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An Inverse Electromagnetic Scattering Problem for an Ellipsoid

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Abstract—The scattering problem of time-harmonic electromagnetic plane waves by an impedance and a dielectric ellipsoid is considered. A low-frequency formulation of the direct scattering problem using the Rayleigh approximation is described. Considering far-field data, an inverse electromagnetic scattering problem is formulated and studied. A finite number of measurements of the leading-order term of the electric far-field pattern in the low-frequency approximation leads to specify the semi-axes of the ellipsoid. The orientation of the ellipsoid is obtained by using the Euler angles. Corresponding results for the sphere, spheroid, needle, and disc can be obtained considering them as geometrically degenerate forms of the ellipsoid for suitable values of its geometrical parameters.

1. INTRODUCTION

The inverse scattering theory has a lot of important applications in medical imaging, geological studies, and detection of buried or underwater objects. An ellipsoid can be considered as a good approximation for the shape of many objects such as spheres, spheroids, needles, and discs. Therefore, inverse scattering problems for ellipsoids are significant.

In a series of papers, Dassios and his coauthors have published results about ellipsoids in inverse scattering problems using far-field data and specifically the low frequency coefficients of the far-field patterns and the scattering cross-sections. Dassios in [6] first introduced a new method for solving inverse scattering problems for an acoustically soft ellipsoid. Later, similar problems were studied for a rigid and a penetrable ellipsoid for acoustic waves as well as an acoustic ellipsoidal boss [8, 12], a rigid ellipsoid for elastic waves [4], an ellipsoidal perfect conductor, and an electromagnetic ellipsoidal boss [9]. Moreover, inverse scattering problems concerning ellipsoids have been studied for acoustic waves in [7], for elastic waves in [13], and for electromagnetic waves in [14]. Also, inverse scattering problems concerning spheres have been treated in [1] and [3].

In the present work, this method is extended for the case of an impedance ellipsoid and the case of a dielectric ellipsoid. The far-field pattern in terms of low-frequency coefficients is used, and then a measurement matrix is constructed whose eigenvalues and eigenvectors provide information for the semi-axes and orientation of the ellipsoid.

In Section 2, the direct scattering problem is formulated, and the formula of the far-field pattern for the impedance ellipsoid as well as for the lossy and lossless dielectric ellipsoid is given. In Section 3, the inverse scattering problem using far-field data for each case is studied. A matrix whose elements are given in terms of measurements of the leading order low-frequency coefficient is constructed. The eigenvalues and eigenvectors of this matrix provide information about the size and orientation of the ellipsoid, respectively. In Section 4, corresponding results are presented for the spheroid and the sphere considering them as geometrically degenerate forms of the ellipsoid. In Section 5, a numerical example is presented. In Section 6, a conclusion to this paper is given. Finally, a flowchart corresponding to the algorithm of the method is given in the Appendix.

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2. THE DIRECT SCATTERING PROBLEM

The surface of an ellipsoid centered at the origin with principal semi-axes α_n along the axes $\mathbf{x}_n$ has the equation:

$$S : \sum_{n=1}^{3} \frac{x_n^2}{\alpha_n^2} = 1, \tag{1}$$

where (x_1, x_2, x_3) is the Cartesian coordinates and $\alpha_1 > \alpha_2 > \alpha_3 > 0$. The ellipsoidal coordinates (ρ, μ, ν) satisfy the inequalities:

$$-h_3 \leq \nu \leq h_3 \leq \mu \leq h_2 \leq \rho < +\infty. \tag{2}$$

where h_1, h_2, h_3 are the semi-interfocal distances given by:

$$h_1^2 = \alpha_2^2 - \alpha_3^2, \quad h_2^2 = \alpha_1^2 - \alpha_3^2, \quad h_3^2 = \alpha_1^2 - \alpha_2^2. \tag{3}$$

Let V^+ be the exterior, V^- the interior region of S, and $\hat{\mathbf{n}}$ the outward unit normal vector on S. In the ellipsoidal coordinate system, the surface S is defined by $\rho = \alpha_1$, the exterior by $\rho > \alpha_1$, and the interior by $\sqrt{\alpha_1^2 - \alpha_3^2} \leq \rho < \alpha_1$ (Figure 1).

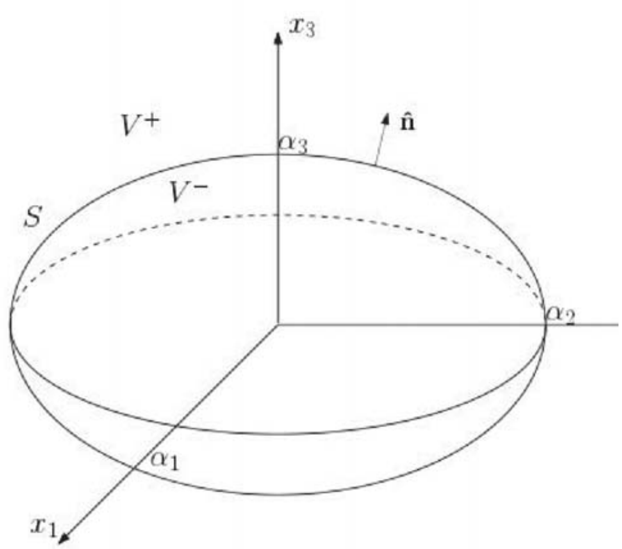


Figure 1. Ellipsoidal scatterer of surface S.

The superscripts $\pm$ denote parameters or vectors in the region $V^\pm$, respectively. A given time-harmonic electromagnetic plane wave of the form:

$$\mathbf{E}^i\left(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{p}}\right) = \hat{\mathbf{p}} e^{ik\hat{d}\cdot r}, \quad \mathbf{H}^i\left(\mathbf{r}; \hat{\mathbf{d}}, \hat{\mathbf{q}}\right) = \frac{1}{Z^+} \hat{\mathbf{q}} e^{ik\hat{d}\cdot r}, \tag{4}$$

where k is the wave number in V^+ and $Z^+ = \sqrt{\mu^+/\varepsilon^+}$ the characteristic impedance with ε^+, μ^+ being the electric permittivity and magnetic permeability in V^+, respectively, and the electromagnetic plane wave of the form in Eq. (4) is incident upon the ellipsoid of surface S. The unit vectors $\hat{\mathbf{d}}, \hat{\mathbf{p}}, \hat{\mathbf{q}}$ describe the direction of propagation, the electric and magnetic polarizations, respectively, and satisfy the relations $\hat{\mathbf{p}} \cdot \hat{\mathbf{d}} = \hat{\mathbf{q}} \cdot \hat{\mathbf{d}} = 0$ and $\hat{\mathbf{d}} \times \hat{\mathbf{p}} = \hat{\mathbf{q}}$.

In the present work, the cases of an impedance ellipsoidal surface and a dielectric surface are studied. In both cases, the total field $\mathbf{E}^+, \mathbf{H}^+$ in V^+ satisfies the Maxwell equations:

$$\nabla \times \mathbf{E}^+ = i\omega\mu^+ \mathbf{H}^+, \quad \nabla \times \mathbf{H}^+ = -i\omega\varepsilon^+ \mathbf{E}^+ \text{ in } V^+, \tag{5}$$

where ω is the angular frequency. Moreover, in the case of the dielectric ellipsoid, the total interior field $\mathbf{E}^-, \mathbf{H}^-$ in V^- satisfies the Maxwell equations:

$$\nabla \times \mathbf{E}^- = i\omega\mu^-\mathbf{H}^-, \quad \nabla \times \mathbf{H}^- = \left(-i\omega\varepsilon^- + \sigma^-\right)\mathbf{E}^- \text{ in } V^-, \tag{6}$$

where σ^- is the conductivity in V^-.

For the case of the impedance surface S, the following boundary condition is satisfied:

$$\hat{\mathbf{n}} \times \mathbf{E}^+ = Z_s Z^+ \hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \mathbf{H}^+\right) \text{ on } S, \tag{7}$$

where Z_s denotes the surface impedance. For the case of the dielectric surface S, the following transmission conditions are satisfied:

$$\hat{\mathbf{n}} \times \mathbf{E}^+ = \hat{\mathbf{n}} \times \mathbf{E}^-, \quad \hat{\mathbf{n}} \times \mathbf{H}^+ = \hat{\mathbf{n}} \times \mathbf{H}^- \text{ on } S. \tag{8}$$

In both the cases, the total field can be written as the superposition of the incident field $\mathbf{E}^i, \mathbf{H}^i$ and scattered field $\mathbf{E}^s, \mathbf{H}^s$ which is assumed to satisfy the Silver-Müller radiation condition:

$$Z^+\hat{\mathbf{r}} \times \mathbf{H}^s + \mathbf{E}^s = o\left(\frac{1}{r}\right), \quad r \to \infty \tag{9}$$

uniformly in all directions $\hat{\mathbf{r}} = \frac{\mathbf{r}}{r}$, $r = |\mathbf{r}|$. For more details on the direct scattering problems, we refer to books [2] and [5]. In what follows, only the electric far-field patterns are used for brevity. Analogous results can be obtained by using the magnetic far-field patterns. Moreover, the superscripts I, D, L will denote the impedance, lossless ($\sigma^- = 0$), and lossy ($\sigma^- > 0$) dielectric ellipsoids, respectively. The superscript $A = I, D, L$ will denote all the above cases. The low-frequency expansion of the electric far-field pattern $\mathbf{E}^{\infty(A)}$, for $A = I, D, L$, is given by [10, 11]:

$$\mathbf{E}^{\infty(A)}\left(\hat{\mathbf{r}}; \hat{\mathbf{d}}, \hat{\mathbf{p}}\right) = -ik^3 \sum_{n=1}^{3} \left[W_n^{(A)} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_n)(\hat{\mathbf{x}}_n \cdot \hat{\mathbf{p}}) + T_n^{(A)} (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_n)(\hat{\mathbf{x}}_n \cdot \hat{\mathbf{q}})\right] + \mathcal{O}(k^4), \tag{10}$$

with

$$W_n^{(D)} = \frac{V}{3} \frac{(\mu^+\eta^2 - \mu^-)}{(\mu^+\eta^2 - \mu^-) V I_1^n + \mu^-}, \quad W_n^{(L)} = W_n^{(I)} = \frac{1}{3I_1^n}, \tag{11}$$

$$T_n^{(D)} = T_n^{(L)} = \frac{V}{3} \frac{(\mu^+ - \mu^-)}{(\mu^+ - \mu^-) V I_1^n - \mu^+}, \quad T_n^{(I)} = \frac{1}{3} \tag{12}$$

where $V = \alpha_1\alpha_2\alpha_3$, η is the relative index of refraction and I_1^n the elliptic integral of degree 1 and order $n = 1, 2, 3$, where:

$$I_1^n(\rho) = \frac{1}{2} \int_{\rho^2 - \alpha_1^2}^{\infty} \frac{du}{(u + \alpha_n^2)\sqrt{u + \alpha_1^2}\sqrt{u + \alpha_2^2}\sqrt{u + \alpha_3^2}}, \tag{13}$$

with

$$I_1^1(\rho) + I_1^2(\rho) + I_1^3(\rho) = \frac{1}{\rho\sqrt{\rho^2 - h_2^2}\sqrt{\rho^2 - h_3^2}}, \tag{14}$$

and $I_1^n = I_1^n(\alpha_1)$ with

$$I_1^1 + I_1^2 + I_1^3 = \frac{1}{V}. \tag{15}$$

Based on the following relations:

$$\sum_{n=1}^{3} W_n^{(A)} \hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_n)(\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{p}}) = -\hat{\mathbf{r}} \times \left(W^{(A)}\hat{\mathbf{p}}\right) \times \hat{\mathbf{r}} \tag{16}$$

and

$$\sum_{n=1}^{3} T_n^{(A)} (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n})(\hat{\mathbf{x}}_\mathbf{n} \cdot \hat{\mathbf{q}}) = \sum_{n=1}^{3} T_n^{(A)} (\hat{\mathbf{r}} \times \hat{\mathbf{x}}_\mathbf{n})\left(\hat{\mathbf{x}}_\mathbf{n} \cdot (\hat{\mathbf{d}} \times \hat{\mathbf{p}})\right) = \hat{\mathbf{r}} \times \left(T^{(A)}\left(\hat{\mathbf{d}} \times \hat{\mathbf{p}}\right)\right), \tag{17}$$

the far-field pattern can be written as:

$$\mathbf{E}^{\infty(A)}\left(\hat{\mathbf{r}};\hat{\mathbf{d}},\hat{\mathbf{p}}\right)=ik^3\mathbf{f}^{(A)}(\hat{\mathbf{r}};\hat{\mathbf{d}},\hat{\mathbf{p}})+\mathcal{O}(k^4),\tag{18}$$

for $k\to 0$, with the leading-order coefficient given by:

$$\mathbf{f}^{(A)}\left(\hat{\mathbf{r}};\hat{\mathbf{d}},\hat{\mathbf{p}}\right)=\hat{\mathbf{r}}\times\left[\left(W^{(A)}\hat{\mathbf{p}}\right)\times\hat{\mathbf{r}}-T^{(A)}\left(\hat{\mathbf{d}}\times\hat{\mathbf{p}}\right)\right],\tag{19}$$

where $W^{(A)}=\mathrm{diag}(W_n^{(A)})$ and $T^{(A)}=\mathrm{diag}(T_n^{(A)})$ are 3×3 diagonal matrices with elements $W_n^{(A)}$ and $T_n^{(A)}$ given in Eqs. (11) and (12) respectively for $n=1,2,3$ and $A=I,D,L$.

3. THE INVERSE SCATTERING PROBLEM

Consider two Cartesian systems with the same origin and their corresponding orthonormal bases $\{\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3\}$ and $\{\hat{\mathbf{x}}'_1,\hat{\mathbf{x}}'_2,\hat{\mathbf{x}}'_3\}$. The system $\{\hat{\mathbf{x}}_1,\hat{\mathbf{x}}_2,\hat{\mathbf{x}}_3\}$ coincides with the principal directions of the unknown ellipsoid while the system $\{\hat{\mathbf{x}}'_1,\hat{\mathbf{x}}'_2,\hat{\mathbf{x}}'_3\}$ is a known reference system. In order to specify the orientation and size of the ellipsoid, the $\mathbf{x}'_i$ system is transformed to $\mathbf{x}_i$ system with the use of orthogonal rotation matrix P whose elements are given in terms of Euler angles (α,β,γ) (depicted in Figure 2) as follows [11]:

$$P=\begin{bmatrix}\cos\alpha\cos\gamma-\cos\beta\sin\alpha\sin\gamma & \sin\alpha\cos\gamma+\cos\beta\cos\alpha\sin\gamma & \sin\beta\sin\gamma\\ -\cos\alpha\sin\gamma-\cos\beta\sin\alpha\cos\gamma & -\sin\alpha\sin\gamma+\cos\beta\cos\alpha\cos\gamma & \sin\beta\cos\gamma\\ \sin\beta\sin\alpha & -\sin\beta\cos\alpha & \cos\beta\end{bmatrix}\tag{20}$$

Therefore, vectors $\mathbf{r},\hat{\mathbf{r}},\hat{\mathbf{d}},\hat{\mathbf{p}}$ satisfy the following rotation relations:

$$\mathbf{r}=P\mathbf{r}',\quad \hat{\mathbf{r}}=P\hat{\mathbf{r}}',\quad \hat{\mathbf{d}}=P\hat{\mathbf{d}}',\quad \hat{\mathbf{p}}=P\hat{\mathbf{p}}'.\tag{21}$$

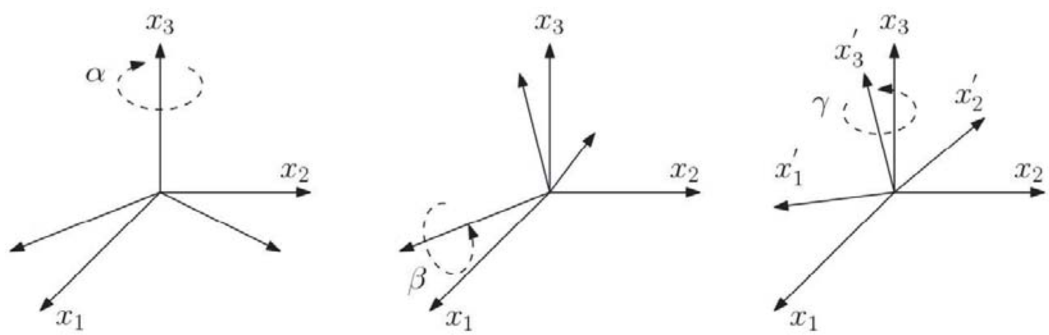


Figure 2. Euler angles.

Inserting the rotation relations in Eq. (21) for the directions in the coefficient $\mathbf{f}^{(A)}$, they are transformed from the $\mathbf{x}'_i$ system to $\mathbf{x}_i$. Then by multiplying $\mathbf{f}^{(A)}$ with $P^\top$, we go back to the reference system $\mathbf{x}'_i$ where the measurements will be taken. The superscript $\top$ denotes transposition and $P^\top=P^{-1}$ since P is orthogonal. Let

$$\mathbf{g}^{(A)}\left(\hat{\mathbf{r}}';\hat{\mathbf{d}}',\hat{\mathbf{p}}'\right):=P^\top\mathbf{f}^{(A)}\left(P\hat{\mathbf{r}}';P\hat{\mathbf{d}}',P\hat{\mathbf{p}}'\right)=P^\top\mathbf{f}^{(A)}\left(\hat{\mathbf{r}};\hat{\mathbf{d}},\hat{\mathbf{p}}\right).\tag{22}$$

Therefore

$$\mathbf{g}^{(A)}\left(\hat{\mathbf{r}}';\hat{\mathbf{d}}',\hat{\mathbf{p}}'\right)=P^\top\left\{P\hat{\mathbf{r}}'\times\left[W^{(A)}P\hat{\mathbf{p}}'\times P\hat{\mathbf{r}}'-T^{(A)}\left(P\hat{\mathbf{d}}'\times P\hat{\mathbf{p}}'\right)\right]\right\}.\tag{23}$$

Letting the directions $\hat{\mathbf{r}}' = \varepsilon_{ijk}\hat{\mathbf{x}}'_k$, $\hat{\mathbf{d}}' = \hat{\mathbf{x}}'_i$ and $\hat{\mathbf{p}}' = \hat{\mathbf{x}}'_j$, where ε_{ijk} the permutation symbol with $\varepsilon_{ijk} = \varepsilon_{jki} = \varepsilon_{kij} = 1$ and $\varepsilon_{ikj} = \varepsilon_{jik} = \varepsilon_{kji} = -1$, for $1 \le i, j, k \le 3$ with i, j, k distinct, leads to:

$$\mathbf{g}^{(A)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}'_k; \hat{\mathbf{x}}'_i, \hat{\mathbf{x}}'_j\right) = \varepsilon_{ijk}\hat{\mathbf{x}}'_k \times \left[P^\top W^{(A)} P\hat{\mathbf{x}}'_j \times \varepsilon_{ijk}\hat{\mathbf{x}}'_k - P^\top T^{(A)} P\left(\hat{\mathbf{x}}'_i \times \hat{\mathbf{x}}'_j\right)\right], \tag{24}$$

which after calculations can be written as:

$$\mathbf{g}^{(A)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}'_k; \hat{\mathbf{x}}'_i, \hat{\mathbf{x}}'_j\right) = \left[\left(\mathbf{P}_i^\top W^{(A)}\mathbf{P}_j + \varepsilon_{ijk}\mathbf{P}_j^\top T^{(A)}\mathbf{P}_k\right)\hat{\mathbf{x}}'_i + \left(\mathbf{P}_j^\top W^{(A)}\mathbf{P}_j - \varepsilon_{ijk}\mathbf{P}_i^\top T^{(A)}\mathbf{P}_k\right)\hat{\mathbf{x}}'_j\right], \tag{25}$$

where $\mathbf{P}_j = P\hat{\mathbf{x}}'_j$ is the j-th column of matrix P and $\mathbf{P}_j^\top = \hat{\mathbf{x}}_j'^\top P^\top$ the j-th row of matrix $P^\top$.

Since $W^{(A)}$ is diagonal for $A = I, D, L$, the following relations, for $1 \le i, j \le 3$, are obtained:

$$\mathbf{P}_i^\top W^{(A)}\mathbf{P}_j = \mathbf{P}_j^\top W^{(A)}\mathbf{P}_i. \tag{26}$$

Taking measurements of the vectors $\mathbf{g}^{(I)}, \mathbf{g}^{(D)}$ and $\mathbf{g}^{(L)}$, for direction combinations along the axes $\mathbf{x}'_i$ and specifically 6 direction combinations which are used to solve for $\mathbf{P}_i^\top W^{(A)}\mathbf{P}_j$, leads to:

$$2\mathbf{P}_j^\top W^{(A)}\mathbf{P}_j = \left[\mathbf{g}^{(A)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}'_k; \hat{\mathbf{x}}'_i, \hat{\mathbf{x}}'_j\right) + \mathbf{g}^{(A)}\left(\varepsilon_{kji}\hat{\mathbf{x}}'_i; \hat{\mathbf{x}}'_k, \hat{\mathbf{x}}'_j\right)\right] \cdot \hat{\mathbf{x}}'_j, \tag{27}$$

$$\begin{aligned} 2\mathbf{P}_i^\top W^{(A)}\mathbf{P}_j &= \left[2\mathbf{g}^{(A)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}'_k; \hat{\mathbf{x}}'_i, \hat{\mathbf{x}}'_j\right) - \mathbf{g}^{(A)}\left(\varepsilon_{jik}\hat{\mathbf{x}}'_k; \hat{\mathbf{x}}'_j, \hat{\mathbf{x}}'_i\right) + \mathbf{g}^{(A)}\left(\varepsilon_{kij}\hat{\mathbf{x}}'_j; \hat{\mathbf{x}}'_k, \hat{\mathbf{x}}'_i\right)\right] \cdot \hat{\mathbf{x}}'_i \\ &= \left[2\mathbf{g}^{(A)}\left(\varepsilon_{jik}x\hat{\mathbf{x}}'_k; \hat{\mathbf{x}}'_j, \hat{\mathbf{x}}'_i\right) - \mathbf{g}^{(A)}\left(\varepsilon_{ijk}\hat{\mathbf{x}}'_k; \hat{\mathbf{x}}'_i, \hat{\mathbf{x}}'_j\right) + \mathbf{g}^{(A)}\left(\varepsilon_{kji}\hat{\mathbf{x}}'_i; \hat{\mathbf{x}}'_k, \hat{\mathbf{x}}'_j\right)\right] \cdot \hat{\mathbf{x}}'_j, \end{aligned} \tag{28}$$

The measurement matrices $M^{(A)}$, for the impedance, the lossless and lossy dielectric cases are constructed as follows:

$$M^{(A)} = \left(M_{ij}^{(A)}\right), \quad M_{ij}^{(A)} = 2\mathbf{P}_i^\top W^{(A)}\mathbf{P}_j \tag{29}$$

for $1 \le i, j \le 3$. The above relation can be written in matrix form as:

$$M^{(A)} = 2P^\top W^{(A)} P, \tag{30}$$

or since the rotation matrix P is orthogonal:

$$PM^{(A)}P^\top = 2W^{(A)}, \tag{31}$$

which is an orthogonal similarity relation between the measurement matrix $M^{(A)}$ and the diagonal matrix $W^{(A)}$, for $A = I, D, L$.

The measurement matrix $M^{(A)}$ is real and symmetric, and each has three real eigenvalues $\lambda_n^{(A)}$ and three corresponding orthonormal eigenvectors $\mathbf{v}_n^{(A)}$ for $A = I, D, L$. Therefore, based on the orthogonal similarity relation, it is concluded that for $n = 1, 2, 3$:

$$\begin{cases} \lambda_n^{(A)} = 2W_n^{(A)}, \\ \mathbf{v_n} = \left(P_{n1}^{(A)}, P_{n2}^{(A)}, P_{n3}^{(A)}\right), \end{cases} \tag{32}$$

where $P_{n1}^{(A)}, P_{n2}^{(A)}, P_{n3}^{(A)}$ are elements of the nth row of rotation matrix P, and the superscript A denotes the case of the impedance (I), lossless dielectric ellipsoid (D) and lossy dielectric ellipsoid (L), respectively. From elements of the three orthogonal eigenvectors in each case, the elements of matrix P are obtained, and therefore the Euler angles are specified by using the following relations [11]:

$$\alpha^{(A)} = \sin^{-1}\left(\frac{P_{31}^{(A)}}{\sqrt{1 - P_{33}^{2(A)}}}\right), \quad \beta^{(A)} = \sin^{-1}\left(\sqrt{1 - P_{33}^{2(A)}}\right), \quad \gamma^{(A)} = \sin^{-1}\left(\frac{P_{13}^{(A)}}{\sqrt{1 - P_{33}^{2(A)}}}\right). \tag{33}$$

These angles show the orientation of the ellipsoid in the impedance, the lossless dielectric and lossy dielectric cases, respectively.

From the system of equations that connect the eigenvalues with the elements of the diagonal matrices $W^{(I)}, W^{(D)}, W^{(L)}$, the following system of equations for $n = 1, 2, 3$ is obtained:

$$\lambda_n^{(I)} = \frac{2}{3}\frac{1}{I_1^n}, \tag{34}$$

$$\lambda_n^{(D)} = \frac{2V^{(D)}}{3}\left(\frac{\mu^+\eta^2 - \mu^-}{(\mu^+\eta^2 - \mu^-)\, V^{(D)} I_1^n + \mu^-}\right), \tag{35}$$

$$\lambda_n^{(L)} = \frac{2}{3}\frac{1}{I_1^n}. \tag{36}$$

From these equations, the semi-axes $\alpha_1, \alpha_2, \alpha_3$ are specified, and therefore, the size of the impedance, lossless and lossy dielectric ellipsoids is obtained. Specifically, improper elliptic integrals are specified directly from the eigenvalues via the following relations:

$$I_1^n = \frac{2}{3\lambda_n^{(I)}}, \tag{37}$$

$$I_1^n = \frac{2}{3\lambda_n^{(D)}} - \frac{2\mu^-\left(\lambda_1^{(D)}\lambda_2^{(D)} + \lambda_2^{(D)}\lambda_3^{(D)} + \lambda_1^{(D)}\lambda_3^{(D)}\right)}{3\left(\mu^+\eta^2 + 2\mu^-\right)\lambda_1^{(D)}\lambda_2^{(D)}\lambda_3^{(D)}}. \tag{38}$$

$$I_1^n = \frac{2}{3\lambda_n^{(L)}}. \tag{39}$$

Moreover, the quantities $V^{(A)}$ can also be found in terms of the eigenvalues using the relation in Eq. (15). Having found the quantities I_1^n for $n = 1, 2, 3$ and using the relations that connect the improper elliptic integrals with the incomplete elliptic integrals of the first kind (F) and the second kind (E) ([11], p. 381):

$$I_1^1 = \frac{1}{h_2 h_3^2}\left(F(\phi, \alpha) - E(\phi, \alpha)\right), \tag{40}$$

$$I_1^2 = \frac{h_2}{(h_2^2 - h_3^2)h_3^2}E(\phi, \alpha) - \frac{1}{h_2 h_3^2}F(\phi, \alpha) - \frac{1}{(h_2^2 - h_3^2)}\frac{\sqrt{\alpha_1^2 - h_2^2}}{\alpha_1\sqrt{\alpha_1^2 - h_3^2}}, \tag{41}$$

$$I_1^3 = -\frac{1}{(h_2^2 - h_3^2)h_2}E(\phi, \alpha) + \frac{1}{(h_2^2 - h_3^2)}\frac{\sqrt{\alpha_1^2 - h_3^2}}{\alpha_1\sqrt{\alpha_1^2 - h_2^2}}, \tag{42}$$

leads to a system of three equations for the three unknowns α_1, h_2, h_3 which can be solved using various softwares (e.g., Matlab), and the three semi-axes can be obtained via the relations $\alpha_2 = \sqrt{\alpha_1^2 - h_3^2}$ and $\alpha_3 = \sqrt{\alpha_1^2 - h_2^2}$. Alternatively, one can proceed similar to the method described in [9].

Note here that the inverse problem can also be solved using directions of observation in the forward and backward directions of propagation instead of the directions of magnetic polarization as in the method described above. In this case, letting $\hat{\mathbf{r}}' = c\hat{\mathbf{x}}_i'$ (with $c = \pm 1$), $\hat{\mathbf{d}}' = \hat{\mathbf{x}}_i'$ and $\hat{\mathbf{p}}' = \hat{\mathbf{x}}_j'$ for $1 \le i, j \le 3$, leads to:

$$\mathbf{g}^{(A)}\left(c\hat{\mathbf{x}}_i'; \hat{\mathbf{x}}_i', \hat{\mathbf{x}}_j'\right) = \left[\left(\mathbf{P}_j^\top W^{(A)}\mathbf{P}_j + c\mathbf{P}_k^\top T^{(A)}\mathbf{P}_k\right)\hat{\mathbf{x}}_j' + \left(\mathbf{P}_k^\top W^{(A)}\mathbf{P}_j - c\mathbf{P}_j^\top T^{(A)}\mathbf{P}_k\right)\hat{\mathbf{x}}_k'\right], \tag{43}$$

for $A = I, D, L$. Taking measurements for direction combinations similar to the previous method and solving for $\mathbf{P}_k^\top W^{(A)}\mathbf{P}_j$ leads to:

$$2\mathbf{P}_j^\top W^{(A)}\mathbf{P}_j = \left[\mathbf{g}^{(A)}\left(\hat{\mathbf{x}}_i'; \hat{\mathbf{x}}_i', \hat{\mathbf{x}}_j'\right) + \mathbf{g}^{(A)}\left(-\hat{\mathbf{x}}_i'; \hat{\mathbf{x}}_i', \hat{\mathbf{x}}_j'\right)\right]\cdot\hat{\mathbf{x}}_j' \tag{44}$$

$$\begin{aligned} 2\mathbf{P}_k^\top W^{(A)}\mathbf{P}_j &= \left[\mathbf{g}^{(A)}\left(\hat{\mathbf{x}}_i'; \hat{\mathbf{x}}_i', \hat{\mathbf{x}}_j'\right) + \mathbf{g}^{(A)}\left(-\hat{\mathbf{x}}_i'; \hat{\mathbf{x}}_i', \hat{\mathbf{x}}_j'\right)\right]\cdot\hat{\mathbf{x}}_k' \\ &= \left[\mathbf{g}^{(A)}\left(\hat{\mathbf{x}}_i'; \hat{\mathbf{x}}_i', \hat{\mathbf{x}}_k'\right) + \mathbf{g}^{(A)}\left(-\hat{\mathbf{x}}_i'; \hat{\mathbf{x}}_i', \hat{\mathbf{x}}_k'\right)\right]\cdot\hat{\mathbf{x}}_j' \end{aligned} \tag{45}$$

Therefore, the following matrix is constructed:

$$M^{(A)} = (M_{kj}^{(A)}), \quad M_{kj}^{(A)} = \mathbf{P}_k^\top W^{(A)}\mathbf{P}_j, \tag{46}$$

for $A = I, D, L$, and its eigenvalues and eigenvectors are used to specify the orientation and size of the unknown ellipsoidal scatterer as done previously.

4. GEOMETRICALLY DEGENERATE FORMS

The cases of spheroids and spheres can be considered as geometrically degenerate forms of the ellipsoid.

Specifically, for the sphere, the following relations are valid:

$$\alpha_1 = \alpha_2 = \alpha_3, \quad h_2 = h_3 = \mu = \nu = 0, \quad I_1^n(\rho) = \frac{1}{3\rho^3}, \tag{47}$$

for $n = 1, 2, 3$.

For the case of spheroids, the following relations are valid:

$$\text{prolate spheroid:} \quad \alpha_1 > \alpha_2 = \alpha_3, \quad h_2 = h_3, \tag{48}$$

$$\text{oblate spheroid:} \quad \alpha_1 = \alpha_2 > \alpha_3, \quad h_3 = 0. \tag{49}$$

Based on the relation:

$$I_1^i - I_1^j = \frac{\alpha_j^2 - \alpha_i^2}{2} \int_0^\infty \frac{dx}{(x+\alpha_i^2)(x+\alpha_j^2)R(x)}, \tag{50}$$

with $R(x) = \sqrt{(x+\alpha_1^2)(x+\alpha_2^2)(x+\alpha_3^2)}$, it can be observed that when $\lambda_1 = \lambda_2 = \lambda_3 \Rightarrow \alpha_1 = \alpha_2 = \alpha_3$ which means that the unknown scatterer is a sphere independent of orientation and radius $\alpha_1 = V^{\frac{1}{3}}$.

In the case that two out of the three eigenvalues are equal, i.e., $\lambda_2 = \lambda_3$, the scatterer is a spheroid. Thus, based on [9] and the fact that $M_2 = M_3$, the following equation is valid:

$$\frac{1}{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2} \left[1 - \frac{\left(\frac{\alpha_1}{\alpha_2}\right) \cos^{-1}\left(\frac{\alpha_1}{\alpha_2}\right)}{\sqrt{1 - \left(\frac{\alpha_1}{\alpha_2}\right)^2}} \right] = M_1, \tag{51}$$

and the semi-axes are given by:

$$\alpha_1 = \left(\frac{\alpha_1^2}{\alpha_2^2} V\right)^{\frac{1}{3}}, \quad \alpha_2 = \alpha_3 = \left(\frac{V\alpha_2}{\alpha_1}\right)^{\frac{1}{3}}, \tag{52}$$

where V can be found in terms of eigenvalues λ_1 and $\lambda_2 = \lambda_3$ using identity (15) since I_1^n can also be found in terms of them via relations in Eqs. (37)–(39) for each case, respectively.

5. NUMERICAL RESULTS

In this section, numerical results following this method are presented:

Numerical example

Taking far-field measurements and following the numerical algorithm depicted in the flowchart given in the Appendix, the corresponding measurement matrix is constructed, and based on its eigenvalues and eigenvectors the following numerical example is presented:

For measurements that give the measurement matrix:

$$M = \begin{bmatrix} 96.1185 & -0.4228 & 17.8598 \\ -0.4228 & 88.0748 & -0.3351 \\ 17.8598 & -0.3351 & 87.7339 \end{bmatrix},$$

with eigenvalues and eigenvectors:

$$\lambda_1 = 110.2845, \quad \lambda_2 = 88.0617, \quad \lambda_3 = 73.5810,$$

$$\mathbf{v}_1 = (0.7835, -0.0243, 0.6209), \quad \mathbf{v}_2 = (-0.0190, -0.9997, -0.0151), \quad \mathbf{v}_3 = (-0.6211, 0.0000, 0.7837),$$

the following results are obtained which are depicted in Figure 3.

Where the Euler angles are $\alpha = -1.5708$, $\beta = 0.6701$, $\gamma = 1.5465$ and its semi-axes are $\alpha_1 = 4.2002$, $\alpha_2 = 3.5004$ and $\alpha_3 = 2.9962$.

Special thanks to PhD student G. Kounadis for his help with the numerical example.

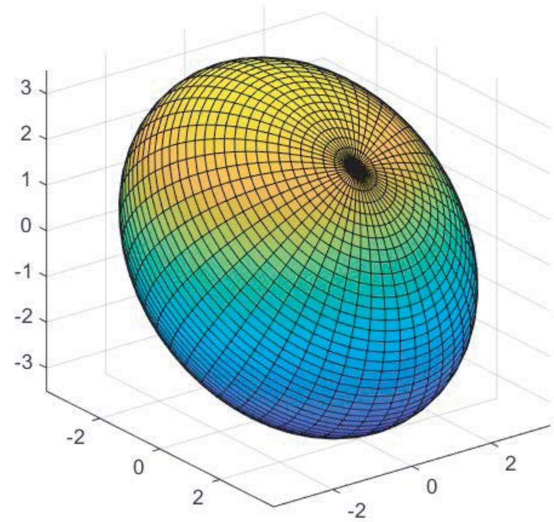
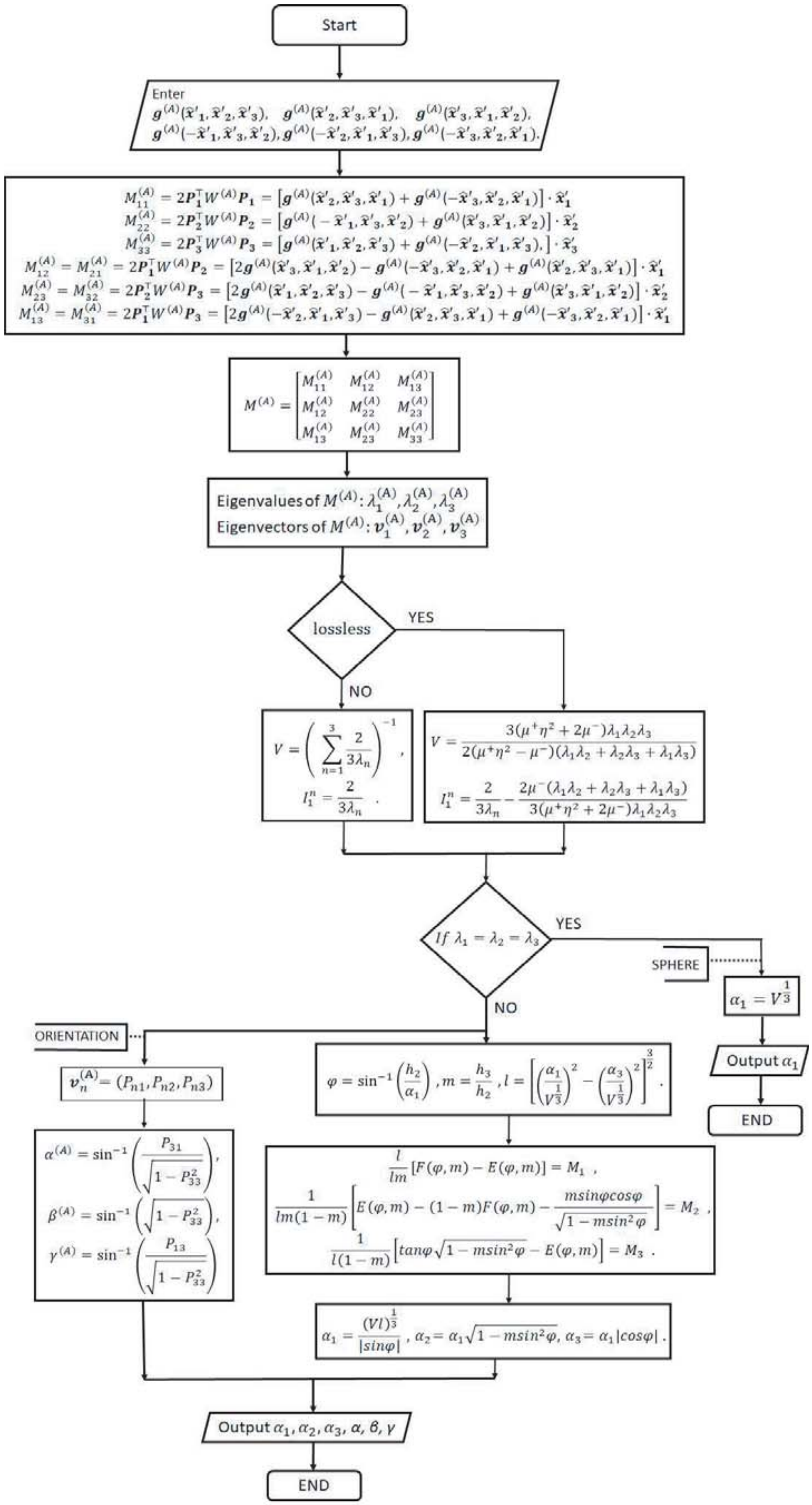


Figure 3. Reconstruction of an ellipsoid using far-field data.

6. CONCLUSION

In the preceding work, the theoretical results concerning a method for solving an inverse electromagnetic scattering problem for the cases of an impedance or a dielectric ellipsoid are presented. In particular, using six measurements of the leading order low-frequency coefficient of the electric far-field pattern, a measurement matrix is constructed whose eigenvalues and eigenvectors are used to specify the size and orientation of the unknown ellipsoid, respectively. Finally, a numerical example is presented. Therefore, the presented method can lead to an efficient algorithm.

APPENDIX A.


Start
Enter
lossless
YES
NO
If $\lambda_1 = \lambda_2 = \lambda_3$
YES
SPHERE
$\alpha_1 = V^{\frac{1}{3}}$
Output α_1
END
NO
ORIENTATION
Output $\alpha_1, \alpha_2, \alpha_3, \alpha, \beta, \gamma$
END

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