

Triple Two-Level Nested Array with Improved Degrees of Freedom

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Abstract—A triple two-level nested array (TTNA) configuration is proposed for direction-of-arrival (DOA) estimation of multiple time-space signals. The proposed TTNA consists of multiple two-level nested arrays, and the distance between two adjacent nested arrays is also given according to a nested array. As traditional nested arrays, it can generate a hole-free different co-array. Compared with some preexisting nested arrays, the proposed nested array can offer more degrees of freedom (DOFs). The closed-form expression of DOFs and the array configuration are given. Moreover, the detailed process for the construction of extended covariance matrix also is obtained. The simulation results show that the proposed method offers improved performance in the precision of DOA estimation due to the increase of virtual sensors.

1. INTRODUCTION

Direction-of-arrival (DOA) estimation of multiple time-space signals based on antenna array has got a lot of attention because of its widespread application in wireless communication and multiple input multiple output (MIMO) radar system. Different from the uniform linear array (ULA), the inter-element spacing of sparse arrays can be variable and larger than the half wavelength of incident signal. Exploiting the location difference between two sensors, more virtual sensors can be obtained from sparse linear arrays. Hence sparse linear array can offer higher degrees of freedom (DOFs) than ULA. Minimum-redundancy array (MRA) [1] is one of the earliest sparse linear arrays, and its difference co-array (DCA) can be seen as a ULA with the most possible consecutive virtual sensors. So, for the same number of sensors, MRA can provide more DOFs than any other sparse array configurations. However, it is difficult to obtain the specific array configuration of MRA as the number of the sensors is larger.

Recently, two kinds of sparse linear arrays, called as co-prime arrays [2–7] and nested arrays [8–21], have gained wide attention. In addition, concentric-ring isophoric sparse array [22] is another important array structure which is used widely for optimal power synthesis of beams. The original co-prime array [2] consists of an M -element uniform linear array with the inter-element spacing being N units and an N -element uniform linear array with the inter-element spacing being M units, where M and N are two given co-prime positive integers. Toward improving the performance of co-prime array, many modified co-prime arrays including generalized co-prime array [3], multi-period co-prime array [4], and reduced-sensors co-prime array [5] have been proposed. In addition, some co-prime MIMO radar configurations [6, 7] have also been presented based on the co-prime array. The attractive advantage of co-prime array is shown in reducing the mutual coupling between sensors. However, compared with MRA and nested array, co-prime array shows a distinct disadvantage in DOFs.

Two-level nested array (TNA) was firstly developed in [8]. Original TNA is constructed by an M -element uniform linear array with the inter-element spacing being one unit and an N -element uniform linear array with the inter-element spacing being M units, where M and N are two given positive

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integers. TNA has more DOFs than co-prime arrays and simpler structure than MRA. Hence, TNA has been improved constantly and used widely since it was proposed. Combining the construction of TNA with RMA, the nested MRA [9] was proposed. Since this array consists of multiple RMAs, it is still difficult to get the array configuration for larger number of sensors. In [10, 11], two kinds of improved TNA configurations have been proposed by adjusting the inter-element spacing of the second uniform linear array. The nested array [11] can provide 2 more DOFs than the TNA [8], and the nested array [10] can provide $L-2$ or $L-3$ more DOFs than the TNA [8], where L is the number of sensors. Moving part sensors from the first sub-array to construct the third sub-array, an augmented nested array has been proposed in [12]. It can offer the same number of DOFs as the nested array [10], while reducing the mutual coupling between sensors. The generalized nested array presented in [13] can also work for reducing mutual coupling, but it cannot increase the DOFs. In [14], the authors used many preexisting arrays to construct some new sparse arrays including a double two-level nested array (DTNA) construction. Compared with the nested arrays in [8, 10–13], the nested arrays in [15, 16] can offer more DOFs. However, the two kinds of nested arrays show advantages only in the DOA estimation of periodic stationary signals because of the existence of “holes”. In addition to this, many other types of arrays have been proposed based on nested array, such as L-shaped nested array [17], nested arrays based on fourth-order cumulant [18, 19], and nested MIMO radars [20, 21].

In this paper, we present a hole-free nested array called triple two-level nested array (TTNA). The proposed nested array consists of multiple nested arrays [10], and it can offer more DOFs than some preexisting multiple nested arrays. For many preexisting nested arrays, the authors have given general expressions of the array configurations, but they did not give the closed-form method to construct the extended covariance matrix. Compared with these arrays, another contribution of this work is that we have given a detailed process to construct extended covariance matrix.

Notation: $[\bullet]^T$, $[\bullet]^*$, $[\bullet]^H$, and $E[\bullet]$ indicate transpose, conjugate, conjugate transpose, and statistical expectation, respectively. $|L|$ denotes the number of elements in set L . $\text{Min}\{L\}$ and $\text{Max}\{L\}$ stand for the minimum and maximum of set L , respectively. $\text{vec}(\mathbf{R})$ represents the vectorization of matrix \mathbf{R} , and \mathbf{J} denotes a matrix with 1 on the back diagonal and 0 on other positions.

2. THE RECEIVED DATA MODEL

Suppose that K narrowband, uncorrelated and far-field signals impinge on an L -element linear array, and θ_k , $k = 1, 2, 3, \dots, K$ is the DOA of the k th signal. Denoting d_l , $l = 2, 3, \dots, L$ as the distance between the l th sensor and the reference sensor, the received data vector $\mathbf{x}(t) = [x_1(t), x_2(t), \dots, x_L(t)]^T \in C^{L \times 1}$ is presented as

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t) \quad (1)$$

where $\mathbf{A} = [\mathbf{a}(\theta_1), \mathbf{a}(\theta_2), \dots, \mathbf{a}(\theta_K)] \in C^{L \times K}$ is the array manifold matrix with $\mathbf{a}(\theta_k) = [1, e^{-i\frac{2\pi}{\lambda}d_2 \sin(\theta_k)}, \dots, e^{-i\frac{2\pi}{\lambda}d_L \sin(\theta_k)}]^T \in C^{L \times 1}$ and λ being the wavelength. $\mathbf{s}(t) = [s_1(t), s_2(t), \dots, s_K(t)]^T \in C^{K \times 1}$ indicates the signal vector, and $\mathbf{n}(t) \in C^{L \times 1}$ represents the noise vector.

3. CONSTRUCTION OF TTNA

In order to avoid direction ambiguity caused by the proposed sparse array in DOA estimation process, we denote $d = \lambda/2$ as the unit inter-element spacing of nested array [10]. Firstly, we construct an NM -element double two-level nested array (DTNA) by using NM -element nested arrays [10], and the configuration is shown in Fig. 1. For convenient expression, we call the nested array [10] as fundamental nested array (FNA). Place the N FNAs discretely in a line, and make the N first sensors to construct a large-interval nested array (LNA) with the similar construction as FNA. However, the unit inter-element spacing of the LNA is Dd , where D is the DOFs of FNA. Then, we construct the HNM -element triple two-level nested array (TTNA) by using H NM -element DTNAs, whose the configuration is also shown in Fig. 1. As the DTNA, we place the H DTNAs discretely in a line and make H first sensors to construct a super large-interval nested array (SLNA) with a similar construction as FNA [10]. However, the unit inter-element spacing of the SLNA is D_1d , where D_1 is the DOFs of DTNA.

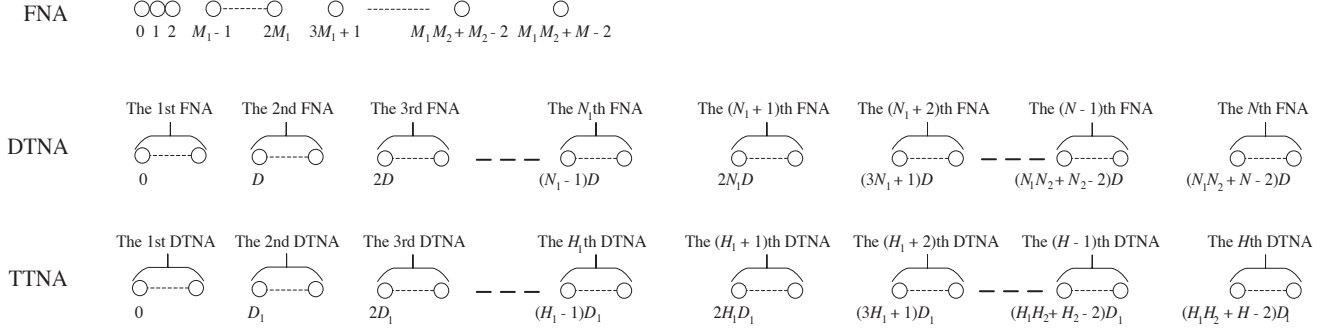

Figure 1. Construction of DTNA and TTNA.

Table 1. Positions of a 96-element TTNA ($M = 4$, $N = 4$, $H = 6$).

	The 1st FNA	The 2nd FNA	The 3rd FNA	The 4th FNA
The 1st DTNA	0, 1, 4, 6	13, 14, 17, 19	52, 53, 56, 58	78, 79, 82, 84
The 2nd DTNA	169, 170, 173, 175	182, 183, 186, 188	221, 222, 225, 227	247, 248, 251, 253
The 3rd DTNA	338, 339, 342, 344	351, 352, 355, 357	390, 391, 394, 396	416, 417, 420, 422
The 4th DTNA	1014, 1015, 1018, 1020	1027, 1028, 1031, 1033	1066, 1067, 1070, 1072	1092, 1093, 1096, 1098
The 5th DTNA	1690, 1691, 1694, 1696	1703, 1704, 1707, 1709	1742, 1743, 1746, 1748	1768, 1769, 1772, 1774
The 6th DTNA	2197, 2198, 2201, 2203	2210, 2211, 2214, 2216	2249, 2250, 2252, 2540	2275, 2276, 2279, 2281

Here we consider that the FNA consists of an M_1 -element uniform array and an M_2 -element sparse array, where $M_1 + M_2 = M$. The LNA consists of an N_1 -element large-interval uniform array and an N_2 -element large-interval sparse array, where $N_1 + N_2 = N$. The SLNA consists of an H_1 -element super large-interval uniform array and an H_2 -element superlarge-interval sparse array, where $H_1 + H_2 = H$. For given M , N , and H , the optimal M_1 , M_2 , N_1 , N_2 and H_1 , H_2 can be obtained as [10]. Taking a 96-element TTNA ($M = 4$, $N = 4$, $H = 6$) as example, the positions of this TTNA are depicted in Table 1.

Remark 1: It should be clear that the authors consider the $(M_1 + M_2)$ -element nested array as an $(M_1 - 1)$ -element uniform array, M_2 -element large-interval uniform array, and an isolated sensor in [10]. We use the new description only for the convenience of expression in the following page. In addition, the DTNA is first proposed by Yang et al. in [14], but the DTNA [14] consists of multiple nested arrays [8]. In [21], the authors have proposed a nested MIMO array, whose equivalent array construction is the same as the DTNA in Fig. 1.

According to the construction of TTNA, the positions of the n th FNA in the h th DTNA are denoted by

$$\left\{ da_{n,m}^h / m = 1, 2, \dots, M \right\} \quad (2)$$

where $da_{n,m}^h$ is the location of the m th sensor in the n th FNA of the h th DTNA.

The expression of $a_{n,m}^1$ can be given by

$$a_{n,m}^1 = \begin{cases} (m-1) + D(n-1), & \text{when } m \leq M_1, n \leq N_1 \\ M_1 m - M_1^2 + m - 1 + D(n-1), & \text{when } M_1 < m < M, n \leq N_1 \\ M_2 M_1 + M - 2 + D(n-1), & \text{when } m = M, n \leq N_1 \\ (m-1) + D(N_1 n - N_1^2 + n - 1), & \text{when } m \leq M_1, N_1 < n < N \\ M_1 m - M_1^2 + m - 1 + D(N_1 n - N_1^2 + n - 1), & \text{when } M_1 < m < M, N_1 < n < N \\ M_2 M_1 + M - 2 + D(N_1 n - N_1^2 + n - 1), & \text{when } m = M, N_1 < n < N \\ (m-1) + D(N_2 N_1 + N - 2), & \text{when } m \leq M_1, n = N \\ M_1 m - M_1^2 + m - 1 + D(N_2 N_1 + N - 2), & \text{when } M_1 < m < M, n = N \\ M_2 M_1 + M - 2 + D(N_2 N_1 + N - 2), & \text{when } m = M, n = N \end{cases} \quad (3)$$

where $D = 2M + 2M_2M_1 - 3$, then $a_{n,m}^h$ can be expressed as

$$a_{n,m}^h = \begin{cases} a_{n,m}^1 + D_1(h-1), & \text{when } h \leq H_1 \\ a_{n,m}^1 + D_1(H_1h - H_1^2 + h - 1), & \text{when } H_1 < h < H \\ a_{n,m}^1 + D_1(H_2H_1 + H - 2), & \text{when } h = H \end{cases} \quad (4)$$

where D_1 is the DOFs of DTNA.

Omitting the symbol of unit inter-element spacing d , we denote the position set of the n th FNA in the h th DTNA as

$$L_n^h = \{a_{n,m}^h/m = 1, 2, \dots, M\} \quad (5)$$

From Eqs. (3)–(5), we can know that

$$\text{Min}\{L_{n_1}^h\} - \text{Max}\{L_{n_2}^h\} > 0 \quad (6)$$

where $n_1 > n_2$.

Denote the nonnegative cross-lap set between the n_1 th FNA and n_2 th FNA in the h th DTNA as L_{n_1, n_2}^h which can be expressed as

$$L_{n_1, n_2}^h = \begin{cases} \{a_{n_1 m_1}^h - a_{n_2 m_2}^h/m_1 = 1, 2, \dots, M, m_2 = 1, 2, \dots, M\}, & n_1 > n_2 \\ \{a_{n_1 m_1}^h - a_{n_2 m_2}^h/m_1 = 1, 2, \dots, M, m_2 = 1, 2, \dots, M, m_1 \geq m_2\}, & n_1 = n_2 \end{cases} \quad (7)$$

Then, the nonnegative self-lap set of the h th DTNA can be described as

$$L^h = \left(\bigcup_{n_1 > n_2} L_{n_1, n_2}^h \right) \cup L_{n, n}^h \quad (8)$$

where n is an arbitrary integer from 1 to N .

Denote the nonnegative cross-lap set between the h_1 th DTNA and h_2 th DTNA as L^{h_1, h_2} which can be expressed as

$$L^{h_1, h_2} = \begin{cases} \{a_{n_1 m_1}^{h_1} - a_{n_2 m_2}^{h_2}/m_1, m_2 = 1, 2, \dots, M; n_1, n_2 = 1, 2, \dots, N\}, & \text{when } h_1 > h_2 \\ L^{h_1}, & \text{when } h_1 = h_2 \end{cases} \quad (9)$$

Then, the nonnegative lap set of TTNA can be described as

$$L = \left(\bigcup_{h_1 > h_2} L^{h_1, h_2} \right) \cup L^h \quad (10)$$

where h is an arbitrary integer from 1 to H .

In order to get the DOFs of the proposed TTNA and drive the detailed process for constructing covariance matrix, we generalize the properties of L_{n_1, n_2}^h , L^h and L , which can be listed as follows.

Proposition 1: As $n_1 > n_2$, following descriptions hold for the cross-lap set L_{n_1, n_2}^h .

- L_{n_1, n_2}^h contains all the contiguous integers from $\text{Min}\{L_{n_1, n_2}^h\}$ to $\text{Max}\{L_{n_1, n_2}^h\}$.
- $|\text{Min}\{L_{n_1, n_2}^h\}| = 2M + 2M_2M_1 - 3 = D$.

The proof can be found in Appendix A.

Proposition 2: L^h contains all the contiguous integers from $\text{Min}\{L^h\}$ to $\text{Max}\{L^h\}$, where $\text{Min}\{L^h\} = 0$ and $\text{Max}\{L^h\} = [N + N_1N_2 - 2][2M - 3 + 2M_2M_1] + M + M_2M_1 - 2$.

The proof can be found in Appendix B.

Proposition 3: L contains all the contiguous integers from $\text{Min}\{L\}$ to $\text{Max}\{L\}$, where $\text{Min}\{L\} = 0$ and $\text{Max}\{L\} = (D_1 - 1)/2 + D_1(H + H_2H_1 - 2)$, where $D_1 = [2N + 2N_1N_2 - 3][2M + 2M_2M_1 - 3]$.

The proof can be found in Appendix C.

According to Proposition 3 and the symmetry of laps, we can know that the negative lap set L^- contains all the contiguous integers from $-[(D_1 - 1)/2 + D_1(H + H_2H_1 - 2)]$ to -1 . Then, it is easy to

Table 2. DOFs of three multiple-nested array configurations.

Number of sensors	Proposed TTNA	DTNA	DTNA [14]
64	2197 ($M = 4, N = 4, H = 4$)	2025 ($M = 8, N = 8$)	1521 ($M = 8, N = 8$)
80	3211 ($M = 4, N = 4, H = 5$)	3015 ($M = 8, N = 10$)	2301 ($M = 8, N = 10$)
96	4563 ($M = 4, N = 4, H = 6$)	4185 ($M = 8, N = 12$)	3237 ($M = 8, N = 12$)
100	4693 ($M = 4, N = 5, H = 5$)	4489 ($M = 10, N = 10$)	3481 ($M = 10, N = 10$)
112	5915 ($M = 4, N = 4, H = 7$)	5915 ($M = 8, N = 14$)	4329 ($M = 8, N = 14$)
120	6669 ($M = 4, N = 5, H = 6$)	6231 ($M = 10, N = 12$)	4897 ($M = 10, N = 12$)
125	6859 ($M = 5, N = 5, H = 5$)	6821 ($M = 5, N = 25$)	5729 ($M = 5, N = 25$)
150	9747 ($M = 5, N = 5, H = 6$)	9313 ($M = 10, N = 15$)	7493 ($M = 10, N = 15$)

obtain that the DOFs of TTNA are $D_1(2H + 2H_2H_1 - 3)$. Table 2 shows the DOFs of three multiple-nested arrays under different numbers of sensors. From Table 2, we can see clearly that the proposed multiple-nested array can provide more DOFs than DTNA [14] and DTNA.

Remark 2: We must notice that the number of sensors in nested array [10] is no less than 4. Hence, the number of sensors in proposed TTNA should be written as the product of three integers greater than 4. Just for this case, we only give the expression of DOFs on certain number of sensors, such as 64, 80, and 96. When the number of sensors is smaller than 64, we can see the DTNA as the particular TTNA with $H = 1$. DOFs of five nested arrays with small number of sensors are listed in Table 3.

Table 3. DOFs of five nested array configurations.

Number of sensors	TNA [8]	Nested array [11]	Nested array [12] (ANAI-1)	Nested array [10]	DTNA
16	143	145	157	157	169 ($M = 4, N = 4$)
20	219	221	237	237	247 ($M = 4, N = 5$)
24	297	299	319	319	351 ($M = 4, N = 6$)
25	337	339	359	359	361 ($M = 5, N = 5$)
28	419	421	445	445	455 ($M = 4, N = 7$)
30	479	481	507	507	513 ($M = 5, N = 6$)
32	545	547	575	575	585 ($M = 4, N = 8$)

4. CONSTRUCTION OF EXTENDED COVARIANCE MATRIX

Constructing extended covariance matrix is the key point to increase the potential DOFs of a sparse array. Spatial smoothing (SS) [23, 24] is a well-known technique to construct an extended full-rank covariance matrix. The principle of SS algorithm is briefly introduced as follows.

Denote the conventional covariance matrix $\mathbf{R}_{xx} = E\{\mathbf{x}\mathbf{x}^H\}$, where \mathbf{x} is the received data vector described in Eq. (1). Then we can obtain a vector $\mathbf{z} = \text{vec}(\mathbf{R}_{xx})$. Picking out all the consecutive lags samples of \mathbf{z} , then we can construct a new vector \mathbf{z}_{new} . Suppose that the length of \mathbf{z}_{new} is $2L_\epsilon + 1$, and two kinds of extended covariance matrix can be constructed as [23, 24], respectively

$$\mathbf{R}'_{xx} = \frac{1}{L_\epsilon + 1} \sum_{i=1}^{L_\epsilon+1} \mathbf{z}_{\text{new}}(L_\epsilon + 2 - i : 2L_\epsilon + 2 - i) \mathbf{z}_{\text{new}}^H(L_\epsilon + 2 - i : 2L_\epsilon + 2 - i) \quad (11)$$

or

$$\mathbf{R}''_{xx} = [\mathbf{z}_{\text{new}}(L_\epsilon + 1 : 2L_\epsilon + 1) \quad \mathbf{z}_{\text{new}}(L_\epsilon : 2L_\epsilon) \quad \cdots \quad \mathbf{z}_{\text{new}}(1 : L_\epsilon + 1)] \quad (12)$$

where $\mathbf{z}_{\text{new}}(L_\varepsilon + 2 - i : 2L_\varepsilon + 2 - i)$ stands for a vector composed by the $(L_\varepsilon + 2 - i)$ th component to the $(2L_\varepsilon + 2 - i)$ th component of \mathbf{z}_{new} .

Performing EVD of \mathbf{R}'_{xx} or \mathbf{R}''_{xx} , the DOA can be estimated by the MUSIC [25] or ESPRIT algorithm [26].

In fact, only $L_\varepsilon + 1$ elements are exploited to form the vector \mathbf{z}_{new} ; therefore, we do not need to obtain all the elements of vector \mathbf{z} . Then, we introduce the detailed process for constructing the vector \mathbf{z}_{new} according to the special structure characteristic of proposed TTNA.

Based on Eq. (1), the received data of the m th sensor in the n th FNA of the h th DTNA can be expressed as

$$x_{nm}^h(t) = \left[e^{-i\frac{2\pi}{\lambda}da_{m,n}^h \sin(\theta_1)} \quad \dots \quad e^{-i\frac{2\pi}{\lambda}da_{m,n}^h \sin(\theta_K)} \right] \mathbf{s}(t) + n_{n,m}^h(t) \quad (13)$$

We first define the continuous sampling covariance vector between $\mathbf{x}_{n_1}^{h_1}$ and $\mathbf{x}_{n_2}^{h_2}$ as $\mathbf{z}_{n_1, n_2}^{h_1, h_2}$, where $\mathbf{x}_{n_1}^{h_1}$ and $\mathbf{x}_{n_2}^{h_2}$ are the data vector of the n_1 th FNA of the h_1 th DTNA and the n_2 th FNA of the h_2 th DTNA, respectively.

When $\mathbf{x}_{n_1}^{h_1} = \mathbf{x}_{n_2}^{h_2} = \mathbf{x}_n^h$, $\mathbf{z}_{n,n}^{h,h}$ can be expressed as

$$\begin{aligned} \mathbf{z}_{n,n}^{h,h} = & \left[E\{x_{n,M_1}^h (x_{n,M_1}^h)^*\}, E\{x_{n,M_1}^h (x_{n,M_1-1}^h)^*\}, \dots, E\{x_{n,M_1}^h (x_{n,1}^h)^*\}, E\{x_{n,M}^h (x_{n,M-1}^h)^*\} \right. \\ & E\{x_{n,M_1+1}^h (x_{n,M_1}^h)^*\}, E\{x_{n,M_1+1}^h (x_{n,M_1-1}^h)^*\}, \dots, E\{x_{n,M_1+1}^h (x_{n,1}^h)^*\}, E\{x_{n,M}^h (x_{n,M-2}^h)^*\} \\ & \vdots \\ & E\{x_{n,M-1}^h (x_{n,M_1}^h)^*\}, E\{x_{n,M-1}^h (x_{n,M_1-1}^h)^*\}, \dots, E\{x_{n,M-1}^h (x_{n,1}^h)^*\}, E\{x_{n,M}^h (x_{n,M_1}^h)^*\} \\ & \left. E\{x_{n,M}^h (x_{n,M_1-1}^h)^*\}, E\{x_{n,M}^h (x_{n,M_1-2}^h)^*\}, \dots, E\{x_{n,M}^h (x_{n,1}^h)^*\} \right]^T \in C^{(M+M_1M_2-1) \times 1} \quad (14) \end{aligned}$$

When $\mathbf{x}_{n_1}^{h_1} \neq \mathbf{x}_{n_2}^{h_2}$, we denote $\mathbf{z}_{n_1, n_2}^{h_1, h_2} = [(\mathbf{z}_{1n_1, n_2}^{h_1, h_2})^T \quad (\mathbf{z}_{2n_1, n_2}^{h_1, h_2})^T \quad (\mathbf{z}_{3n_1, n_2}^{h_1, h_2})^T \quad (\mathbf{z}_{4n_1, n_2}^{h_1, h_2})^T \quad (\mathbf{z}_{5n_1, n_2}^{h_1, h_2})^T]^T$, where $\mathbf{z}_{1n_1, n_2}^{h_1, h_2}$, $\mathbf{z}_{2n_1, n_2}^{h_1, h_2}$, $\mathbf{z}_{3n_1, n_2}^{h_1, h_2}$, $\mathbf{z}_{4n_1, n_2}^{h_1, h_2}$ and $\mathbf{z}_{5n_1, n_2}^{h_1, h_2}$ can be expressed as

$$\begin{aligned} \mathbf{z}_{1n_1, n_2}^{h_1, h_2} = & \left[E\{x_{n_1,1}^{h_1} (x_{n_2, M}^{h_2})^*\}, E\{x_{n_1,2}^{h_1} (x_{n_2, M}^{h_2})^*\}, \dots, E\{x_{n_1, M_1}^{h_1} (x_{n_2, M}^{h_2})^*\}, E\{x_{n_1,1}^{h_1} (x_{n_2, M-1}^{h_2})^*\}, \right. \\ & \left. E\{x_{n_1,2}^{h_1} (x_{n_2, M-1}^{h_2})^*\}, \dots, E\{x_{n_1, M_1}^{h_1} (x_{n_2, M-1}^{h_2})^*\} \right]^T \in C^{2M_1 \times 1} \quad (15) \end{aligned}$$

$$\begin{aligned} \mathbf{z}_{2n_1, n_2}^{h_1, h_2} = & \left[E\{x_{n_1, M_1+1}^{h_1} (x_{n_2, M}^{h_2})^*\}, E\{x_{n_1,1}^{h_1} (x_{n_2, M-2}^{h_2})^*\}, E\{x_{n_1,2}^{h_1} (x_{n_2, M-2}^{h_2})^*\}, \dots, E\{x_{n_1, M_1}^{h_1} (x_{n_2, M-2}^{h_2})^*\}, \right. \\ & E\{x_{n_1, M_1+2}^{h_1} (x_{n_2, M}^{h_2})^*\}, E\{x_{n_1,1}^{h_1} (x_{n_2, M-3}^{h_2})^*\}, E\{x_{n_1,2}^{h_1} (x_{n_2, M-3}^{h_2})^*\}, \dots, E\{x_{n_1, M_1}^{h_1} (x_{n_2, M-3}^{h_2})^*\}, \\ & \vdots \\ & E\{x_{n_1, M-2}^{h_1} (x_{n_2, M}^{h_2})^*\}, E\{x_{n_1,1}^{h_1} (x_{n_2, M_1+1}^{h_2})^*\}, E\{x_{n_1,2}^{h_1} (x_{n_2, M_1+1}^{h_2})^*\}, \dots, E\{x_{n_1, M_1}^{h_1} (x_{n_2, M_1+1}^{h_2})^*\}, \\ & \left. E\{x_{n_1, M-1}^{h_1} (x_{n_2, M}^{h_2})^*\} \right]^T \in C^{[(M_2-2)M_1+M_2-1] \times 1} \quad (16) \end{aligned}$$

$$\begin{aligned} \mathbf{z}_{3n_1, n_2}^{h_1, h_2} = & \left[E\{x_{n_1,1}^{h_1} (x_{n_2, M_1}^{h_2})^*\}, E\{x_{n_1,2}^{h_1} (x_{n_2, M_1}^{h_2})^*\}, \dots, E\{x_{n_1, M_1}^{h_1} (x_{n_2, M_1}^{h_2})^*\}, E\{x_{n_1, M_1}^{h_1} (x_{n_2, M_1-1}^{h_2})^*\}, \right. \\ & \left. E\{x_{n_1, M_1}^{h_1} (x_{n_2, M_1-2}^{h_2})^*\}, \dots, E\{x_{n_1, M_1}^{h_1} (x_{n_2, 1}^{h_2})^*\} \right]^T \in C^{(2M_1-1) \times 1} \quad (17) \end{aligned}$$

$$\begin{aligned}
\mathbf{z}_{4n_1n_2}^{h_1,h_2} = & \left[E\{x_{n_1,M}^{h_1}(x_{n_2,M-1}^{h_2})^*\}, E\{x_{n_1,M+1}^{h_1}(x_{n_2,M_1}^{h_2})^*\}, E\{x_{n_1,M+1}^{h_1}(x_{n_2,M_1-1}^{h_2})^*\}, \right. \\
& \cdots, E\{x_{n_1,M+1}^{h_1}(x_{n_2,1}^{h_2})^*\}, E\{x_{n_1,M}^{h_1}(x_{n_2,M-2}^{h_2})^*\}, E\{x_{n_1,M+2}^{h_1}(x_{n_2,M_1}^{h_2})\}, \\
& E\{x_{n_1,M+2}^{h_1}(x_{n_2,M_1-1}^{h_2})\}, \cdots, E\{x_{n_1,M+2}^{h_1}(x_{n_2,1}^{h_2})\}, \\
& \vdots \\
& E\{x_{n_1,M}^{h_1}(x_{n_2,M+2}^{h_2})^*\}, E\{x_{n_1,M-2}^{h_1}(x_{n_2,M_1}^{h_2})^*\}, E\{x_{n_1,M-2}^{h_1}(x_{n_2,M_1-1}^{h_2})^*\}, \\
& \left. \cdots, E\{x_{n_1,M-2}^{h_1}(x_{n_2,1}^{h_2})^*\}, E\{x_{n_1,M}^{h_1}(x_{n_2,M+1}^{h_2})^*\} \right]^T \in C^{[(M_2-2)M_1+M_2-1] \times 1} \quad (18)
\end{aligned}$$

$$\begin{aligned}
\mathbf{z}_{5n_1n_2}^{h_1,h_2} = & \left[E\{x_{n_1,M-1}^{h_1}(x_{n_2,M_1}^{h_2})^*\}, E\{x_{n_1,M-1}^{h_1}(x_{n_2,M_1-1}^{h_2})^*\}, \cdots, E\{x_{n_1,M-1}^{h_1}(x_{n_2,1}^{h_2})^*\}, \right. \\
& \left. E\{x_{n_1,M}^{h_1}(x_{n_2,M_1}^{h_2})^*\}, E\{x_{n_1,M}^{h_1}(x_{n_2,M_1-1}^{h_2})^*\}, \cdots, E\{x_{n_1,M}^{h_1}(x_{n_2,1}^{h_2})^*\} \right]^T \in C^{2M_1 \times 1} \quad (19)
\end{aligned}$$

Then, we denote \mathbf{z}^{h_1,h_2} as continuous lap sampling covariance vector between \mathbf{x}^{h_1} and \mathbf{x}^{h_2} , where \mathbf{x}^{h_1} and \mathbf{x}^{h_2} are the data vector of the h_1 th DTNA and h_2 th DTNA, respectively.

When $\mathbf{x}^{h_1} = \mathbf{x}^{h_2} = \mathbf{x}^h$, $\mathbf{z}^{h,h}$ can be expressed as

$$\begin{aligned}
\mathbf{z}^{h,h} = & \left[(\mathbf{z}_{N_1,N_1}^{h,h})^T, (\mathbf{z}_{N_1,N_1-1}^{h,h})^T, (\mathbf{z}_{N_1,N_1-2}^{h,h})^T, \cdots, (\mathbf{z}_{N_1,1}^{h,h})^T, (\mathbf{z}_{N,N-1}^{h,h})^T \right. \\
& (\mathbf{z}_{N_1+1,N_1}^{h,h})^T, (\mathbf{z}_{N_1+1,N_1-1}^{h,h})^T, \cdots, (\mathbf{z}_{N_1+1,1}^{h,h})^T, (\mathbf{z}_{N,N-2}^{h,h})^T, \\
& \vdots \\
& (\mathbf{z}_{N-1,N_1}^{h,h})^T, (\mathbf{z}_{N-1,N_1-1}^{h,h})^T, \cdots, (\mathbf{z}_{N-1,1}^{h,h})^T, (\mathbf{z}_{N,N_1}^{h,h})^T \\
& \left. (\mathbf{z}_{N,N_1-1}^{h,h})^T, (\mathbf{z}_{N,N_1-2}^{h,h})^T, \cdots, (\mathbf{z}_{N,1}^{h,h})^T \right]^T \quad (20)
\end{aligned}$$

When $\mathbf{x}^{h_1} \neq \mathbf{x}^{h_2}$, we denote $\mathbf{z}^{h_1,h_2} = [(\mathbf{z}_1^{h_1,h_2})^T (\mathbf{z}_2^{h_1,h_2})^T (\mathbf{z}_3^{h_1,h_2})^T (\mathbf{z}_4^{h_1,h_2})^T (\mathbf{z}_5^{h_1,h_2})^T]^T$, where $\mathbf{z}_1^{h_1,h_2}$, $\mathbf{z}_2^{h_1,h_2}$, $\mathbf{z}_3^{h_1,h_2}$, $\mathbf{z}_4^{h_1,h_2}$ and $\mathbf{z}_5^{h_1,h_2}$ can be expressed as

$$\mathbf{z}_1^{h_1,h_2} = \left[(\mathbf{z}_{1,N}^{h_1,h_2})^T, (\mathbf{z}_{2,N}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1,N}^{h_1,h_2})^T, (\mathbf{z}_{1,N-1}^{h_1,h_2})^T, (\mathbf{z}_{2,N-1}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1,N-1}^{h_1,h_2})^T \right]^T \quad (21)$$

$$\begin{aligned}
\mathbf{z}_2^{h_1,h_2} = & \left[(\mathbf{z}_{N_1+1,N}^{h_1,h_2})^T, (\mathbf{z}_{1,N-2}^{h_1,h_2})^T, (\mathbf{z}_{2,N-2}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1,N-2}^{h_1,h_2})^T, \right. \\
& (\mathbf{z}_{N_1+2,N}^{h_1,h_2})^T, (\mathbf{z}_{1,N-3}^{h_1,h_2})^T, (\mathbf{z}_{2,N-3}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1,N-3}^{h_1,h_2})^T, \\
& \vdots \\
& \left. (\mathbf{z}_{N-2,N}^{h_1,h_2})^T, (\mathbf{z}_{1,N_1+1}^{h_1,h_2})^T, (\mathbf{z}_{2,N_1+1}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1,N_1+1}^{h_1,h_2})^T, (\mathbf{z}_{N-1,N}^{h_1,h_2})^T \right]^T \quad (22)
\end{aligned}$$

$$\mathbf{z}_3^{h_1,h_2} = \left[(\mathbf{z}_{1,N_1}^{h_1,h_2})^T, (\mathbf{z}_{2,N_1}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1,N_1}^{h_1,h_2})^T, (\mathbf{z}_{N_1,N_1-1}^{h_1,h_2})^T, (\mathbf{z}_{N_1,N_1-2}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1,1}^{h_1,h_2})^T \right]^T \quad (23)$$

$$\begin{aligned}
\mathbf{z}_4^{h_1,h_2} = & \left[(\mathbf{z}_{N,N-1}^{h_1,h_2})^T, (\mathbf{z}_{N_1+1,N_1}^{h_1,h_2})^T, (\mathbf{z}_{N_1+1,N_1-1}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1+1,1}^{h_1,h_2})^T \right. \\
& \left. (\mathbf{z}_{N,N-2}^{h_1,h_2}, \mathbf{z}_{N_1+2,N_1}^{h_1,h_2})^T, (\mathbf{z}_{N_1+2,N_1-1}^{h_1,h_2})^T, \cdots, (\mathbf{z}_{N_1+2,1}^{h_1,h_2})^T \right]^T
\end{aligned}$$

$$\begin{aligned} & \vdots \\ & \left[\left(\mathbf{z}_{N,N_1+2}^{h_1,h_2} \right)^T, \left(\mathbf{z}_{N-2,N_1}^{h_1,h_2} \right)^T, \left(\mathbf{z}_{N-2,N_1-1}^{h_1,h_2} \right)^T, \dots, \left(\mathbf{z}_{N-2,1}^{h_1,h_2} \right)^T, \left(\mathbf{z}_{N,N_1+1}^{h_1,h_2} \right)^T \right]^T \end{aligned} \quad (24)$$

$$\mathbf{z}_5^{h_1,h_2} = \left[\left(\mathbf{z}_{N-1,N_1}^{h_1,h_2} \right)^T, \left(\mathbf{z}_{N-1,N_1-1}^{h_1,h_2} \right)^T, \dots, \left(\mathbf{z}_{N-1,1}^{h_1,h_2} \right)^T, \left(\mathbf{z}_{N,N_1}^{h_1,h_2} \right)^T, \left(\mathbf{z}_{N,N_1-1}^{h_1,h_2} \right)^T, \dots, \left(\mathbf{z}_{N,1}^{h_1,h_2} \right)^T \right]^T \quad (25)$$

Then we can construct a vector as

$$\begin{aligned} \mathbf{z}_+ = & \left[\left(\mathbf{z}^{H_1,H_1} \right)^T, \left(\mathbf{z}^{H_1,H_1-1} \right)^T, \left(\mathbf{z}^{H_1,H_1-2} \right)^T, \dots, \left(\mathbf{z}^{H_1,1} \right)^T, \left(\mathbf{z}^{H,H-1} \right)^T \right. \\ & \left. \left(\mathbf{z}^{H_1+1,H_1} \right)^T, \left(\mathbf{z}^{H_1+1,H_1-1} \right)^T, \dots, \left(\mathbf{z}^{H_1+1,1} \right)^T, \left(\mathbf{z}^{H,H-2} \right)^T, \right. \\ & \vdots \\ & \left. \left(\mathbf{z}^{H-1,H_1} \right)^T, \left(\mathbf{z}^{H-1,H_1-1} \right)^T, \dots, \left(\mathbf{z}^{H-1,1} \right)^T, \left(\mathbf{z}^{H,H_1} \right)^T, \right. \\ & \left. \left(\mathbf{z}^{H,H_1-1} \right)^T, \left(\mathbf{z}^{H,H_1-2} \right)^T, \dots, \left(\mathbf{z}^{H,1} \right)^T \right]^T \end{aligned} \quad (26)$$

According to the proofs of Appendix A, Appendix B, and Appendix C, we know that \mathbf{z}_+ consists of all the non-negative consecutive lap samples. According to the symmetry of lap, we can obtain \mathbf{z}_{new} as

$$\mathbf{z}_{\text{new}} = \begin{bmatrix} \mathbf{J}\mathbf{z}_+^* \left(2 : \frac{\text{DOFs}+1}{2} \right) \\ \mathbf{z}_+ \end{bmatrix} \quad (27)$$

According to Eqs. (11) and (12), we can obtain the extended covariance matrix \mathbf{R}'_{xx} or \mathbf{R}''_{xx} . The flowchart about constructing the covariance matrix is shown in Fig. 2.

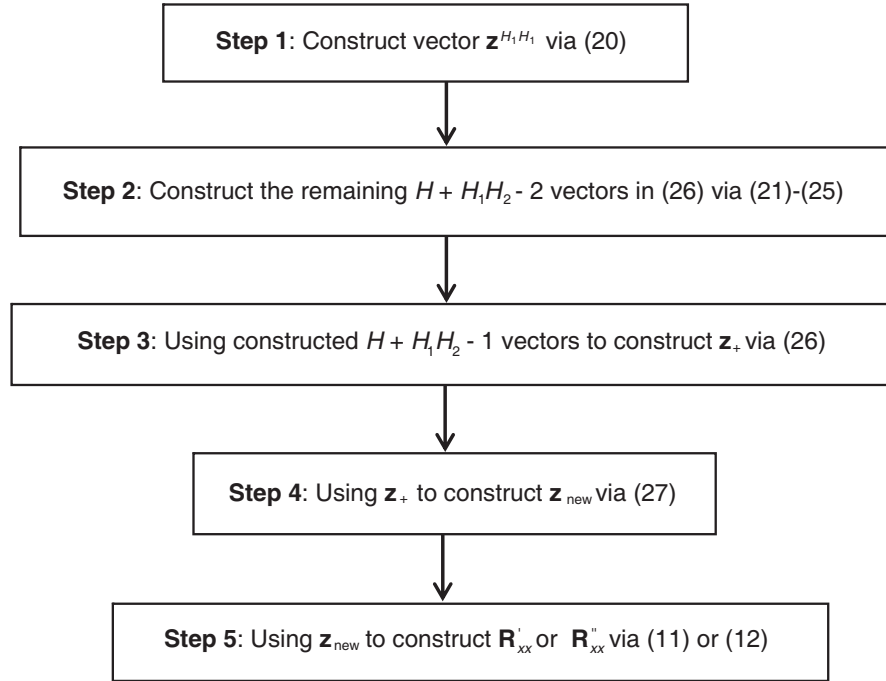


Figure 2. Flowchart of the process to construct covariance matrix.

Remark 3: In [9–13], although the authors have proved the consecutiveness of the laps for the proposed nested array, they did not give the detail for how to construct the extended covariance matrix. In this subsection, we give the detailed process to construct the extended covariance matrix.

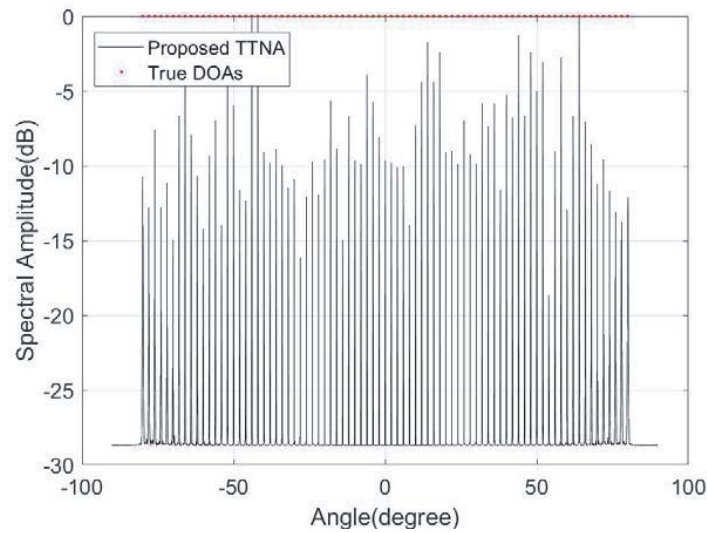


Figure 3. MUSIC spectra of proposed TTNA for 81 signals.

5. SIMULATION

In this section, we present some experiments to examine the effectiveness of proposed TTNA for DOA estimation. For all nested arrays in each experiment, MUSIC algorithm [25] is used to perform DOA estimation.

5.1. Comparison of Space Spectra

Firstly, we compare the space spectra of three multiple-nested arrays for larger number of sensors. We suppose that the total number of sensors is 64, and SNR is 0 dB. 200 snapshots are used to estimate the extended covariance matrix. The searching range of MUSIC algorithm is from -90° to 90° with the grid of 0.1° . Fig. 3, Fig. 4, and Fig. 5 show the MUSIC spectra of three arrays for 81 signals distributed uniformly from -80° to 80° . From Fig. 3, we can find that the proposed TTNA can distinguish the 81 signals clearly. From Fig. 4 and Fig. 5, we can see clearly that a few signals cannot be discriminated by

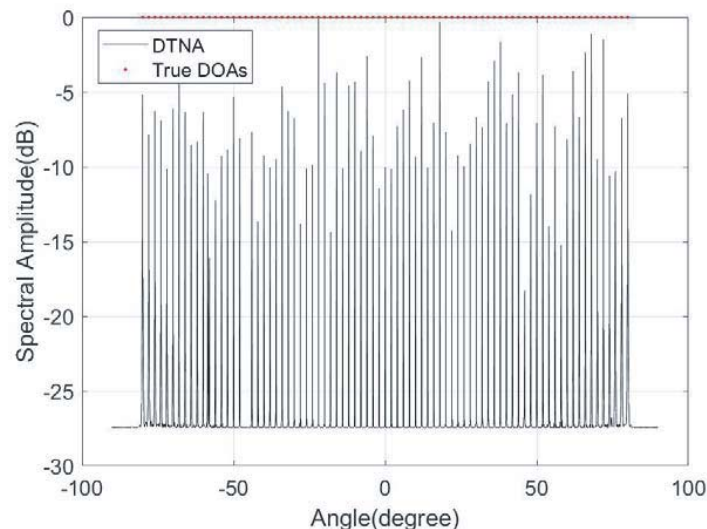


Figure 4. MUSIC spectra of DTNA for 81 signals.

the other two DTNAs.

Secondly, we compare the space spectra of different nested arrays for smaller number of sensors. Suppose that the total number of sensors is 20, and SNR is 0 dB. 500 snapshots are used to estimate the extended covariance matrix. Because the number of sensors for the common TTNA should be larger than 64, we take a 20-element DTNA as a particular TTNA with $H = 1$. In [21], some comparison experiments of two equivalent DTNAs with smaller number of sensors have been presented. Hence, we only compare the space spectra of DTNA with other three nested arrays [8, 10, 11]. Fig. 6 shows the MUSIC spectra of 15 signals distributed uniformly between -35° and 35° . Fig. 7 shows the MUSIC spectra of 41 signals distributed uniformly from -80° to 80° . From Fig. 6 and Fig. 7, we can find that DTNA shows higher resolution than the other three nested arrays.

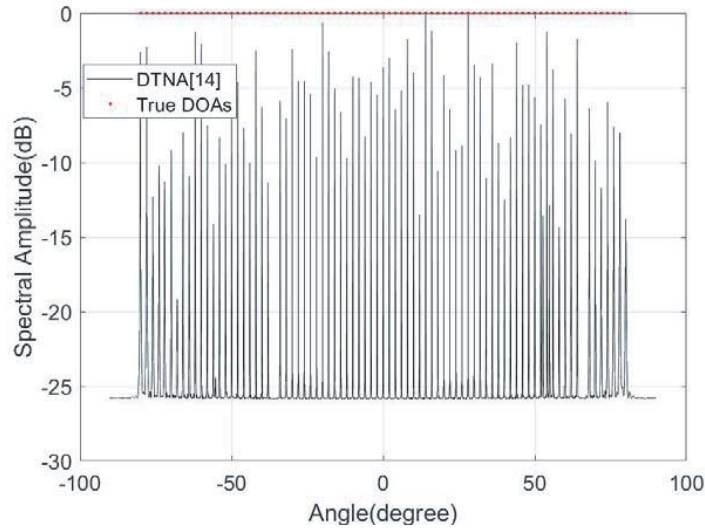


Figure 5. MUSIC spectra of DTNA [14] for 81 signals.

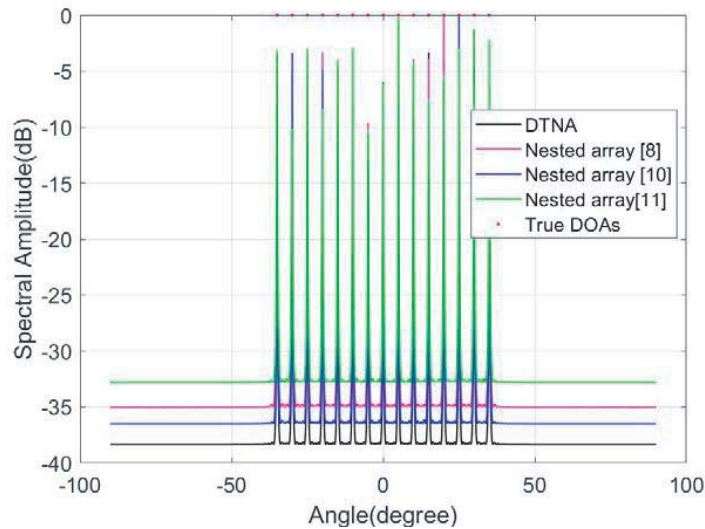


Figure 6. MUSIC spectra of four nested arrays for 15 signals.

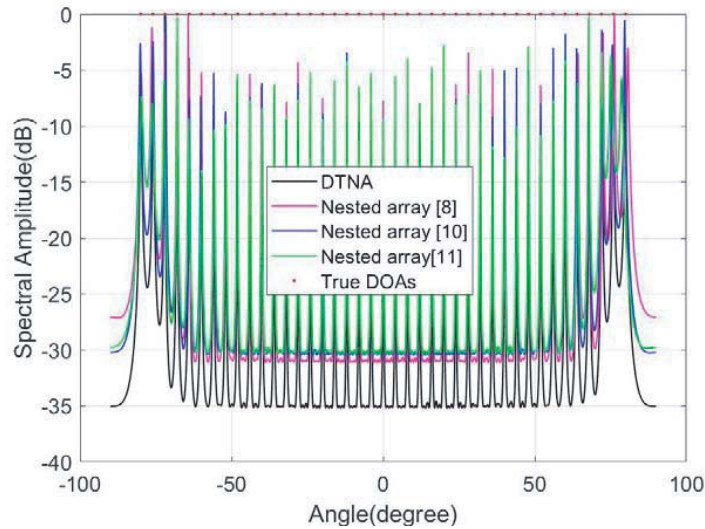


Figure 7. MUSIC spectra of four nested arrays for 41 signals.

5.2. Comparison of RMSE

The root-mean-square error (RMSE) of DOA estimation as the performance measurement is given by

$$RMSE = \sqrt{\frac{1}{KJ} \sum_{j=1}^J \sum_{k=1}^K (\hat{\theta}_{kj} - \theta_k)^2} \tag{28}$$

where $J = 200$, and $\hat{\theta}_{kj}$ is the estimation of θ_k in the j th Monte Carlo trial.

Firstly, we compare the RMSE of DOA estimation for three multiple-nested arrays with larger number of sensors. We suppose that the total number of sensors is 64 for the three multiple-nested arrays. Suppose that 41 signals are uniformly distributed from -80° to 80° . Fig. 8 shows the RMSE of DOA estimation versus SNR with $T = 200$. From Fig. 8, we can see clearly that the RMSE of MUSIC

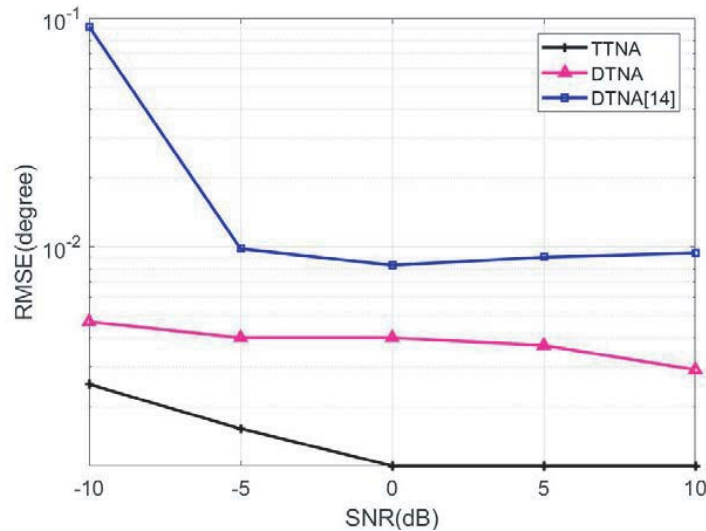


Figure 8. RMSE against SNR for three multiple nested arrays.

algorithm with the proposed TTNA is far lower than the other two DTNAs, particularly when the SNR is larger than 0 dB.

Secondly, we compare the RMSE of DOA estimation for different nested arrays with smaller number of sensors. We suppose that the total number of sensors is 20 for the four nested arrays. The used 20-element DTNA is composed by 4 5-element nested arrays. Suppose that 15 signals are uniformly distributed from -70° to -70° . Fix the snapshots at $T = 500$, and Fig. 9 shows the RMSE of MUSIC algorithm versus SNR for four nested arrays. Then, we fix SNR at 5 dB, and Fig. 10 shows the RMSE of MUSIC algorithm versus snapshots for four nested arrays. From the two figures, it is clear to find that the RMSE of MUSIC algorithm with the DTNA is lower than the other three nested arrays.

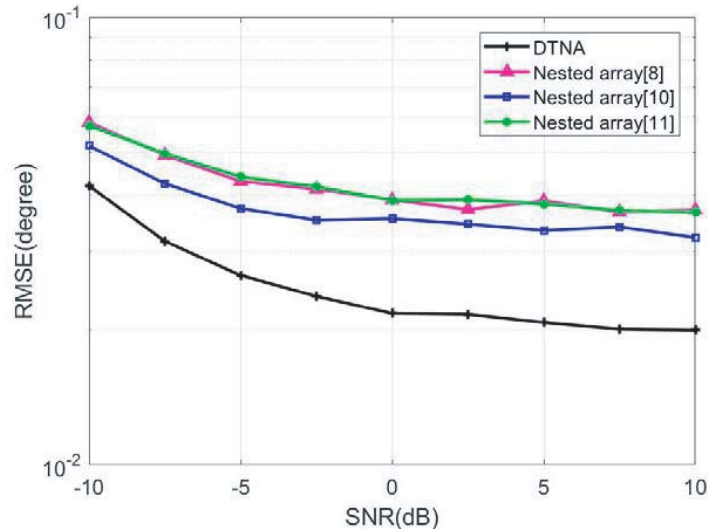


Figure 9. RMSE against SNR for four nested arrays.

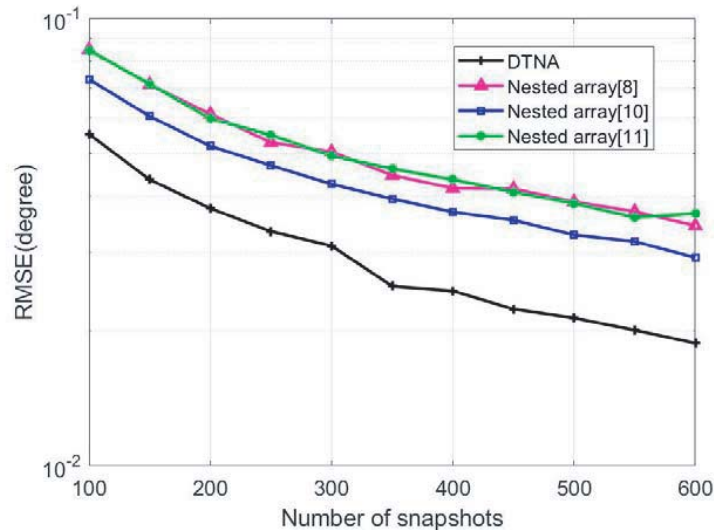


Figure 10. RMSE against snapshots for three nested arrays.

6. CONCLUSION

In this paper, we present a new hole-free nested array which consists of multiple fundamental nested arrays. The positions of these fundamental nested arrays are obtained according to the other given nested array. The closed-form expression of DOFs and the detailed process for the construction of extended covariance matrix are given. Compared with many preexisting nested arrays, the proposed nested array can provide more degrees of freedom (DOFs). Because of the increase of DOFs, the proposed array shows higher resolution in DOA estimation. Lots of simulation results certify that the proposed array has better performance for DOA estimation.

APPENDIX A.

Proof of Proposition 1

Observing the set L_{n_1, n_2}^h , we can find that many repeating elements appear in the set. If we want to know the characteristic of the set L_{n_1, n_2}^h , we only need to pick out all unique elements. Giving enough thought to the construction of FNA, we denote five sub-sets of L_{n_1, n_2}^h as

$$L_{1n_1n_2}^h = \{a_{n_1,1}^h - a_{n_2,M}^h, a_{n_1,2}^h - a_{n_2,M}^h, \dots, a_{n_1,M_1}^h - a_{n_2,M}^h, a_{n_1,1}^h - a_{n_2,M-1}^h, a_{n_1,2}^h - a_{n_2,M-1}^h, \dots, a_{n_1,M_1}^h - a_{n_2,M-1}^h\} \quad (\text{A1})$$

$$L_{2n_1n_2}^h = \{a_{n_1,M_1+1}^h - a_{n_2,M}^h, a_{n_1,1}^h - a_{n_2,M-2}^h, a_{n_1,2}^h - a_{n_2,M-2}^h, \dots, a_{n_1,M_1}^h - a_{n_2,M-2}^h, a_{n_1,M_1+2}^h - a_{n_2,M}^h, a_{n_1,1}^h - a_{n_2,M-3}^h, a_{n_1,2}^h - a_{n_2,M-3}^h, \dots, a_{n_1,M_1}^h - a_{n_2,M-3}^h, \dots, a_{n_1,M-2}^h - a_{n_2,M}^h, a_{n_1,1}^h - a_{n_2,M_1+1}^h, a_{n_1,2}^h - a_{n_2,M_1+1}^h, \dots, a_{n_1,M_1}^h - a_{n_2,M_1+1}^h, a_{n_1,M-1}^h - a_{n_2,M}^h\} \quad (\text{A2})$$

$$L_{3n_1n_2}^h = \{a_{n_1,1}^h - a_{n_2,M_1}^h, a_{n_1,2}^h - a_{n_2,M_1}^h, \dots, a_{n_1,M_1}^h - a_{n_2,M_1}^h, a_{n_1,M_1}^h - a_{n_2,M_1-1}^h, a_{n_1,M_1}^h - a_{n_2,M_1-2}^h, \dots, a_{n_1,M_1}^h - a_{n_2,1}^h\} \quad (\text{A3})$$

$$L_{4n_1n_2}^h = \{a_{n_1,M}^h - a_{n_2,M-1}^h, a_{n_1,M_1+1}^h - a_{n_2,M_1}^h, a_{n_1,M_1+1}^h - a_{n_2,M_1-1}^h, \dots, a_{n_1,M_1+1}^h - a_{n_2,1}^h, a_{n_1,M}^h - a_{n_2,M-2}^h, a_{n_1,M_1+2}^h - a_{n_2,M_1}^h, a_{n_1,M_1+2}^h - a_{n_2,M_1-1}^h, \dots, a_{n_1,M_1+2}^h - a_{n_2,1}^h, \dots, a_{n_1,M}^h - a_{n_2,M_1+2}^h, a_{n_1,M-2}^h - a_{n_2,M_1}^h, a_{n_1,M-2}^h - a_{n_2,M_1-1}^h, \dots, a_{n_1,M-2}^h - a_{n_2,1}^h, a_{n_1,M-2}^h - a_{n_2,1}^h, a_{n_1,M}^h - a_{n_2,M_1+1}^h\} \quad (\text{A4})$$

$$L_{5n_1n_2}^h = \{a_{n_1,M-1}^h - a_{n_2,M_1}^h, a_{n_1,M-1}^h - a_{n_2,M_1-1}^h, \dots, a_{n_1,M-1}^h - a_{n_2,1}^h, a_{n_1,M}^h - a_{n_2,M_1}^h, a_{n_1,M}^h - a_{n_2,M_1-1}^h, \dots, a_{n_1,M}^h - a_{n_2,1}^h\} \quad (\text{A5})$$

According to the rule of the five subsets from Eqs. (A1)–(A5), we have

$$\begin{cases} |L_{1n_1n_2}^h| = |L_{5n_1n_2}^h| = 2M_1 \\ |L_{2n_1n_2}^h| = |L_{4n_1n_2}^h| = (M_2 - 2)M_1 + M_2 - 1 \\ |L_{3n_1n_2}^h| = 2M_1 - 1 \end{cases} \quad (\text{A6})$$

Comparing any two adjacent elements in subset $L_{in_1n_2}^h$, $i = 1, 2, 3, 4, 5$, we can find that the elements increase strictly. Using Equations (3)–(4), we can calculate the first element and last element

of $L_{in_1n_2}^h$. Comparing the last element of $L_{in_1n_2}^h$ with the first element of $L_{(i+1)n_1n_2}^h$, $i = 1, 2, 3, 4$, yields

$$\begin{cases} a_{n_1, M_1}^h - a_{n_2, M-1}^h < a_{n_1, M_1+1}^h - a_{n_2, M}^h \\ a_{n_1, M-1}^h - a_{n_2, M}^h < a_{n_1, 1}^h - a_{n_2, M_1}^h \\ a_{n_1, M_1}^h - a_{n_2, 1}^h < a_{n_1, M}^h - a_{n_2, M-1}^h \\ a_{n_1, M}^h - a_{n_2, M_1+1}^h < a_{n_1, M-1}^h - a_{n_2, M_1}^h \end{cases} \quad (\text{A7})$$

Combining the progressive increase of $L_{in_1n_2}^h$ with Eq. (A7), we can know that any two elements in $\bigcup_{i=1}^5 L_{in_1n_2}^h$ are unequal. Then, it is easy to know

$$\left| L_{n_1, n_2}^h \right| \geq \left| L_{1n_1n_2}^h \right| + \left| L_{2n_1n_2}^h \right| + \left| L_{3n_1n_2}^h \right| + \left| L_{4n_1n_2}^h \right| + \left| L_{5n_1n_2}^h \right| = 2M + 2M_2M_1 - 3 = D \quad (\text{A8})$$

$$\text{Min}(L_{n_1, n_2}^h) = \text{Min}\{L_{n_1}^h\} - \text{Max}\{L_{n_2}^h\} = a_{n_1, 1}^h - a_{n_2, M}^h \quad (\text{A9})$$

$$\text{Max}(L_{n_1, n_2}^h) = \text{Max}\{L_{n_1}^h\} - \text{Min}\{L_{n_2}^h\} = a_{n_1, M}^h - a_{n_2, 1}^h \quad (\text{A10})$$

$$\left| L_{n_1, n_2}^h \right| \leq \text{Max}(L_{n_1, n_2}^h) - \text{Min}(L_{n_1, n_2}^h) + 1 = (a_{n_1, M}^h - a_{n_2, 1}^h) - (a_{n_1, 1}^h - a_{n_2, M}^h) + 1 = 2M + 2M_2M_1 - 3 \quad (\text{A11})$$

We also need to notice the fact that equality in Eq. (A8) holds if and only if L_{n_1, n_2}^h contains all the contiguous integers from $\text{Min}(L_{n_1, n_2}^h)$ to $\text{Max}(L_{n_1, n_2}^h)$.

Combining Eq. (A8) with Eq. (A11), we have

$$\left| L_{n_1, n_2}^h \right| = 2M + 2M_2M_1 - 3 = \text{Max}(L_{n_1, n_2}^h) - \text{Min}(L_{n_1, n_2}^h) + 1 \quad (\text{A12})$$

Then, we can prove the two facts in Proposition 1 simultaneously.

APPENDIX B.

Proof of Proposition 2

Consider $N_2 + 1$ groups of cross-lap sets, which are expressed as

$$\begin{cases} \text{Group 1: } L_{N_1, N_1}^h, L_{N_1, N_1-1}^h, \dots, L_{N_1, 1}^h, L_{N, N-1}^h \\ \text{Group 2: } L_{N_1+1, N_1}^h, L_{N_1+1, N_1-1}^h, \dots, L_{N_1+1, 1}^h, L_{N, N-2}^h \\ \qquad \qquad \qquad \vdots \\ \text{Group } N_2: L_{N-1, N_1}^h, L_{N-1, N_1-1}^h, \dots, L_{N-1, 1}^h, L_{N, N_1}^h \\ \text{Group } N_2 + 1: L_{N, N_1-1}^h, L_{N, N_1-2}^h, \dots, L_{N, 1}^h \end{cases} \quad (\text{B1})$$

Obviously, the last group contains $N_1 - 1$ sets, and there are $N_1 + 1$ sets in any other group. Hence, it is easy to know that the total number of sets in the $N_2 + 1$ groups is $N + N_1N_2 - 1$.

Because similar rules exist in the top N_2 groups, we first consider these groups. In each group, comparing the maximum values of adjacent sets via Eqs. (3), (4), and (7) yields

$$\begin{aligned} \text{Max}(L_{n_1, n_2-1}^h) - \text{Max}(L_{n_1, n_2}^h) &= \left[\text{Max}(L_{n_1}^h) - \text{Min}(L_{n_2-1}^h) \right] - \left[\text{Max}(L_{n_1}^h) - \text{Min}(L_{n_2}^h) \right] \\ &= \text{Min}(L_{n_2}^h) - \text{Min}(L_{n_2-1}^h) = D \end{aligned} \quad (\text{B2})$$

where $N - 1 \geq n_1 \geq N_1$, $2 \leq n_2 \leq N_1$, and

$$\begin{aligned} \text{Max}(L_{N, n_2}^h) - \text{Max}(L_{N+N_1-1-n_2, 1}^h) &= \left[\text{Max}(L_N^h) - \text{Min}(L_{n_2}^h) \right] - \left[\text{Max}(L_{N+N_1-1-n_2}^h) - \text{Min}(L_1^h) \right] \\ &= D \end{aligned} \quad (\text{B3})$$

where $N - 1 \geq n_2 \geq N_1$.

Then, we consider the $N - 1$ sets $L_{N,N-1}^h, L_{N,N-2}^h, \dots, L_{N,1}^h$. Comparing the maximum values of adjacent sets, we have

$$\begin{aligned} \text{Max}(L_{N,n_2}^h) - \text{Max}(L_{N,n_2+1}^h) &= \left[\text{Max}(L_N^h) - \text{Min}(L_{n_2}^h) \right] - \left[\text{Max}(L_N^h) - \text{Min}(L_{n_2+1}^h) \right] \\ &= \text{Min}(L_{n_2+1}^h) - \text{Min}(L_{n_2}^h) = \begin{cases} D, & N_1 - 1 \geq n_2 \geq 1 \\ (N_1+1)D, & N_1 \leq n_2 \leq N - 2 \end{cases} \quad (\text{B4}) \end{aligned}$$

From Eqs. (B2), (B3), and the first case of Eq. (B4), we can confirm that the maximum of each set in the same group grows uniformly with D being common difference.

Next, we consider all the sets in the $N_2 + 1$ groups and sort them based on the order in Eq. (B1). For example, we call $L_{N,N-1}^h$ as the $(N_1 + 1)$ th set, and L_{N_1+1,N_1}^h as the $(N_1 + 2)$ th set. According to the second case of Eq. (B4), we can further confirm that the $N + N_1N_2 - 1$ maximums of corresponding $N + N_1N_2 - 1$ sets increase with D being common difference. Based on Proposition 1, L_{n_1,n_2}^h contains all the contiguous integers from $\text{Min}(L_{n_1,n_2}^h)$ to $\text{Max}(L_{n_1,n_2}^h)$ and $|L_{n_1,n_2}^h| = D$ when $n_1 > n_2$. It is indicated that $\text{Max}\{L_{n_1,n_2}^h\} - \text{Min}\{L_{n_1,n_2}^h\} = D - 1$. Then, it is easy to know that the minimum of the i th set is one more than the maximum of $(i - 1)$ th set when $N + N_1N_2 - 1 \geq i > 1$. Denote the union of the $N + N_1N_2 - 1$ sets in Eq. (B1) as

$$\bar{L}^h = \left(\bigcup_{n_1=N_1, n_2=1}^{n_1=N-1, n_2=N_1} L_{n_1, n_2}^h \right) \cup \left(\bigcup_{n_2=1}^{N-1} L_{N, n_2}^h \right). \quad (\text{B5})$$

It is clear that \bar{L}^h contains all the contiguous integers from $\text{Min}(L_{N_1,N_1}^h)$ to $\text{Max}(L_{N,1}^h)$. Because $\text{Min}(L_{N_1,N_1}^h) = \text{Min}(L^h)$, $\text{Max}(L_{N,1}^h) = \text{Max}(L^h)$ and $\bar{L}^h \subset L^h$, we know that L^h contains all the contiguous integers from $\text{Min}(L^h)$ to $\text{Max}(L_{N,1}^h)$. Since $\text{Min}(L^h) = 0$ and $\text{Max}(L^h) = [N + N_1N_2 - 2][2M + 2M_2M_1 - 3] + M + M_2M_1 - 2$, L contains all the integers from 0 to $[N + N_1N_2 - 2][2M + 2M_2M_1 - 3] + M + M_2M_1 - 2$.

APPENDIX C.

Proof of Proposition 3

Denoting the position set of the h th DTNA as $P^h = \{L_1^h, L_2^h, \dots, L_N^h\}$, the number of integers between $\text{Max}(P^h)$ and $\text{Max}(P^{h+1})$ can be expressed as

$$W_{h,h+1} = \begin{cases} D_1 - 1, & \text{when } 1 \leq h \leq H_1 - 1 \\ (H_1 + 1)D_1 - 1, & \text{when } H_1 \leq h < H - 1 \\ H_1D_1 - 1, & \text{when } h = H - 1 \end{cases} \quad (\text{C1})$$

Denoting $L_{n_1,n_2}^{h_1,h_2}$ as non-negative cross-lap set between $L_{n_1}^{h_1}$ and $L_{n_2}^{h_2}$, it is easy to know that the non-negative cross-lapset $L^{h_1,h_2}(h_1 > h_2)$ can be seen as the union of some different $L_{n_1,n_2}^{h_1,h_2}$.

As proposition 1, considering $L_1^h, L_2^h, \dots, L_N^h$ as N numbers, we can find that the number of different $L_{n_1,n_2}^{h_1,h_2}$ is $2N + 2N_2N_1 - 3$, and the intersection of any two different $L_{n_1,n_2}^{h_1,h_2}$ is empty. According to Eq. (A12), we can know that $|L_{n_1,n_2}^{h_1,h_2}| = 2M + 2M_2M_1 - 3$, so we have

$$|L^{h_1,h_2}| = (2M + 2M_2M_1 - 3)(2N + 2N_2N_1 - 3) = D_1, \quad (h_1 > h_2) \quad (\text{C2})$$

After computing $\text{Max}(P^h)$ and $\text{Max}(P^{h+1})$, we can drive that

- 1) $L^{h+1,1}$ contains all the integers between $\text{Max}(P^h)$ and $\text{Max}(P^{h+1})$, when $1 \leq h \leq H_1 - 1$;
- 2) The $H_1 + 1$ sets $L^{h+1,H_1}, L^{h+1,H_1-1}, \dots, L^{h+1,1}$ and $L^{H,H-(h-H_1+1)}$ contain all the integers between $\text{Max}(P^h)$ and $\text{Max}(P^{h+1})$, when $H_1 \leq h < H - 1$;
- 3) The H_1 sets $L^{H,H_1}, L^{H,H_1-1}, \dots, L^{H,1}$ contain all the integers between $\text{Max}(P^h)$ and $\text{Max}(P^{h+1})$, when $H_1 \leq h \leq H - 1$.

Combining 1), 2), and 3) with Proposition 2, we can know the correctness of Proposition 3.

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