUSIC spectrum would resolve the above stated problem. In HoS, fourth-order cumulant is utilized, because of its special property of insensitiveness to Gaussian noise as the kurtosis of Gaussian process is zero. However, it is observed that the fourth-order cumulant based MUSIC spectrum fails to resolve the sources which are closely spaced to each other. So, in this work a fourth-order cumulant based modified MUSIC spectrum is proposed which involves in the product of second-order differentiation on fourth-order Cumulant MUSIC spectrum and fourth-order Cumulant MUSIC spectrum. The detailed mathematical analysis of the proposed spectrum is presented with illustrations. The proposed work is validated with several computer simulations such as RMSE with respect to the number of elements, snapshots, and SNR. From this, the proposed method is observed to be computationally efficient for hardware implementation applications due to the reduced requirement of number of elements and

snapshots. The comparison of the proposed spectrum with MUSIC, Cumulant MUSIC is also presented. Finally, the resolution of the proposed spectrum is evaluated with respect to the angular separation and 3dB spectral width. From all the simulation studies, it is observed that the proposed spectrum outperforms the existing Cumulant MUSIC, Propagator, and Propagator-Cumulant methods in terms of estimation accuracy, resolution, and computational complexity. This approach has the potential to be extended for addressing the near-field DOA estimation problem.

## REFERENCES

1. Shu, F., et al., "Directional modulation: A physical-layer security solution to B5G and future wireless networks," *IEEE Network*, Vol. 34, No. 2, 210–216, Mar./Apr. 2020.
2. Mohammad Razavizadeh, S., M. Ahn, and I. Lee, "Three-dimensional beamforming: A new enabling technology for 5G wireless networks," *IEEE Signal Process. Mag.*, Vol. 31, No. 6, 94–101, Nov. 2014.
3. 3GPP, "Report of 3GPP RAN Workshop on Release 12 and onwards," RWS-120052, 3GPP Workshop on Release 12 Onward, Jun. 2012.
4. Foutz, J., A. Spanias, and M. K. Banavar, "Narrowband direction of arrival estimation of antenna arrays, Synthesis lectures on antennas#8," *Synthesis Lectures on Antennas*, Morgan and Claypool Publishers, San Rafael, CA, 2008.
5. Evers, C., E. A. P. Habets, S. Gannot, and P. A. Naylor, "DOA reliability for distributed acoustic tracking," *IEEE Signal Processing Letters*, Vol. 25, No. 9, 1320–1324, Sept. 2018.
6. Schmidt, R. O., "Multiple emitter location and signal parameter estimation," *IEEE Trans. Antennas Propag.*, Vol. 34, 276–280, Mar. 1986.
7. Roy, R. and T. Kailath, "ESPRIT-estimation of signal parameters via rotational invariance techniques," *IEEE Trans. Acoust., Speech, Signal Processing*, Vol. 37, 984–995, Jul. 1989.
8. Capon, J., "High resolution frequency-wavenumber spectral analysis," *Proc. IEEE*, Vol. 57, No. 8, 1408–1418, 1969.
9. Ziskind, I. and M. Wax, "Maximum likelihood localization of multiple sources by alternating projection," *IEEE Trans. Acous., Speech and Signal Processing*, Vol. 36, No. 10, 1553–1560, 1988.
10. Krim, H. and M. Viberg, "Two decades of array signal processing research," *IEEE Signal Process. Mag.*, Vol. 13, No. 7, 67–94, Jul. 1996.
11. Zhang, Y., G. Zhang, and H. Leung, "Grid-less coherent DOA estimation based on fourth-order cumulants with Gaussian coloured noise," *IET Radar, Sonar & Navigation*, Vol. 14, No. 5, 677–685, 2020.
12. Chen, G., X. Zeng, S. Jiao, A. Yu, and Q. Luo, "High accuracy near-field localization algorithm at low SNR using fourth-order cumulant," *IEEE Communications Letters*, Vol. 24, No. 3, 553–557, Mar. 2020.
13. Porat, B. and B. Friedlander, "Direction finding algorithms based on higher order statistics," *IEEE Transactions on Signal Processing*, Vol. 39, No. 9, 2016–2024, Sept. 1991.
14. Liao, B. and S.-C. Chan, "A cumulant-based method for direction finding in uniform linear arrays with mutual coupling," *IEEE Antennas and Wireless Propagation Letters*, Vol. 13, 1717–1720, 2014.
15. Dougan, M. C. and J. M. Mendel, "Applications of cumulants to array processing Part II: Non-Gaussian noise suppression," *IEEE Transactions on Signal Processing*, Vol. 43, No.7, 1663–1676, 1995.
16. Palanisamy, P. and N. Rao, "Direction of arrival estimation based on fourth-order cumulant using propagator method," *Progress In Electromagnetics Research B*, Vol. 18, 83–99, 2009.
17. Ebrahimi, A. A., H. R. Abutalebi, and M. Karimi, "Generalised two stage cumulants-based MUSIC algorithm for passive mixed sources localisation," *IET Signal Processing*, Vol. 13, No. 4, 409–414, 2019.

18. Kumar, L., A. Tripathy, and R. M. Hegde, "Robust multi-source localization over planar arrays using music-group delay spectrum," *IEEE Transactions on Signal Processing*, Vol. 62, No. 17, 4627–4636, Sept. 2014.
19. Ichige, K., Y. Ishikawa, and H. Arai, "Accurate direction-of-arrival estimation using second-order differential of music spectrum," *Proc. IEEE Int. Symp. Intell. Signal Process. Commun. (ISPACS)*, 995–998, 2006.
20. Ichige, K., K. Saito, and H. Arai, "High resolution DOA estimation using unwrapped phase information of music-based noise subspace," *IEICE Trans. Fundam. Electron. Commun. Comput. Sci.*, Vol. E91-A, 1990–1999, Aug. 2008.
21. Ahmed, T., X. Zhang, and W. U. Hassan, "A higher-order propagator method for 2D-DOA estimation in massive MIMO systems," *IEEE Communications Letters*, Vol. 24, No. 3, 543–547, Mar. 2020.