

Shielding of a Perfectly Conducting Circular Disk: Exact and Static Analytical Solution

Giampiero Lovat^{1, *}, Paolo Burghignoli², Rodolfo Araneo¹, Salvatore Celozzi¹,
Amedeo Andreotti³, Dario Assante⁴, and Luigi Verolino³

Abstract—The problem of the shielding evaluation of an infinitesimally thin perfectly conducting circular disk against a vertical magnetic dipole is here addressed. The problem is reduced to a set of dual integral equations and solved in an exact form through the application of the Galerkin method in the Hankel transform domain. It is shown that a second-kind Fredholm infinite matrix-operator equation can be obtained by selecting a complete set of orthogonal eigenfunctions of the static part of the integral operator as expansion basis. A static solution is finally extracted in a closed form which is shown to be accurate up to remarkably high frequencies.

1. INTRODUCTION

The interaction of electromagnetic waves with a circular metal disk constitutes a classical diffraction problem that, along with its Babinet-complementary problem of diffraction by a circular hole in an infinite metal plate, has received considerable interest in the literature of the last decades (see, e.g., [1–16] and references therein). Such a canonical configuration is in fact of interest for scattering (e.g., radar cross-section evaluation), antennas, as well as electromagnetic-shielding problems.

In this paper we consider the incidence of spherical waves produced by a point source placed at a finite distance from a circular perfectly-conducting (PEC) disk, namely a Vertical Magnetic Dipole (VMD) placed along the axis of azimuthal symmetry of the structure. This canonical source constitutes a valid representation for a practical small electric-loop radiator parallel to the disk and co-axial with it.

The formulation of the problem is first presented in Section 2, and operating in the Hankel transform domain, a set of dual integral equations (whose unknown is the surface current density induced on the PEC disk) is derived. An exact numerical solution, valid in any frequency range, is obtained in Section 3 through the application of Galerkin's Method of Moments choosing a set of orthogonal eigenfunctions of the static part of the involved integral operator as expansion functions for the surface current induced on the disk. It is shown that such functions allow for a rapidly convergent representation of the unknown since they reconstruct the physical behavior of the surface current density both at the center and at the edges of the disk. The rapidly converging properties of such a numerical solution are also improved by a suitable series representation of the elements of the impedance matrix. One very interesting result of the present work (presented in Section 4) is that, thanks to different integral identities, the static-limit solution can be extracted in a closed form, and such a closed form is shown to be accurate up to remarkably high frequencies, depending on the involved geometric parameters of the problem.

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* Corresponding author: Giampiero Lovat (giampiero.lovat@uniroma1.it).

¹ Department of Astronautical, Electrical, and Energetic Engineering, Sapienza University of Rome, Italy. ² Department of Information Engineering, Electronics and Telecommunications, Sapienza University of Rome, Italy. ³ Department of Electrical Engineering and Information Technology, University of Naples "Federico II", Italy. ⁴ Faculty of Engineering, International Telematic University Uninettuno, Italy.

It is important to point out, as rigorously shown in Section 5, that the proposed solution scheme fits into the more general method of analytical regularization already proposed with success in the literature [9, 12, 15, 16]. Finally, in Section 6, the conclusions of the present investigation are drawn.

2. FORMULATION OF THE PROBLEM

The configuration under analysis consists of an infinitesimally thin, perfectly conducting (PEC) circular disk of radius a placed on the plane $z = 0$ of a cylindrical coordinate system (ρ, ϕ, z) with center at the origin and a vertical magnetic dipole (VMD) with magnetic dipole moment \mathbf{m} placed along the z axis at a height $z = h$ (see Fig. 1) and coaxial with it. The vertical magnetic dipole can effectively model a small current loop parallel to the plane $z = 0$ and coaxial with the disk. The electromagnetic problem is axially symmetric so that all the fields depend only on ρ and z . Time-harmonic sources and fields are assumed with an implicit $e^{j\omega t}$ dependence.

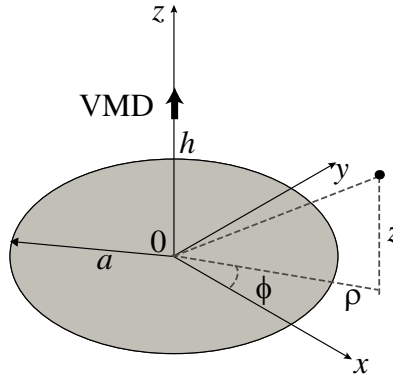


Figure 1. Configuration under analysis: a vertical magnetic dipole (VMD) radiates in the presence of a circular perfectly conducting (PEC) disk with radius a and of negligible thickness. The VMD is axially symmetric with respect to the disk and placed at a distance h from it.

2.1. Electric Field of a Current Loop

We first consider an electric current loop of radius ρ_0 placed over the plane $z = z_0$. Therefore

$$\mathbf{J}(\rho, z) = J_\phi(\rho, z)\mathbf{u}_\phi = P_0 \frac{\delta(\rho - \rho_0)}{\rho} \delta(z - z_0)\mathbf{u}_\phi \quad (1)$$

where P_0 is a suitable coefficient. The vector potential \mathbf{A} has only the component A_ϕ which must satisfy the Helmholtz equation, which in cylindrical coordinates reads

$$\frac{\partial^2 A_\phi}{\partial z^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial A_\phi}{\partial \rho} \right) - \frac{A_\phi}{\rho^2} + k_0^2 A_\phi = -\mu_0 J_\phi \quad (2)$$

where k_0 is the free-space wavenumber. By introducing the Hankel transform of order 1 defined as [17]

$$\begin{aligned} \tilde{F}(\lambda) &= \mathcal{H}_1\{F(\rho)\} = \int_0^\infty \rho F(\rho) J_1(\lambda\rho) d\rho \\ F(\rho) &= \mathcal{H}_1^{-1}\{\tilde{F}(\lambda)\} = \int_0^\infty \lambda \tilde{F}(\lambda) J_1(\lambda\rho) d\lambda \end{aligned} \quad (3)$$

where $J_1(\cdot)$ is the first-kind Bessel function of order 1, we have [17]

$$\frac{\partial^2 \tilde{A}_\phi}{\partial z^2} - \lambda^2 \tilde{A}_\phi + k_0^2 \tilde{A}_\phi = -\mu_0 \tilde{J}_\phi \quad (4)$$

By letting $k_z = \sqrt{k_0^2 - \lambda^2} = -j\sqrt{\lambda^2 - k_0^2}$ and by Hankel-transforming the current (1) (which samples the Bessel function J_1 in ρ_0) we obtain

$$\frac{\partial^2 \tilde{A}_\phi}{\partial z^2} + k_z^2 \tilde{A}_\phi = -\mu_0 P_0 J_1(\lambda \rho_0) \delta(z - z_0) \quad (5)$$

whose solution is

$$\tilde{A}_\phi(\lambda, z) = \mu_0 P_0 J_1(\lambda \rho_0) \frac{e^{-j k_z |z - z_0|}}{2j k_z} \quad (6)$$

By Hankel inverse-transforming, we thus have

$$A_\phi(\rho, z) = \frac{\mu_0 P_0}{2j} \int_0^\infty \frac{e^{-j k_z |z - z_0|}}{k_z} J_1(\lambda \rho_0) J_1(\lambda \rho) \lambda d\lambda \quad (7)$$

Since $\nabla \cdot \mathbf{A} = 0$, it follows that $\mathbf{E} = -j\omega \mathbf{A}$ and therefore

$$E_\phi(\rho, z) = -\frac{k_0 \zeta_0 P_0}{2} \int_0^\infty \frac{e^{-j k_z |z - z_0|}}{k_z} J_1(\lambda \rho_0) J_1(\lambda \rho) \lambda d\lambda \quad (8)$$

where ζ_0 is the free-space characteristic impedance.

2.2. Electric Field of a Current Disk

By considering a disk of radius a placed in $z_0 = 0$, with a current density $\mathbf{J}(\rho, z) = J_{S\phi}(\rho)\delta(z)\mathbf{u}_\phi$, by integrating (8) with $P_0 = \rho_0 J_{S\phi}(\rho_0)$, we obtain

$$E_\phi(\rho, z) = -\frac{k_0 \zeta_0}{2} \int_0^a \rho_0 J_{S\phi}(\rho_0) \int_0^\infty \frac{e^{-j k_z |z|}}{k_z} J_1(\lambda \rho_0) J_1(\lambda \rho) \lambda d\lambda d\rho_0 \quad (9)$$

and rearranging

$$E_\phi(\rho, z) = -\frac{k_0 \zeta_0}{2} \int_0^\infty \left(\int_0^a \rho_0 J_{S\phi}(\rho_0) J_1(\lambda \rho_0) d\rho_0 \right) \frac{e^{-j k_z |z|}}{k_z} J_1(\lambda \rho) \lambda d\lambda \quad (10)$$

By introducing the Hankel transform of the current density, we have

$$E_\phi(\rho, z) = -\frac{k_0 \zeta_0}{2} \int_0^\infty \tilde{J}_{S\phi}(\lambda) \frac{e^{-j k_z |z|}}{k_z} J_1(\lambda \rho) \lambda d\lambda \quad (11)$$

2.3. Electric Field of a Magnetic Dipole

By indicating the moment of a current loop placed in $z_0 = h$ with $|\mathbf{m}| = I\pi\rho_0^2$, the relevant electric field is expressed as

$$E_\phi^{\text{inc}}(\rho, z) = -\frac{k_0 \zeta_0 |\mathbf{m}|}{2\pi\rho_0} \int_0^\infty \frac{e^{-j k_z |z - h|}}{k_z} J_1(\lambda \rho_0) J_1(\lambda \rho) \lambda d\lambda \quad (12)$$

In the limit $\rho_0 \rightarrow 0$ (since $J_1(x) \simeq x/2$), we obtain

$$E_\phi^{\text{inc}}(\rho, z) = -\frac{k_0 \zeta_0 |\mathbf{m}|}{4\pi} \int_0^\infty \frac{e^{-j k_z |z - h|}}{k_z} J_1(\lambda \rho) \lambda^2 d\lambda \quad (13)$$

2.4. Boundary Condition

Since we assume a perfectly conducting disk, the tangential component (i.e., the ϕ component) of the total electric field \mathbf{E} vanishes at $z = 0$ for $\rho < a$, i.e.,

$$E_\phi^{\text{scat}}(\rho, z = 0) + E_\phi^{\text{inc}}(\rho, z = 0) = 0 \quad (14)$$

By using Eqs. (11) and (13) for $z = 0$ and $\rho < a$, we thus have

$$-k_0 \zeta_0 \int_0^\infty \frac{1}{2k_z} J_1(\lambda \rho) \tilde{J}_{S\phi}(\lambda) \lambda d\lambda - k_0 \zeta_0 |\mathbf{m}| \int_0^\infty \frac{e^{-j k_z h}}{4\pi k_z} J_1(\lambda \rho) \lambda^2 d\lambda = 0 \quad (15)$$

i.e.,

$$\int_0^\infty \frac{1}{k_z} J_1(\lambda \rho) \tilde{J}_{S\phi}(\lambda) \lambda d\lambda + \frac{|\mathbf{m}|}{2\pi} \int_0^\infty \frac{e^{-j k_z h}}{k_z} J_1(\lambda \rho) \lambda^2 d\lambda = 0 \quad (16)$$

This equation and the condition for which the current density vanishes for $\rho > a$ constitute a system of dual integral equations. By rearranging we obtain

$$\begin{aligned} \int_0^\infty \frac{1}{k_z} \left(\tilde{J}_{S\phi}(\lambda) + \frac{|\mathbf{m}| \lambda}{2\pi} e^{-j k_z h} \right) J_1(\lambda \rho) \lambda d\lambda &= 0, \quad \rho < a \\ \int_0^\infty \tilde{J}_{S\phi}(\lambda) J_1(\lambda \rho) \lambda d\lambda &= 0, \quad \rho > a \end{aligned} \quad (17)$$

3. GALERKIN METHOD-OF-MOMENTS SOLUTION

The unknown current density $J_{S\phi}$ can be expanded through a set of basis functions $b_n(\rho)$ whose transform $\tilde{b}_n(\lambda)$ should automatically satisfy the second in Eq. (17) and correctly reproduce the singular behavior of the current in $\rho = a$. Therefore we must have

$$\int_0^\infty \tilde{b}_n(\lambda) J_1(\lambda \rho) \lambda d\lambda = 0 \quad (18)$$

for $\rho > a$ and

$$b_n(\rho) = \int_0^\infty \tilde{b}_n(\lambda) J_1(\lambda \rho) \lambda d\lambda \propto \frac{1}{\sqrt{a^2 - \rho^2}} \quad (19)$$

for $\rho < a$. Moreover, in the origin the value of b_n has to be finite or, better, identically zero.

A possible set of basis functions is therefore provided by the following integral identity [18]

$$\int_0^{+\infty} \frac{J_{2m-1+k}(\lambda a)}{(\lambda a)^k} J_1(\lambda \rho) \lambda d\lambda = \begin{cases} 0 & \rho > a \\ \frac{B(m, k)}{a^{2k+1}} \rho (a^2 - \rho^2)^{k-1} P_{m-1}^{(1, k-1)} \left(1 - \frac{2\rho^2}{a^2} \right) & \rho < a \end{cases} \quad (20)$$

where

$$B(m, k) = \frac{(m-1)!}{2^{k-1} \Gamma(m+k-1)} \quad (21)$$

and $P_n^{(\alpha, \beta)}(\cdot)$ are the Jacobi polynomials of order n and $\Gamma(\cdot)$ is the Gamma function. Therefore

$$\mathcal{H}_1^{-1} \left\{ \frac{J_{2m-1+k}(\lambda a)}{(\lambda a)^k} \right\} = \frac{B(m, k)}{a^{2k+1}} \rho (a^2 - \rho^2)^{k-1} P_{m-1}^{(1, k-1)} \left(1 - \frac{2\rho^2}{a^2} \right) u_{-1}(a - \rho) \quad (22)$$

We can thus adopt the following set of basis functions:

$$b_n(\rho) = \begin{cases} 0 & \rho > a \\ \frac{\sqrt{2}(n-1)!}{\Gamma(n-1/2)} \frac{\rho}{a \sqrt{a^2 - \rho^2}} P_{n-1}^{(1, -1/2)} \left(1 - \frac{2\rho^2}{a^2} \right), \quad n = 1, 2, \dots & \rho < a \end{cases} \quad (23)$$

which satisfy all the required conditions and whose Hankel transforms are

$$\tilde{b}_n(\lambda) = \sqrt{\frac{a}{\lambda}} J_{2n-1/2}(\lambda a) \quad (24)$$

We can thus express

$$J_{S\phi}(\rho) = \sum_{n=1}^{+\infty} i_n b_n(\rho) \quad (25)$$

and

$$\tilde{J}_{S\phi}(\lambda) = \sum_{n=1}^{+\infty} i_n \tilde{b}_n(\lambda) \tag{26}$$

By using the basis-function expansion and projecting on the generic basis function $b_m(\rho)$ we obtain

$$\int_0^a \rho b_m(\rho) \int_0^\infty \frac{1}{k_z} \sum_{n=1}^\infty i_n \tilde{b}_n(\lambda) J_1(\lambda\rho) \lambda d\lambda d\rho = - \int_0^a \rho b_m(\rho) \int_0^\infty \frac{|\mathbf{m}|\lambda}{2\pi k_z} e^{-j k_z h} J_1(\lambda\rho) \lambda d\lambda d\rho \tag{27}$$

By truncating the expansion of the current density to N basis functions we have

$$\sum_{n=1}^N i_n \int_0^\infty \tilde{b}_m(\lambda) \frac{1}{k_z} \tilde{b}_n(\lambda) \lambda d\lambda = - \int_0^\infty \tilde{b}_m(\lambda) \frac{|\mathbf{m}|\lambda^2}{2\pi k_z} e^{-j k_z h} d\lambda, \quad m = 1, \dots, N \tag{28}$$

i.e.,

$$\sum_{n=1}^N i_n Z_{mn} = V_m, \quad m = 1, \dots, N \tag{29}$$

where

$$Z_{mn} = \int_0^\infty \frac{\tilde{b}_m(\lambda) \tilde{b}_n(\lambda)}{k_z} \lambda d\lambda \tag{30}$$

and

$$V_m = - \int_0^\infty \tilde{b}_m(\lambda) \frac{|\mathbf{m}|\lambda^2}{2\pi k_z} e^{-j k_z h} d\lambda \tag{31}$$

The solution of the algebraic system in Eq. (29) furnishes the coefficients i_n , and the current density $J_{S\phi}$ is recovered through Eq. (25).

In particular, from Eqs. (30) and (24) we have

$$Z_{mn} = \int_0^\infty \sqrt{\frac{a}{\lambda}} J_{2m-1/2}(\lambda a) \sqrt{\frac{a}{\lambda}} J_{2n-1/2}(\lambda a) \frac{1}{\sqrt{k_0^2 - \lambda^2}} \lambda d\lambda = a \int_0^\infty \frac{J_{2m-1/2}(\lambda a) J_{2n-1/2}(\lambda a)}{\sqrt{k_0^2 - \lambda^2}} d\lambda \tag{32}$$

In general, the improper integrals in Eq. (32) are highly oscillating and slowly decaying. However, as shown in [16], they can be transformed in

$$\int_0^\infty \frac{J_{2m-1/2}(\lambda a) J_{2n-1/2}(\lambda a)}{\sqrt{k_0^2 - \lambda^2}} d\lambda = \int_0^{\pi/2} J_{2m-1/2}(k_0 a \sin t) H_{2n-1/2}^{(2)}(k_0 a \sin t) dt \tag{33}$$

if $m \geq n$, where $H_\nu^{(2)}(\cdot)$ is the second-kind Hankel function of order ν , and

$$\int_0^\infty \frac{J_{2m-1/2}(\lambda a) J_{2n-1/2}(\lambda a)}{\sqrt{k_0^2 - \lambda^2}} d\lambda = \int_0^{\pi/2} J_{2n-1/2}(k_0 a \sin t) H_{2m-1/2}^{(2)}(k_0 a \sin t) dt \tag{34}$$

if $n > m$ so that

$$Z_{mn} = a \int_0^{\pi/2} J_{2\max\{m,n\}-1/2}(k_0 a \sin t) H_{2\min\{m,n\}-1/2}^{(2)}(k_0 a \sin t) dt \tag{35}$$

It can immediately be seen that the matrix $[Z_{mn}]$ is symmetric, i.e., $Z_{mn} = Z_{nm}$.

Alternatively, the integrals in Eq. (32) can be evaluated through a rapidly converging series as [19]

$$\int_0^\infty \frac{J_{2m-1/2}(\lambda a) J_{2n-1/2}(\lambda a)}{\sqrt{k_0^2 - \lambda^2}} d\lambda = \frac{(-1)^p}{2k_0 a} \sum_{l=1}^{+\infty} \frac{\binom{-l+1}{2}_p \binom{-l}{2}_q (j k_0 a)^l}{\Gamma\left(\frac{l+1}{2} + p\right) \Gamma\left(\frac{l}{2} + q\right) \Gamma\left(\frac{l+1}{2}\right)} \tag{36}$$

where $p = n - m$, $q = n + m$ and $(x)_y$ is the Pochhammer symbol defined as

$$(x)_y = \frac{\Gamma(x+y)}{\Gamma(x)} \quad (37)$$

so that

$$Z_{mn} = j \frac{(-1)^{n-m} a^{+\infty}}{2} \sum_{l=1}^{+\infty} \frac{\Gamma\left(\frac{l}{2}\right) \Gamma\left(\frac{-l+1}{2} + n - m\right) \Gamma\left(-\frac{l}{2} + n + m\right) (j k_0 a)^{l-1}}{\Gamma\left(\frac{l+1}{2}\right) \Gamma\left(\frac{-l+1}{2}\right) \Gamma\left(\frac{-l+2}{2}\right) \Gamma\left(\frac{l-1}{2} + n - m\right) \Gamma\left(\frac{l}{2} + n + m\right)} \quad (38)$$

As concerns the known term, we have

$$V_m = - \int_0^\infty \sqrt{\frac{a}{\lambda}} J_{2m-1/2}(\lambda a) \frac{|\mathbf{m}| \lambda^2}{2\pi k_z} e^{-j k_z h} d\lambda = - \frac{|\mathbf{m}| \sqrt{a}}{2\pi} \int_0^\infty J_{2m-1/2}(\lambda a) \lambda^{3/2} \frac{e^{-j \sqrt{k_0^2 - \lambda^2} h}}{\sqrt{k_0^2 - \lambda^2}} d\lambda \quad (39)$$

3.1. Electric and Magnetic Fields

Once the i_n coefficients are known (and thus the current $J_{S\phi}$ as well), the radiated electric field is given by

$$E_\phi(\rho, z) = E_\phi^{\text{inc}}(\rho, z) + E_\phi^{\text{scat}}(\rho, z) \quad (40)$$

where

$$E_\phi^{\text{inc}}(\rho, z) = - \frac{k_0 \zeta_0 |\mathbf{m}|}{4\pi} \int_0^\infty \frac{e^{-j k_z |z-h|}}{k_z} J_1(\lambda \rho) \lambda^2 d\lambda \quad (41)$$

and

$$E_\phi^{\text{scat}}(\rho, z) = - \frac{k_0 \zeta_0}{2} \sum_{n=1}^N i_n \int_0^\infty \tilde{b}_n(\lambda) \frac{e^{-j k_z |z|}}{k_z} J_1(\lambda \rho) \lambda d\lambda \quad (42)$$

The magnetic field is instead given by

$$\mathbf{H}(\rho, z) = - \frac{\nabla \times \mathbf{E}}{j k_0 \zeta_0} = \frac{j}{k_0 \zeta_0} \left[- \frac{\partial E_\phi}{\partial z} \mathbf{u}_\rho + \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\phi) \mathbf{u}_z \right] \quad (43)$$

and therefore

$$H_\rho^{\text{inc}}(\rho, z) = \frac{|\mathbf{m}|}{4\pi} \int_0^\infty e^{-j k_z |z-h|} J_1(\lambda \rho) \lambda^2 d\lambda \quad (44)$$

$$H_z^{\text{inc}}(\rho, z) = -j \frac{|\mathbf{m}|}{4\pi} \int_0^\infty \frac{e^{-j k_z |z-h|}}{k_z} J_0(\lambda \rho) \lambda^3 d\lambda \quad (45)$$

and

$$H_\rho^{\text{scat}}(\rho, z) = \frac{1}{2} \sum_{n=1}^N i_n \int_0^\infty \tilde{b}_n(\lambda) e^{-j k_z |z|} J_1(\lambda \rho) \lambda d\lambda \quad (46)$$

$$H_z^{\text{scat}}(\rho, z) = - \frac{j}{2} \sum_{n=1}^N i_n \int_0^\infty \tilde{b}_n(\lambda) \frac{e^{-j k_z |z|}}{k_z} \lambda^2 J_0(\lambda \rho) d\lambda \quad (47)$$

We thus have

$$E_\phi^{\text{scat}}(\rho, z) = - \frac{k_0 \zeta_0 \sqrt{a}}{2} \sum_{n=1}^N i_n \int_0^\infty J_{2n-1/2}(\lambda a) J_1(\lambda \rho) \sqrt{\lambda} \frac{e^{-j \sqrt{k_0^2 - \lambda^2} |z|}}{\sqrt{k_0^2 - \lambda^2}} d\lambda \quad (48)$$

$$H_\rho^{\text{scat}}(\rho, z) = \frac{\sqrt{a}}{2} \sum_{n=1}^N i_n \int_0^\infty J_{2n-1/2}(\lambda a) J_1(\lambda \rho) \sqrt{\lambda} e^{-j \sqrt{k_0^2 - \lambda^2} |z|} d\lambda \quad (49)$$

$$H_z^{\text{scat}}(\rho, z) = - \frac{j \sqrt{a}}{2} \sum_{n=1}^N i_n \int_0^\infty J_{2n-1/2}(\lambda a) J_0(\lambda \rho) \lambda^{3/2} \frac{e^{-j \sqrt{k_0^2 - \lambda^2} |z|}}{\sqrt{k_0^2 - \lambda^2}} d\lambda \quad (50)$$

4. STATIC SOLUTION

It is interesting to note that in the static limit (i.e., $k_0 \rightarrow 0$) the elements Z_{mn} in Eq. (32) become

$$Z_{mn} = j a \int_0^\infty \frac{J_{2m-1/2}(\lambda a) J_{2n-1/2}(\lambda a)}{\lambda} d\lambda \tag{51}$$

Such an integral can be evaluated in a closed form using the identity [20, 6.574]

$$\int_0^\infty J_\nu(\alpha t) J_\mu(\alpha t) t^{-\lambda} dt = \frac{\alpha^{\lambda-1} \Gamma(\lambda) \Gamma\left(\frac{\nu + \mu - \lambda + 1}{2}\right)}{2^\lambda \Gamma\left(\frac{-\nu + \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{\nu + \mu + \lambda + 1}{2}\right) \Gamma\left(\frac{\nu - \mu + \lambda + 1}{2}\right)} \tag{52}$$

valid for $\nu + \mu + 1 > \lambda > 0$ and $\alpha > 0$. In fact, by letting $\nu = 2m - 1/2$, $\mu = 2n - 1/2$, $\alpha = a$, $t = \lambda$, and $\lambda = 1$, we have

$$Z_{mn} = \frac{j a \Gamma\left(m + n - \frac{1}{2}\right)}{2 \Gamma(n - m + 1) \Gamma\left(m + n + \frac{1}{2}\right) \Gamma(m - n + 1)} \tag{53}$$

On the other hand, the integral in Eq. (51) is the product of two orthogonal functions: therefore, as can be verified by observing the denominator in Eq. (53) (which is always infinite except for the case $m = n$), we have $Z_{mn} = 0$ for $m \neq n$ and

$$Z_{nn} = \frac{j a}{2} \frac{\Gamma\left(2n - \frac{1}{2}\right)}{\Gamma\left(2n + \frac{1}{2}\right)} \tag{54}$$

It is worth noting that the result in Eq. (54) can also be obtained from Eq. (38), by observing that in the static limit only the term $l = 1$ of the series is different from zero. By considering the property of the Gamma function for which

$$\Gamma(z + 1) = z \Gamma(z) \tag{55}$$

Eq. (54) can also be expressed in a simpler way as

$$Z_{nn} = \frac{j a}{(4n - 1)} \tag{56}$$

Concerning the known term, from Eq. (39), in the static limit we have

$$V_m = -j \frac{|\mathbf{m}| \sqrt{a}}{2\pi} \int_0^\infty J_{2m-1/2}(\lambda a) \sqrt{\lambda} e^{-\lambda h} d\lambda \tag{57}$$

Also the latter integral can be expressed in a closed form by using the identity [20, 6.621]

$$\int_0^\infty e^{-\alpha x} J_\nu(\beta x) x^{\mu-1} dx = (\alpha^2 + \beta^2)^{-\mu/2} \Gamma(\nu + \mu) P_{\mu-1}^{-\nu} \left[\alpha (\alpha^2 + \beta^2)^{-1/2} \right] \tag{58}$$

valid for $\alpha > 0$, $\beta > 0$, $\nu + \mu > 0$, and where $P_\mu^\nu(\cdot)$ are the associated Legendre functions of the first kind. In fact, by letting $\alpha = h$, $\nu = 2m - 1/2$, $\beta = a$, and $\mu = 3/2$, we thus have

$$V_m = -j \frac{|\mathbf{m}|}{2\pi} \frac{(2m)! \sqrt{a}}{(a^2 + h^2)^{3/4}} P_{1/2}^{-2m+1/2} \left[h (a^2 + h^2)^{-1/2} \right] \tag{59}$$

By collecting all the results, we have

$$\sum_{n=1}^N i_n Z_{mn} = V_m, \quad m = 1, \dots, N \tag{60}$$

where

$$Z_{mn} = \begin{cases} 0 & m \neq n \\ \frac{j a}{(4n-1)} & m = n \end{cases} \quad (61)$$

and

$$V_m = -j \frac{|\mathbf{m}|}{2\pi} \frac{(2m)! \sqrt{a}}{(a^2 + h^2)^{3/4}} P_{1/2}^{-2m+1/2} \left[\frac{h}{\sqrt{a^2 + h^2}} \right] \quad (62)$$

Since the system is diagonal, we immediately obtain

$$i_n = -\frac{|\mathbf{m}|}{2\pi} \frac{(2n)! (4n-1)}{\sqrt{a} (a^2 + h^2)^{3/4}} P_{1/2}^{-2n+1/2} \left[\frac{h}{\sqrt{a^2 + h^2}} \right] \quad (63)$$

so that, from Eq. (25), in the static limit we have

$$J_{S\phi}(\rho) = -|\mathbf{m}| \sum_{n=1}^{+\infty} \frac{(2n)!(n-1)!(4n-1)}{\sqrt{2\pi}\Gamma(n-1/2) a^{3/2} (a^2 + h^2)^{3/4}} P_{1/2}^{-2n+1/2} \left[\frac{h}{\sqrt{a^2 + h^2}} \right] \\ \cdot \frac{\rho}{\sqrt{a^2 - \rho^2}} P_{n-1}^{(1, -1/2)} \left(1 - \frac{2\rho^2}{a^2} \right) \quad (64)$$

The excellent convergence properties of the basis functions and the behavior of the surface current as a function of the ratio r/a for different values of h/a are reported in Fig. 2.

It is also interesting to observe the variation of the surface current as a function of the radius a for a fixed value of $h = 30$ cm, as shown in Fig. 3.

From Eqs. (41), (44), and (48)–(50), in the static limit we also have $E_\phi = 0$ and

$$H_\rho^{\text{inc}}(\rho, z) = \frac{|\mathbf{m}|}{4\pi} \int_0^\infty e^{-\lambda|z-h|} J_1(\lambda\rho) \lambda^2 d\lambda \quad (65)$$

$$H_z^{\text{inc}}(\rho, z) = \frac{|\mathbf{m}|}{4\pi} \int_0^\infty e^{-\lambda|z-h|} J_0(\lambda\rho) \lambda^2 d\lambda \quad (66)$$

and

$$H_\rho^{\text{scat}}(\rho, z) = \frac{\sqrt{a}}{2} \sum_{n=1}^N i_n \int_0^\infty J_{2n-1/2}(\lambda a) J_1(\lambda\rho) \sqrt{\lambda} e^{-\lambda|z|} d\lambda \quad (67)$$

$$H_z^{\text{scat}}(\rho, z) = \frac{\sqrt{a}}{2} \sum_{n=1}^N i_n \int_0^\infty J_{2n-1/2}(\lambda a) J_0(\lambda\rho) \sqrt{\lambda} e^{-\lambda|z|} d\lambda \quad (68)$$

All these integrals can be evaluated in a closed form. In particular, for the integrals in Eqs. (65)–(66), using the identity in Eq. (58), we have

$$H_\rho^{\text{inc}}(\rho, z) = \frac{3|\mathbf{m}|}{2\pi (|z-h|^2 + \rho^2)^{3/2}} P_2^{-1} \left(\frac{|z-h|}{\sqrt{|z-h|^2 + \rho^2}} \right) \quad (69)$$

$$H_z^{\text{inc}}(\rho, z) = \frac{|\mathbf{m}|}{2\pi (|z-h|^2 + \rho^2)^{3/2}} P_2^0 \left(\frac{|z-h|}{\sqrt{|z-h|^2 + \rho^2}} \right) \quad (70)$$

valid for $|z-h| > 0$. By letting

$$\cos \theta = \frac{|z-h|}{\sqrt{|z-h|^2 + \rho^2}}, \quad \sin \theta = \frac{\rho}{\sqrt{|z-h|^2 + \rho^2}} \quad (71)$$

and since

$$P_2^{-1}(x) = \frac{x\sqrt{1-x^2}}{2} \quad (72)$$

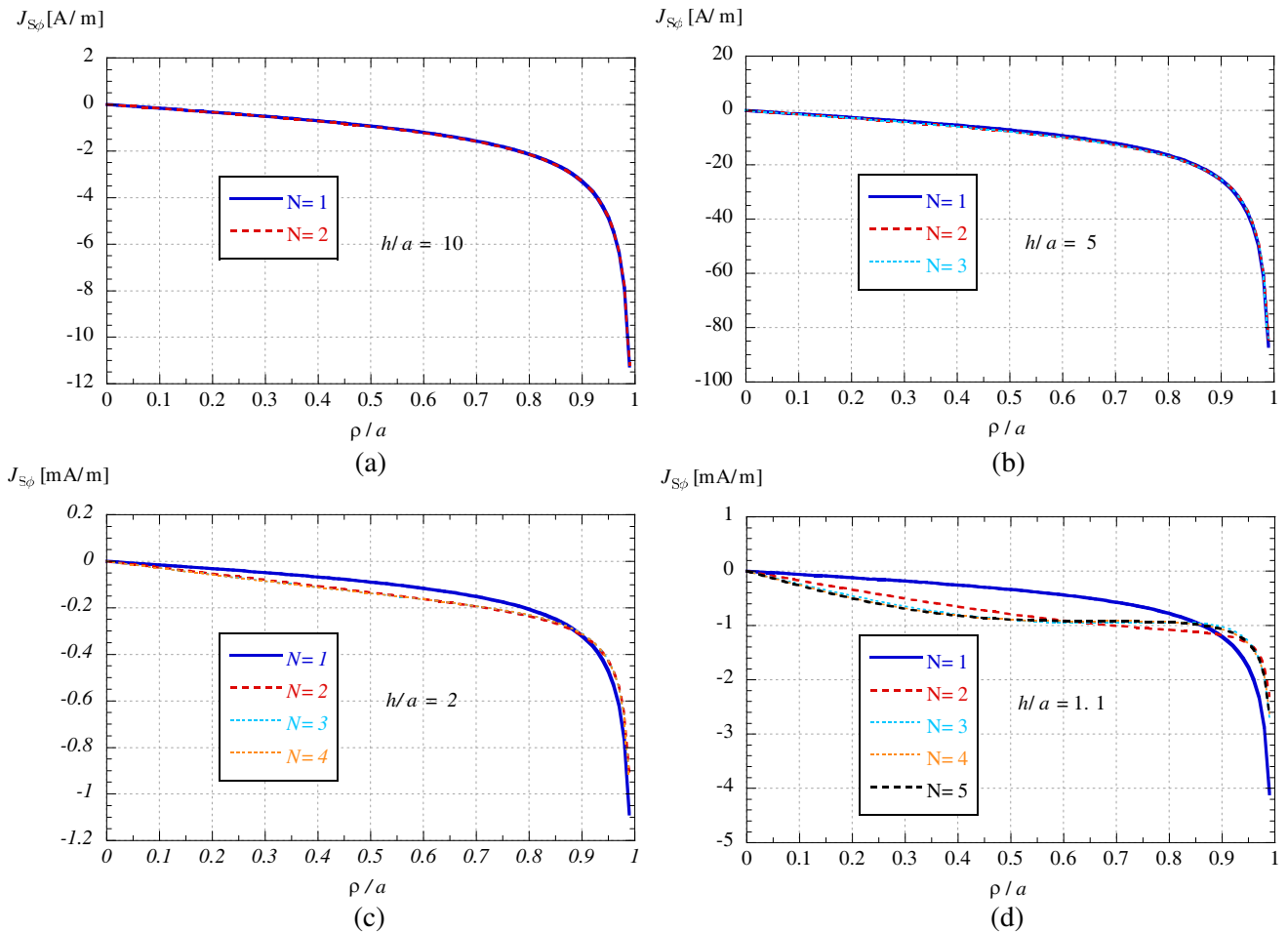


Figure 2. Surface current density as a function of ρ/a for different values of h/a and using a different number N of basis functions. (a) $h/a = 10$; (b) $h/a = 5$; (c) $h/a = 2$; (d) $h/a = 1.1$. The curves are superimposed from $N = 1$ in (a) and (b), from $N = 2$ in (c), and $N = 3$ in (d). Parameters: $a = 5$ cm, $|\mathbf{m}| = 1$ Am².

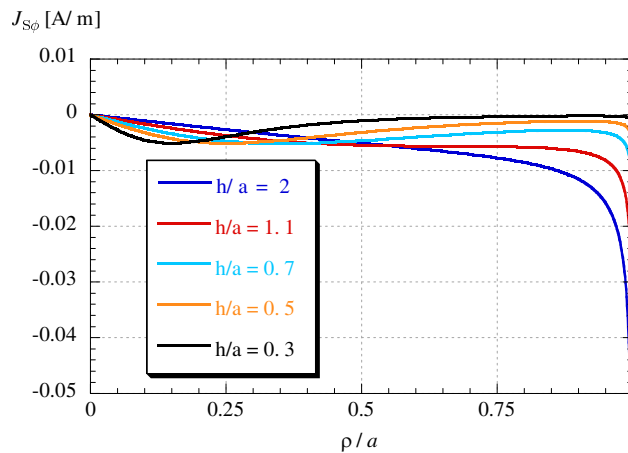


Figure 3. Surface current density as a function of ρ/a for different values of h/a . Parameters: $h = 30$ cm, $|\mathbf{m}| = 1$ Am².

and

$$P_2^0(x) = \frac{3x^2 - 1}{2} \quad (73)$$

we obtain

$$H_\rho^{\text{inc}}(\rho, z) = \frac{3|\mathbf{m}| \cos \theta \sin \theta}{4\pi (|z - h|^2 + \rho^2)^{3/2}} \quad (74)$$

and

$$H_z^{\text{inc}}(\rho, z) = \frac{|\mathbf{m}| (2 \cos^2 \theta - \sin^2 \theta)}{4\pi (|z - h|^2 + \rho^2)^{3/2}} \quad (75)$$

which are well-known results [21].

For the calculation of the scattered field, the following identity can be used [20, 6.626]:

$$\int_0^\infty x^{\lambda-1} e^{-\alpha x} J_\mu(\beta x) J_\nu(\gamma x) dx = \frac{\beta^\mu \gamma^\nu}{\Gamma(\nu+1)} 2^{-\nu-\mu} \alpha^{-\lambda-\mu-\nu} \cdot \sum_{m=0}^\infty \frac{\Gamma(\lambda+\mu+\nu+2m)}{m! \Gamma(\mu+m+1)} F\left(-m, -\mu-m; \nu+1; \frac{\gamma^2}{\beta^2}\right) \left(-\frac{\beta^2}{4\alpha^2}\right)^m \quad (76)$$

valid for $\lambda + \mu + \nu > 0$ and $\alpha > 0$, where $F(\cdot, \cdot; \cdot; \cdot)$ are the Gauss hypergeometric functions. In the integrals in Eqs. (67)–(68) we use $\lambda = 3/2$, $\alpha = |z|$, $\mu = 2n - 1/2$, $\beta = a$, $\gamma = \rho$, and $\nu = 1$ (for Eq. (67)) or 0 (for Eq. (68)). Therefore

$$H_\rho^{\text{scat}}(\rho, z) = \rho \sum_{n=1}^\infty i_n \frac{a^{2n}}{2^{2n+3/2} |z|^{2n+2}} \sum_{m=0}^\infty \frac{(-1)^m (2m+2n+1)!}{m! \Gamma(2n+m+1/2)} F\left(-m, -2n-m+\frac{1}{2}; 2; \frac{\rho^2}{a^2}\right) \left(\frac{a^2}{4|z|^2}\right)^m \quad (77)$$

and

$$H_z^{\text{scat}}(\rho, z) = \sum_{n=1}^\infty i_n \frac{a^{2n}}{2^{2n+1/2} |z|^{2n+1}} \sum_{m=0}^\infty \frac{(-1)^m (2n+2m)!}{m! \Gamma(2n+m+1/2)} F\left(-m, -2n-m+\frac{1}{2}; 1; \frac{\rho^2}{a^2}\right) \left(\frac{a^2}{4|z|^2}\right)^m \quad (78)$$

valid for $|z| > 0$.

Alternatively, it can be recognized that the integrals in Eqs. (67)–(68) are Lipschitz-Hankel type integrals for which different representations exist [22].

For observation points along the z axis (i.e., for $\rho = 0$ and thus $\theta = 0$), the incident field is simply

$$H_\rho^{\text{inc}}(0, z) = 0 \quad (79)$$

$$H_z^{\text{inc}}(0, z) = \frac{|\mathbf{m}|}{2\pi (|z - h|)^3} \quad (80)$$

For the scattered field we instead have

$$H_\rho^{\text{scat}}(0, z) = 0 \quad (81)$$

and

$$H_z^{\text{scat}}(0, z) = \sum_{n=1}^\infty i_n \frac{a^{2n}}{2^{2n+1/2} |z|^{2n+1}} \sum_{m=0}^\infty \frac{(-1)^m (2n+2m)!}{m! \Gamma(2n+m+1/2)} \left(\frac{a^2}{4|z|^2}\right)^m \quad (82)$$

since $F(\cdot, \cdot; \cdot; 0) = 1$. Alternatively, from Eq. (68), when $\rho = 0$ we have

$$H_z^{\text{scat}}(0, z) = \frac{\sqrt{a}}{2} \sum_{n=1}^\infty i_n \int_0^\infty J_{2n-1/2}(\lambda a) \sqrt{\lambda} e^{-\lambda|z|} d\lambda \quad (83)$$

The integral in Eq. (83) can be solved in a closed form by using the identity in Eq. (58) with $x = \lambda$, $\alpha = |z|$, $\nu = 2n - 1/2$, $\beta = a$, and $\mu = 3/2$, thus obtaining

$$H_z^{\text{scat}}(0, z) = \frac{\sqrt{a}}{2} \sum_{n=1}^\infty i_n \frac{(2n)!}{(|z|^2 + a^2)^{3/4}} P_{1/2}^{-(2n-1/2)} \left(\frac{|z|}{\sqrt{|z|^2 + a^2}}\right) \quad (84)$$

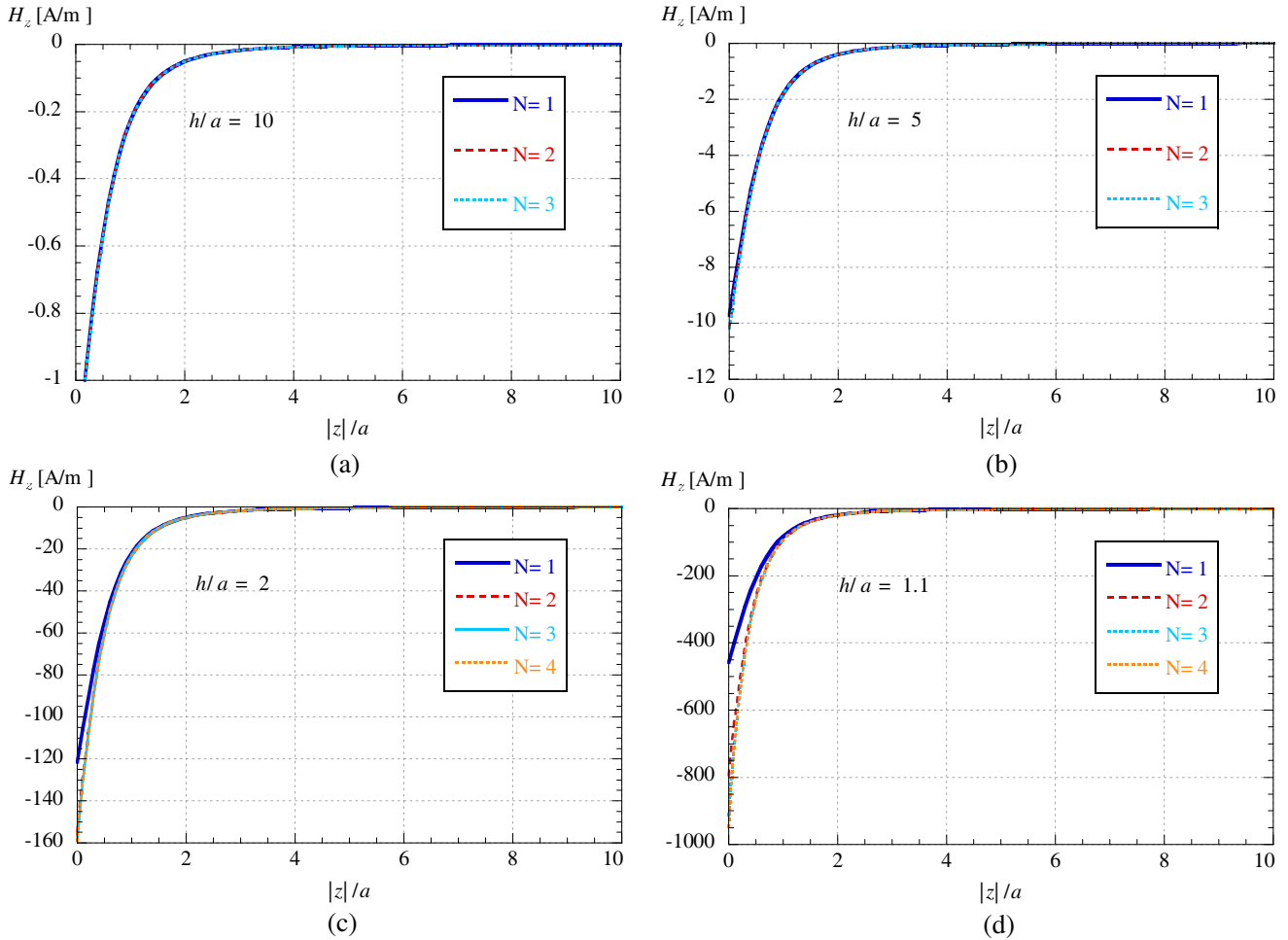


Figure 4. Scattered magnetostatic field along the z axis ($\rho = 0$) as a function of $|z|/a$ for different values of h/a . (a) $h/a = 10$; (b) $h/a = 5$; (c) $h/a = 2$; (d) $h/a = 1.1$. The curves are superimposed from $N = 1$ in (a) and (b), from $N = 2$ in (c), and $N = 3$ in (d). Parameters: $a = 5$ cm, $|\mathbf{m}| = 1$ Am².

The excellent convergence properties of the basis functions and the behavior of the magnetostatic field along the z axis as a function of the ratio $|z|/a$ for different values of h/a are reported in Fig. 4.

It is worth noting that for sources sufficiently far from the disk, *only one basis function* is sufficient to reach an excellent convergence so that in these cases

$$H_z^{\text{scat}}(0, z) \simeq i_1 \frac{\sqrt{a}}{(|z|^2 + a^2)^{3/4}} P_{1/2}^{-3/2} \left(\frac{|z|}{\sqrt{|z|^2 + a^2}} \right) \quad (85)$$

where

$$i_1 = -\frac{|\mathbf{m}|}{\pi} \frac{2\Gamma\left(2 + \frac{1}{2}\right)}{\Gamma\left(2 - \frac{1}{2}\right)} \frac{1}{\sqrt{a}(a^2 + h^2)^{3/4}} P_{1/2}^{-3/2} \left(\frac{h}{\sqrt{|h|^2 + a^2}} \right) \quad (86)$$

By letting

$$\cos \theta_z = \frac{|z|}{\sqrt{|z|^2 + a^2}}, \quad \cos \theta_h = \frac{h}{\sqrt{|h|^2 + a^2}} \quad (87)$$

and using Eq. (55) we thus obtain

$$H_z^{\text{scat}}(0, z) \simeq -3 \frac{|\mathbf{m}|}{\pi} \frac{P_{1/2}^{-3/2}(\cos \theta_h) P_{1/2}^{-3/2}(\cos \theta_z)}{(a^2 + h^2)^{3/4} (|z|^2 + a^2)^{3/4}} \quad (88)$$

so that the total magnetostatic field along the symmetry axis is

$$H_z^{\text{tot}}(0, z) \simeq \frac{|\mathbf{m}|}{2\pi} \left\{ \frac{1}{(|z - h|)^3} - 6 \frac{P_{1/2}^{-3/2}(\cos \theta_h) P_{1/2}^{-3/2}(\cos \theta_z)}{(a^2 + h^2)^{3/4} (|z|^2 + a^2)^{3/4}} \right\} \quad (89)$$

The magnetostatic shielding effectiveness SE_H along the z axis can thus be evaluated as

$$SE_H = 20 \log \frac{|H_z^{\text{inc}}(0, z)|}{|H_z^{\text{tot}}(0, z)|} \quad (90)$$

Some numerical results for a disk with $a = 5$ cm are reported in Fig. 5.

It is very interesting to note that the static approximation provides accurate results up to relatively high frequencies. In Fig. 6 the static and exact frequency-dependent magnetic shielding effectiveness SE_H are reported for different frequencies and observation points for the cases $h/a = 10$ and $h/a = 2$.

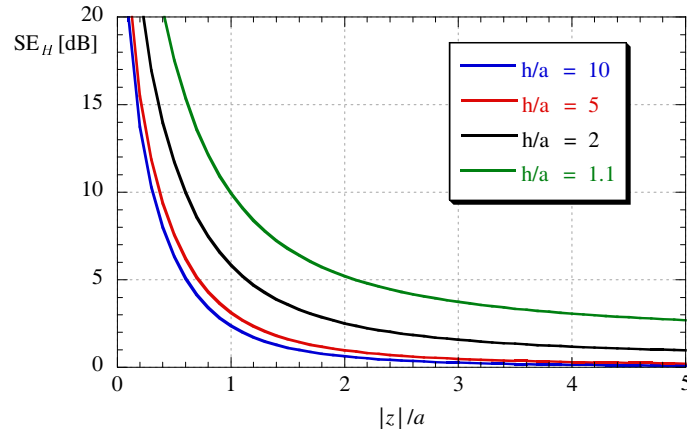


Figure 5. Magnetostatic shielding effectiveness SE_H along the semi-axis $z < 0$ as a function of $|z|/a$ for different values of h/a . Parameters: $a = 5$ cm.

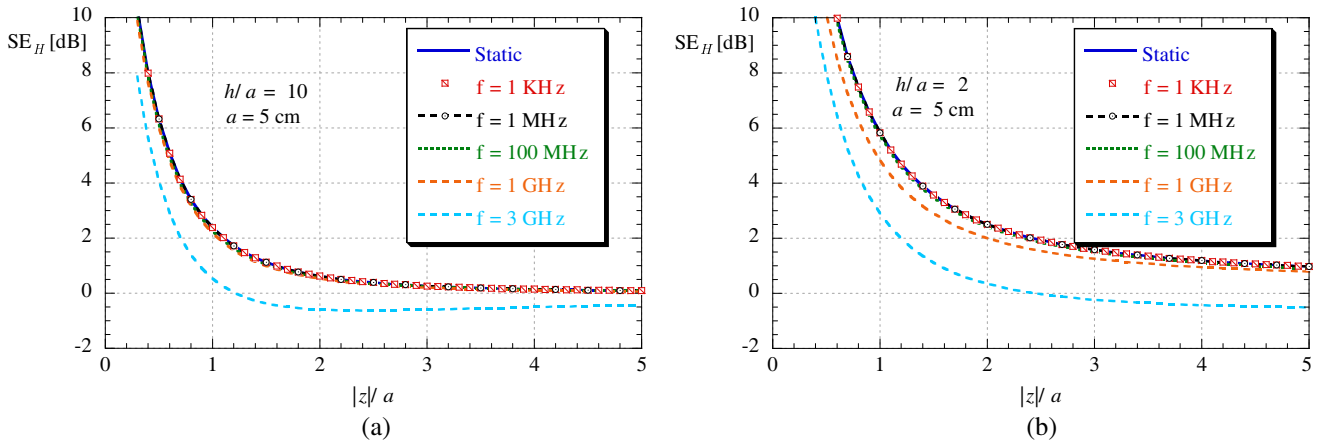


Figure 6. Magnetic shielding effectiveness SE_H along the semi-axis $z < 0$ as a function of $|z|/a$ for different frequencies f . Parameters: $a = 5$ cm, $h/a = 10$ and $h/a = 2$.

The exact frequency-dependent magnetic shielding effectiveness has been obtained through Eqs. (45) and (50) with $\rho = 0$.

5. APPLICATION OF THE METHOD OF ANALYTICAL REGULARIZATION (MAR)

It is interesting to observe that the proposed solution can be effectively put in the framework of the method of analytical regularization (MAR) [9, 15]. In fact, the set of dual integral equations which solve the problem is

$$\begin{aligned} \int_0^\infty \tilde{J}_{S\phi}(\lambda) J_1(\lambda\rho) \lambda d\lambda &= 0, & \rho > a \\ \int_0^\infty \frac{\tilde{J}_{S\phi}(\lambda) J_1(\lambda\rho)}{k_z} \lambda d\lambda &= \frac{|\mathbf{m}|}{2\pi} \int_0^\infty \frac{\lambda}{k_z} e^{-jk_z h} J_1(\lambda\rho) \lambda d\lambda, & \rho < a \end{aligned} \quad (91)$$

with basis functions given by Eq. (23) whose Hankel transform are expressed in Eq. (24). The latter are orthogonal in the range $\lambda \in (0, +\infty)$, i.e.,

$$\int_0^\infty \tilde{b}_m(\lambda) \tilde{b}_n(\lambda) d\lambda = \delta_{mn} \quad (92)$$

Therefore, as already discussed, by letting

$$J_{S\phi}(\lambda) = \sum_{n=1}^{+\infty} i_n b_n(\lambda) \quad (93)$$

the first equation in (91) is automatically satisfied. In the static limit, the weight function in the left-hand side of the second of (91) becomes

$$\frac{1}{k_z} \longrightarrow \frac{j}{\lambda} \quad (94)$$

and the second equation in Eq. (91) can be divided in a static and a dynamic part by expressing

$$\frac{1}{k_z} = \frac{j}{\lambda} + \Omega(\lambda) \quad (95)$$

where

$$\Omega(\lambda) = \left[\frac{1}{\sqrt{k_0^2 - \lambda^2}} - \frac{j}{\lambda} \right] \quad (96)$$

Thanks to such an extraction and to the expansion in Eq. (93), the static part of the second equation in Eq. (91) can be diagonalized and analytically inverted by using the transform in Eq. (24) and the orthogonality property of the Bessel functions in Eq. (92). In particular, following the procedure used in the application of the Method of Moments, we have

$$\begin{aligned} \int_0^a \rho b_m(\rho) \int_0^\infty \frac{j}{\lambda} \sum_{n=1}^N i_n \tilde{b}_n(\lambda) J_1(\lambda\rho) \lambda d\lambda d\rho + \int_0^a \rho b_m(\rho) \int_0^\infty \Omega(\lambda) \sum_{n=1}^N i_n \tilde{b}_n(\lambda) J_1(\lambda\rho) \lambda d\lambda d\rho = \\ - \int_0^a \rho b_m(\rho) \int_0^\infty \frac{|\mathbf{m}| \lambda}{2\pi k_z} e^{-jk_z h} J_1(\lambda\rho) \lambda d\lambda d\rho, \quad m = 1, \dots, N \end{aligned} \quad (97)$$

i.e.,

$$j \sum_{n=1}^N i_n \int_0^\infty \tilde{b}_m(\lambda) \tilde{b}_n(\lambda) d\lambda + \sum_{n=1}^N i_n \int_0^\infty \tilde{b}_m(\lambda) \Omega(\lambda) \tilde{b}_n(\lambda) \lambda d\lambda = - \int_0^\infty \tilde{b}_m(\lambda) \frac{|\mathbf{m}| \lambda^2}{2\pi k_z} e^{-jk_z h} d\lambda \quad (98)$$

for $m = 1, \dots, N$ and therefore

$$j i_m + \sum_{n=1}^N i_n \int_0^\infty \tilde{b}_m(\lambda) \tilde{b}_n(\lambda) \Omega(\lambda) \lambda d\lambda = - \int_0^\infty \tilde{b}_m(\lambda) \frac{|\mathbf{m}| \lambda^2}{2\pi k_z} e^{-jk_z h} d\lambda \quad (99)$$

which can also be expressed as

$$i_m + \sum_{n=1}^N i_n \hat{Z}_{mn} = \hat{V}_m \quad (100)$$

where

$$\hat{Z}_{mn} = -j \int_0^\infty \tilde{b}_m(\lambda) \tilde{b}_n(\lambda) \Omega(\lambda) \lambda d\lambda \quad (101)$$

and

$$\hat{V}_m = j \int_0^\infty \tilde{b}_m(\lambda) \frac{|\mathbf{m}| \lambda^2}{2\pi k_z} e^{-j k_z h} d\lambda \quad (102)$$

We thus have a matrix equation of the kind

$$[i_m] + [\hat{Z}_{mn}] [i_n] = [V_m] \quad (103)$$

Since the asymptotic expansion of the Bessel functions for large orders results in [20] [20]

$$J_\nu(z) \sim \frac{1}{\sqrt{2\pi\nu}} \left(\frac{ez}{2\nu}\right)^\nu \quad (104)$$

it is immediate to recognize that

$$\sum_{m,n=1}^{\infty} |\hat{Z}_{mn}|^2 < +\infty \quad (105)$$

i.e., the operator $[\hat{Z}_{mn}]$ is compact in the space ℓ_2 of the square-summable sequences. Since it is also $[V_m] \in \ell_2$, it follows that (100) is a second-kind Fredholm equation in ℓ_2 of the kind

$$X + AX = B \quad (106)$$

Thanks to the Fredholm theory [23], this means that the exact solution

$$X = (I + A)^{-1} B \quad (107)$$

certainly exists, where I is the identity operator, and that the solution of the discretized problem converges to such an exact solution in the point-wise sense. In practice, the solution of the system truncated to a finite number N of equations converges to the exact solution. This means that if $X^{(N)}$ is the solution of the truncated system

$$X^{(N)} + A^{(N)} X^{(N)} = B^{(N)} \quad (108)$$

then the relative error, by the norm in ℓ_2 , is limited as

$$e(N) = \frac{\|X - X^{(N)}\|}{\|X\|} \leq \|(I + A)^{-1}\| \|A - A^{(N)}\| \quad (109)$$

and vanishes as $N \rightarrow \infty$.

6. CONCLUSION

The electromagnetic field of a vertical magnetic dipole shielded by an infinitesimally thin perfectly conducting disk has been evaluated in an exact form.

The problem has been formulated in the Hankel transform domain obtaining a set of dual integral equations considering the surface current density induced over the disk as unknown. The solution has been achieved by means of a Galerkin's Method-of-Moments approach choosing, as basis functions, the complete set of orthogonal eigenfunctions of the static part of the integral operator as expansion basis. Such functions also reconstruct the physical behavior of the surface current density at the center and the edges of the disk, thus allowing for a rapidly convergent representation of the scattered field. It has been shown that the proposed solution scheme fits into the more general method of analytical

regularization. Finally, the static-limit solution has been extracted in a closed form which has been shown to be accurate up to sufficiently high frequencies.

Possible generalizations of the proposed approach could address the treatment of thin disks with a finite conductivity or/and sources displaced from the symmetry axis of the disk. In both cases the problem would remain planar (whereas a non-negligible thickness of the disk would radically change the mathematical nature of the problem); in the former case, a surface transition impedance boundary condition should be enforced; in the latter case, a representation of the displaced source in terms of continuous, azimuthally phased ring sources should be employed, and the scattered field would be hybrid (TM^z/TE^z). Consideration of such generalized problems will be the subject of future work.

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