

An Accurate Explicit Expression for the Self Inductance of Thin-Wire Round Pancake Coils

Luisa Di Paola¹ and Marco Muzi^{2, *}

Abstract—This paper presents an accurate analytical explicit expression for the self-inductance of a flat pancake round coil made up of concentric turns. The expression is obtained by converting the semi-infinite integral representation for the mutual inductance between two arbitrary turns of the coil into a finite integral, and then by expanding the integrand into a series of Legendre polynomials. As a result, a sum of simpler integrals is obtained, whose analytical evaluation is straightforward. The self inductance is finally expressed as the sum of logarithmic functions, describing the contributions from the self-inductances of the single turns, plus the mutual-inductance terms originating from all the possible pairs of turns of the coil, each one given by a power series of the ratio between the radii of the turns. Numerical simulations are performed to illustrate the advantages of the proposed solution.

1. INTRODUCTION

Thin-wire flat coils of wire are widely used in a number of engineering applications, including wireless power transfer, magnetic resonance imaging, electromagnetic induction heating, and electromagnetic sounding [1–33]. For instance, 3D system integration technologies involve usage of inductive wireless signal/power links as a valuable alternative to wire bonding, and inductive coupling based on planar coils is often the preferred choice because of the compatibility with most planar processes, and the relatively low cost and ease of fabrication of the coils [3]. On the other hand, pancake coils find application in inductive diathermy therapeutic treatments, which consist of positioning an applicator fed by a radiofrequency generator in close proximity to a prescribed region of human body, so as to induce electric fields in subcutaneous tissues by electromagnetic induction [30–32]. Heating that arises from this process can remove vascular occlusion material, relieve muscle spasms, and accelerate wound healing. Furthermore, induction heating coils are also used to heat lignocellulosic biomass up to and well beyond the temperatures required to carry out a pyrolysis reaction [33].

Regardless of the considered application, the a-priori knowledge of the equivalent circuit of the coil is always desired in order to predict its performances, and this requires accurate evaluation of the self inductance. Yet, in spite of the geometrical simplicity, rigorous explicit closed-form expressions describing the inductance of a pancake coil still are not available. In fact, previously published solutions to this problem are in integral form [24], not completely analytical [26, 30–32], or tailored to special cases [7, 28]. These features make their use impractical. For instance, integral solution in [24] has the disadvantage of requiring numerical evaluation, which is both prone to inaccuracies and typically time demanding. In addition, numerical calculation alone does not allow to gain insight in the physics of the problem. On the other hand, quasi-analytical solutions come from evaluating the integral expression for the magnetic field through usage of hybrid analytical-numerical techniques. Well-established examples

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* Corresponding author: Marco Muzi (marco.muzi1983@gmail.com).

¹ Department of Engineering, University Campus Bio-Medico of Rome, Via Alvaro del Portillo 21, Rome 00128, Italy. ² Department of Information Engineering, Electronics and Telecommunications, “La Sapienza” University of Rome, Via Eudossiana 18, Rome 00184, Italy.

are the series-form representations that arise from applying digital filter technique and least squares-based fitting procedures [26, 30–32], which require the a-priori computation of a large set of numerical coefficients and, as a consequence, are computationally expensive. Finally, explicit analytical solutions tailored to special cases can be used only for specific coil configurations. Excellent illustrations of such solutions are the expressions derived in [7, 28], which are valid only for single-turn coils.

The objective of this work is to derive a rigorous explicit expression for the self-inductance of a multi-turn thin-wire round pancake coil, which permits to obtain significant time savings with respect to purely numerical approaches. This alone allows to overcome the limitations implied by the previously published solutions to the same problem. The proposed solution is obtained through an analytical procedure based on expanding the integrand of the integral representation for the self-inductance in terms of Legendre polynomials. Similar approaches have been recently applied to different physical problems like, for instance, the problem of an arbitrarily shaped air inclusion embedded in a conductive ferromagnetic medium illuminated by a current-carrying coil [34]. Formerly, techniques based on Legendre series expansions have been used to solve the scattering problem of a perfectly conducting spheroidal body excited by a magnetic dipole [35] and to investigate the low-frequency interaction of a magnetic dipole with two perfectly conducting spheres buried in a homogeneous conductive medium [36].

The developed analytical procedure consists of two steps. First, the semi-infinite integral representation for the mutual inductance between arbitrary turns is converted into a finite integral, whose integrand is replaced with its Legendre series. Next, the sum is moved outside the integral symbol, and the resulting integrals of the Legendre polynomials are evaluated analytically. This leads to express the generic mutual inductance between two turns as a sum of powers of the ratio between their radii. The derived formula has the advantage of being significantly less time demanding than numerical integration, especially when the turn-to-turn spacing is not negligible with respect to the turn radius. Moreover, its explicit form makes it easy to be implemented, and suitable to be used in the framework of optimization processes that require repeated objective function evaluations. However, it should be noted that the proposed solution is valid as long as the thin-wire assumption, underlying the present derivation, holds. This occurs every time that the wire radius is much smaller than the radii of the turns, that is when the wire radius is much smaller than the inner radius of the pancake coil.

2. THEORY

Consider a round pancake coil composed of N concentric circular turns, with radii a_i ($i = 1, \dots, N$), as shown in Fig. 1. The overall self-inductance of the coil reads

$$L = \sum_{i=1}^N L_t(a_i) + 2 \sum_{i=1}^N \sum_{j=i+1}^N M(a_i, a_j), \quad (1)$$

where $L_t(a)$ is the self-inductance of the generic turn with radius a , while $M(a, b)$ is the mutual inductance of two arbitrary turns with radii a and b ($a < b$). Under the thin-wire assumption, which

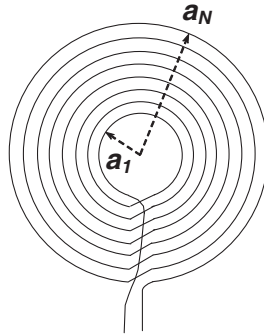


Figure 1. Sketch of a round pancake coil.

implies that the wire radius r_w is much smaller than the radii of turns, $L_t(a)$ may be expressed as [8, 24]

$$L_t(a) = \mu_0 a \left[\log \left(\frac{8a}{r_w} \right) - 2 \right], \tag{2}$$

while $M(a, b)$ is described by the semi-infinite integral [28]

$$M(a, b) = \pi \mu_0 a b \int_0^\infty J_1(\lambda a) J_1(\lambda b) d\lambda, \tag{3}$$

where $J_n(\cdot)$ is the n th-order Bessel function, while μ_0 is the free-space magnetic permeability. The scope of this work is to accurately evaluate the integral representation for $M(a, b)$. To this goal, it is first convenient to introduce the well known identity [37, 11.41.17]

$$J_m(\alpha) J_m(\beta) = \frac{1}{\pi} \int_0^\pi J_0(\gamma) \cos(m\phi) d\phi, \tag{4}$$

with

$$\gamma = \sqrt{\alpha^2 + \beta^2 - 2\alpha\beta \cos \phi}, \tag{5}$$

so as to express Eq. (3) as

$$M(a, b) = \mu_0 a b \int_0^\pi \left[\int_0^\infty J_0(\lambda b q) d\lambda \right] \cos \phi d\phi, \tag{6}$$

being

$$q = \sqrt{1 - 2h \cos \phi + h^2}, \tag{7}$$

and $h = a/b$. The integral within the square brackets of Eq. (6) is the Lipschitz Integral, whose explicit form is given by [16, 38–40]

$$\int_0^\infty J_0(\lambda b q) d\lambda = \frac{1}{bq}. \tag{8}$$

Hence, substitution of Eq. (8) into Eq. (6) leads to the expression

$$M(a, b) = \mu_0 a \int_0^\pi \frac{1}{q} \cos \phi d\phi, \tag{9}$$

which may be evaluated after recognizing that $1/q$ is the generating function of the Legendre polynomials. It reads [41, 3.131]

$$\frac{1}{q} = \sum_{m=0}^\infty P_m(\cos \phi) h^m, \tag{10}$$

with $P_m(x)$ being the m th-order polynomial, defined as [41, 42]

$$P_m(x) = \frac{1}{2^m} \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{m}{k} \binom{2m-2k}{m} x^{m-2k}, \tag{11}$$

where $\lfloor \cdot \rfloor$ denotes the integer part. Use of Eq. (10) in Eq. (9) provides

$$M(a, b) = \mu_0 a \sum_{m=0}^\infty h^m \int_0^\pi P_m(\cos \phi) \cos \phi d\phi \tag{12}$$

where the integral on the right-hand side is non-null only for odd m and given by [41, 3.173]

$$\int_0^\pi P_m(\cos \phi) \cos \phi d\phi = \frac{\Gamma(m/2 + 1) \Gamma(m/2)}{\Gamma(m/2 + 1/2) \Gamma(m/2 + 3/2)} \tag{13}$$

with $\Gamma(\cdot)$ being the Gamma function. Thus, since it holds

$$\Gamma(n/2) = \begin{cases} \sqrt{\pi} 2^{(1-n)/2} (n-2)!!, & \text{odd } n \\ (n/2 - 1)!, & \text{even } n \end{cases} \tag{14}$$

it yields

$$\int_0^\pi P_m(\cos \phi) \cos \phi d\phi = \frac{\pi(m!!)^2}{2^m m [(m-1)/2]! [(m+1)/2]!}, \quad \text{odd } m, \quad (15)$$

and after setting $m=2l-1$, Eq. (12) assumes the form

$$M(a, b) = \pi \mu_0 b \sum_{l=1}^{\infty} \frac{(2l-1)!!(2l-3)!!}{(2l)!!(2l-2)!!} \left(\frac{a}{b}\right)^{2l}. \quad (16)$$

3. NUMERICAL RESULTS

To validate the developed theory, explicit expressions (1), (2), and (16) are first applied to the calculation of the self-inductance of a three-turn coil, as a function of the pitch p of the coil winding, that is the distance between the wire centers of adjacent turns ($p=a_i-a_{i-1}$). $a_1=3$ cm and $r_w=2$ mm are assumed, and the index L at which Eq. (16) is truncated is taken as a parameter. The obtained results, shown in Fig. 2, are compared with those provided by the multipole-accelerated 3-D inductance extraction program Fast-Henry [43]. As can be seen, it suffices to consider only the first five terms of Eq. (16) to reach the accuracy offered by Fast-Henry computational package. This happens regardless of the coil pitch, thus suggesting that the convergence of Eq. (16) does not depend on the turn-to-turn spacing. Furthermore, Fig. 2 also points out that the self-inductance of the coil grows as the pitch is increased, even if the mutual inductances among the turns decrease. This is because the contribution from the mutual inductances is overtaken by the sum of the self-inductances of the turns, which rises proportionally to the winding pitch. The behavior of the mutual inductance between two turns as a function of the turn-to-turn spacing is illustrated by Fig. 3, which depicts profiles of $M(a, b)$ against the ratio between the radii of the turns b/a . It is assumed $a=2$ cm, while b is changed from 2.4 cm to 6 cm. As is noticed, the mutual inductance monotonically diminishes when increasing b/a , and for $b=3a$ it is about three times smaller than the value corresponding to nearly equally sized turns. For the sake of comparison, the outcomes from Fast-Henry simulation tool are also shown, and excellent agreement is observed with the data arising from Eq. (16) with $L=7$. The improvement in accuracy that follows from increasing the length L of the partial sum in Eq. (16) may be appreciated by taking a glance at Fig. 4, which shows the relative error arising from using the proposed analytical method rather than the Fast-Henry program, to calculate the mutual inductance between two turns. The radius of the smaller turn is still $a=2$ cm, while that of the larger turn ranges from 2.4 cm to 6 cm. As highlighted by the curves plotted in Fig. 4, increasing L always leads to the diminution of the relative error, for all the considered values of the ratio b/a . In particular, when b/a is comprised between 1.6 and 3, the relative

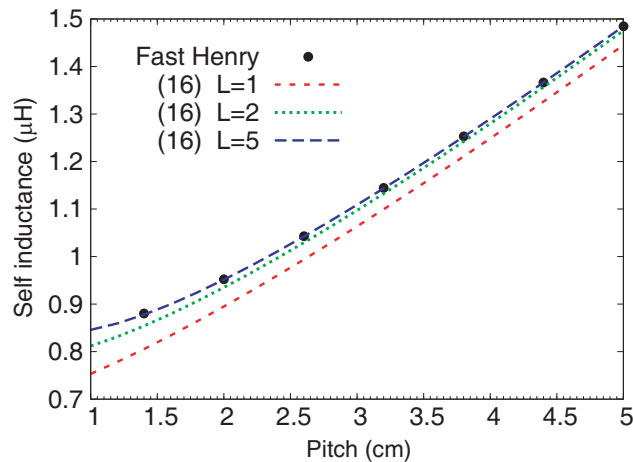


Figure 2. Self-inductance of a three-turn pancake coil, computed against the winding pitch

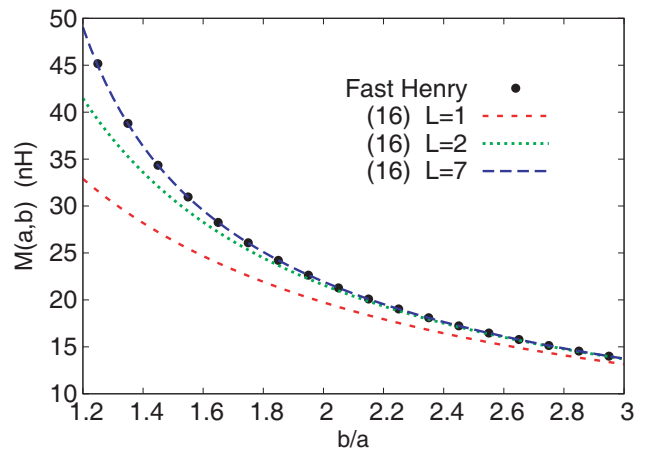


Figure 3. Mutual inductance between two turns with radii a and b , computed against the ratio b/a .

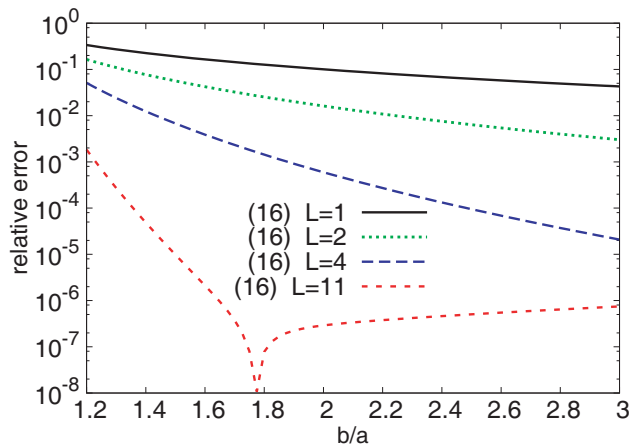


Figure 4. Relative error of the outcomes from (16), as compared to Fast-Henry data.

error associated with $L=11$ falls below 10^{-6} and, as a consequence, at least five digits of precision are ensured. One would ask whether such a level of precision is achieved at the price of a considerable time cost. This aspect is clarified by Table 1, which illustrates the average CPU time taken by Eq. (16) to compute the mutual inductance between the two turns, as well as the speed-up exhibited over Fast-Henry algorithm. As is noticed, the proposed approach allows to obtain significant time savings, since the speed-up is at least equal to 100.

Table 1. CPU time comparisons for the calculation of $M(a, b)$.

Approach	average CPU time [s]	Speed-Up
Fast-Henry	1.07	-
(16) with $L = 1$	$2.13 \cdot 10^{-6}$	$5.02 \cdot 10^5$
(16) with $L = 2$	$1.26 \cdot 10^{-5}$	$8.49 \cdot 10^4$
(16) with $L = 4$	$5.84 \cdot 10^{-4}$	$1.83 \cdot 10^3$
(16) with $L = 11$	$9.21 \cdot 10^{-3}$	$1.16 \cdot 10^2$

4. CONCLUSIONS

This work has proposed an analytical explicit expression for the self-inductance of a round pancake coil. First, the semi-infinite integral representation for the mutual inductance between two arbitrary turns of the coils is turned into a finite integral. Then, the integrand is expanded into a series of Legendre polynomials, which makes it possible to perform analytical integration and express the mutual inductance as a power series of the ratio between the radii of the turns. The self inductance is finally expressed as a sum of logarithmic functions, describing the contributions from the self-inductances of the single turns, plus the mutual-inductance terms originating from all the possible pairs of turns of the coil. Numerical simulations have been performed to show that, accuracy being equal, the proposed explicit expression allows to achieve time savings with respect to inductance extraction program Fast-Henry.

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