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On the Classical Electrodynamics in Dispersive Time-Dependent Linear Isotropic Media

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Abstract—The goal of this study is to conduct an analytical study of the properties (permittivity and permeability or refractive index) of a dispersive time-dependent linear isotropic medium interacting with electromagnetic fields. It is found that the permittivity and permeability of the time-dependent dispersive medium may either have an exponential profile in time or a sinusoidal profile in time. The permittivity and permeability can vanish or can be negative as in metamaterials. Therefore, the refractive index can vanish, so the electromagnetic wave can propagate at an infinite speed ($c \gg 3 \cdot 10^8$ m/s). It is also shown that the permittivity and the permeability can simultaneously be negative as in left-handed metamaterials (LHM). The general electric field and magnetic field solutions are derived, and the electric and magnetic flux densities are evaluated. The wave dispersion relation is also analysed. The obtained solutions can be used to validate experimental results by applying the initial and boundary conditions which are appropriate to the experimental setup.

1. INTRODUCTION

The study of phenomena involving electromagnetic field interactions in dispersive and nondispersive time-dependent material media continues to attract attention of electrodynamists who seek to improve our understanding of the interactions between the matter and radiation in nature [4, 10]. On the other hand, the study of phenomena involving electromagnetic field interactions in dispersive and nondispersive time-dependent material media continues to attract attention of engineers and applied physicists who seek to improve our understanding of the interactions between the matter and the radiation for application purposes [1, 6, 9].

With these objectives in mind, Pedrossa et al. [5] considered three profiles of time-dependent permittivity, a linear function of time, an exponential growth with time, and a sinusoidal function of time, and then obtained solutions to the wave equations in a time-dependent dynamic media in the absence of electric charges. In the two first configurations, they were able to write the wave solutions in terms of Bessel functions, while they have used numerical integration in the configuration with a sinusoidal time-dependent permittivity. Their solutions were limited to dielectric materials (zero conductivity) and non-ferromagnetic materials (constant permeability).

In [3, 4], ionosphere was modeled as an isotropic medium with weakly-random fluctuations in time considering that both the permittivity and permeability are time-dependent weakly-random variables. It was shown in [3, 4] that an isotropic medium with weakly-random fluctuations in time can behave as direct current electric dynamo, and some applications relevant to the ionospheric electrodynamics were given.

It is known that when suddenly photoionized, a gase (dispersive medium) which has a refractive index ~ 1 becomes a plasma, and its refractive index becomes smaller than 1 (~ 0), see Yablonovitch [10]. In that case, the light propagates at a very high speed compared to that in the free space ($c \gg 3{\cdot}10^8$ m/s).

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A similar behavior may take place in semi-conductors whose refractive indices can rapidly drop from 3 to a value smaller than 1 (~ 0) as a result of a sudden creation of electron-hole pairs [10].

Material media with zero refractive index, known as metamaterials, can now be designed and fabricated [1, 7, 9]. Metamaterials are designed to have optical features that ordinary materials found in the nature do not have. They can also have either a negative permittivity or negative permeability. In that case, the refractive index is complex, and the electromagnetic wave is damped as it propagates through the material medium [7]. In the special case where both the permittivity and permeability are negative at the same time, the material is called a left-handed metamaterial (LHM), Veselago [8]. In such a material medium, the refractive index can be negative. As a result, the phase velocity and group velocity become antiparallel [7].

The present work is devoted to the classical electrodynamics in linear dispersive time-dependent media, with or without charge source, which can be either conducting or nonconducting. In Section 2, the wave equations are derived, and these equations are solved in Section 3. It is shown in Section 3 that the permittivity or permeability can be either an exponential function of time or a sinusoidal function of time, and that a dispersive time-dependent medium can exhibit the characteristics of metamaterials depending on how the medium properties' initial conditions are set up. The electric and magnetic flux densities are evaluated as well, and the wave dispersion relation is also investigated. In Section 4, a summary of the results is given.

2. ELECTROMAGNETIC WAVE EQUATIONS IN THE TIME-DEPENDENT DISPERSIVE MEDIA

The equations for electromagnetic fields are derived from Maxwell's equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{3}$$

and

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

complemented by the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}, \tag{5}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field vectors; $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic flux densities; $\mathbf{J}$ is the current density; and ρ is the charge density.

In a time-dependent dispersive medium, the relationships between $\mathbf{D}$ and $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{H}$, and $\mathbf{J}$ and $\mathbf{E}$ take the forms [2],

$$\mathbf{D}(\mathbf{r}, t) = \frac{d}{dt}(\epsilon(t) * \mathbf{E}(\mathbf{r}, t)) = \epsilon(0)\mathbf{E}(\mathbf{r}, t) + \int_0^t \frac{d\epsilon(\tau)}{d\tau}\mathbf{E}(\mathbf{r}, t-\tau)d\tau, \tag{6}$$

$$\mathbf{B}(\mathbf{r}, t) = \frac{d}{dt}(\mu(t) * \mathbf{H}(\mathbf{r}, t)) = \mu(0)\mathbf{H}(\mathbf{r}, t) + \int_0^t \frac{d\mu(\tau)}{d\tau}\mathbf{H}(\mathbf{r}, t-\tau)d\tau \tag{7}$$

and

$$\mathbf{J}(\mathbf{r}, t) = \frac{d}{dt}(\sigma(t) * \mathbf{E}(\mathbf{r}, t)) = \sigma(0)\mathbf{E}(\mathbf{r}, t) + \int_0^t \frac{d\sigma(\tau)}{d\tau}\mathbf{E}(\mathbf{r}, t-\tau)d\tau, \tag{8}$$

where $*$ represents the Laplace convolution integral, ϵ the permittivity of the medium, μ the permeability of the medium, and σ the conductivity of the medium.

The convolution integrals have to reach some stationary states $t \to \infty$ in order to satisfy the principle of causality. We consider that the properties of medium ϵ, μ ,and σ evolve with time and are unknown dependent variables of time which have to be evaluated along with the electromagnetic field properties $\mathbf{E}$ and $\mathbf{H}$ in order to completely describe the electromagnetic field.

For simplification purpose, approximate expressions (without convolution integrals), for $\frac{\partial \mathbf{D}}{\partial t}$, $\frac{\partial^2 \mathbf{D}}{\partial t^2}$, $\frac{\partial \mathbf{B}}{\partial t}$ and $\frac{\partial^2 \mathbf{B}}{\partial t^2}$, are derived in Appendix A and are given by Eqs. (A5), (A6), (A7), and (A8) respectively. They are used in the derivation of the wave equations instead of their exact ones. Therefore, the solutions of the wave equations are approximate solutions. The principle of causality has been used to get rid of the convolution integrals.

Now, applying the curl operator to Eq. (1) and using Eqs. (2) and (A7) yields

$$\begin{aligned}\nabla\times\nabla\times\mathbf{E} &= -\nabla\times\frac{\partial\mathbf{B}}{\partial t}\\ &= -\nabla\times\left[\mu(0)\frac{\partial\mathbf{H}}{\partial t}(\mathbf{r},t)+\frac{d\mu(t)}{dt}\mathbf{H}(\mathbf{r},0)+\frac{d\mu(0)}{dt}\mathbf{H}(\mathbf{r},t)\right]\\ &= -\mu(0)\frac{\partial\nabla\times\mathbf{H}}{\partial t}(\mathbf{r},t)-\frac{d\mu(t)}{dt}\nabla\times\mathbf{H}(\mathbf{r},0)-\frac{d\mu(0)}{dt}\nabla\times\mathbf{H}(\mathbf{r},t)\\ &= -\mu(0)\frac{\partial}{\partial t}\left[\mathbf{J}(\mathbf{r},t)+\frac{\partial\mathbf{D}}{\partial t}(\mathbf{r},t)\right]-\frac{d\mu(t)}{dt}\left[\mathbf{J}(\mathbf{r},0)+\frac{\partial\mathbf{D}}{\partial t}(\mathbf{r},0)\right]-\frac{d\mu(0)}{dt}\left[\mathbf{J}(\mathbf{r},t)+\frac{\partial\mathbf{D}}{\partial t}(\mathbf{r},t)\right]\end{aligned}\tag{9}$$

Expanding the right hand side of Eq. (9) and using the vector formula $\nabla\times\nabla\times\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$ for some vector field $\mathbf{A}$ yields

$$\begin{aligned}\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E} &= -\mu(0)\frac{\partial^2\mathbf{D}}{\partial t^2}(\mathbf{r},t)-\frac{d\mu(0)}{dt}\frac{\partial\mathbf{D}}{\partial t}(\mathbf{r},t)-\frac{d\mu(t)}{dt}\frac{\partial\mathbf{D}}{\partial t}(\mathbf{r},0)\\ &\quad -\mu(0)\frac{\partial\mathbf{J}}{\partial t}(\mathbf{r},t)-\frac{d\mu(0)}{d\tau}\mathbf{J}(\mathbf{r},t)-\frac{d\mu(t)}{dt}\mathbf{J}(\mathbf{r},0).\end{aligned}\tag{10}$$

Using the expression for $\frac{\partial \mathbf{D}}{\partial t}$ given by (A5) and the expression for $\frac{\partial^2 \mathbf{D}}{\partial t^2}$ given by Eq. (A6) and rearranging terms gives the wave equation for the electric field,

$$\begin{aligned}\nabla(\nabla\cdot\mathbf{E})-\nabla^2\mathbf{E} &= -\mu(0)\epsilon(0)\frac{\partial^2\mathbf{E}}{\partial t^2}(\mathbf{r},t)-\left[\epsilon(0)\frac{d\mu(0)}{dt}+\mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\partial\mathbf{E}}{\partial t}(\mathbf{r},t)-\frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},t)\\ &\quad -\frac{d\mu(t)}{dt}\left[\epsilon(0)\frac{\partial\mathbf{E}}{\partial t}(\mathbf{r},0)+\frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},0)\right]-\left[\mu(0)\frac{d^2\epsilon(t)}{dt^2}+\frac{d\mu(0)}{dt}\frac{d\epsilon(t)}{dt}\right]\mathbf{E}(\mathbf{r},0)\\ &\quad -\mu(0)\frac{\partial\mathbf{J}}{\partial t}(\mathbf{r},t)-\frac{d\mu(0)}{dt}\mathbf{J}(\mathbf{r},t)-\frac{d\mu(t)}{dt}\mathbf{J}(\mathbf{r},0).\end{aligned}\tag{11}$$

Applying the curl operator to Eq. (2) and using Eq. (1) yields

$$\begin{aligned}\nabla\times\nabla\times\mathbf{H} &= \nabla\times\mathbf{J}+\nabla\times\frac{\partial\mathbf{D}}{\partial t}\\ &= \nabla\times\mathbf{J}+\nabla\times\left[\epsilon(0)\frac{\partial\mathbf{E}}{\partial t}(\mathbf{r},t)+\frac{d\epsilon(t)}{dt}\mathbf{E}(\mathbf{r},0)+\frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},t)\right]\\ &= \nabla\times\mathbf{J}+\epsilon(0)\frac{\partial\nabla\times\mathbf{E}}{\partial t}(\mathbf{r},t)+\frac{d\epsilon(t)}{dt}\nabla\times\mathbf{E}(\mathbf{r},0)+\frac{d\epsilon(0)}{dt}\nabla\times\mathbf{E}(\mathbf{r},t)\\ &= \nabla\times\mathbf{J}+\epsilon(0)\frac{\partial}{\partial t}\left[-\frac{\partial\mathbf{B}}{\partial t}(\mathbf{r},t)\right]+\frac{d\epsilon(t)}{dt}\left[-\frac{\partial\mathbf{B}}{\partial t}(\mathbf{r},0)\right]+\frac{d\epsilon(0)}{dt}\left[-\frac{\partial\mathbf{B}}{\partial t}(\mathbf{r},t)\right].\end{aligned}\tag{12}$$

Applying the vector formula $\nabla\times\nabla\times\mathbf{A}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$ as before yields

$$\nabla(\nabla\cdot\mathbf{H})-\nabla^2\mathbf{H}=-\epsilon(0)\frac{\partial^2\mathbf{B}}{\partial t^2}(\mathbf{r},t)-\frac{d\epsilon(0)}{dt}\frac{\partial\mathbf{B}}{\partial t}(\mathbf{r},t)-\frac{d\epsilon(t)}{dt}\frac{\partial\mathbf{B}}{\partial t}(\mathbf{r},0)+\nabla\times\mathbf{J}.\tag{13}$$

Using the expression for $\frac{\partial \mathbf{B}}{\partial t}$ given by Eq. (A7) and that for $\frac{\partial^2 \mathbf{B}}{\partial t^2}$ given by Eq. (A8), taking into consideration Eq. (4) and rearranging terms gives the wave equation for the magnetic field

$$\nabla^2 \mathbf{H} = \mu(0)\epsilon(0)\frac{\partial^2 \mathbf{H}}{\partial t^2}(\mathbf{r},t) + \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\partial \mathbf{H}}{\partial t}(\mathbf{r},t) + \frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}\mathbf{H}(\mathbf{r},t) + \frac{d\epsilon(t)}{dt}\left[\mu(0)\frac{\partial \mathbf{H}}{\partial t}(\mathbf{r},0) + \frac{d\mu(0)}{dt}\mathbf{H}(\mathbf{r},0)\right] + \left[\epsilon(0)\frac{d^2\mu(t)}{dt^2} + \frac{d\epsilon(0)}{dt}\frac{d\mu(t)}{dt}\right]\mathbf{H}(\mathbf{r},0) - \nabla \times \mathbf{J}. \tag{14}$$

Without charge source ($\rho = 0$) and if the medium is not conducting ($\mathbf{J} = 0$), Equations (11) and (14) become

$$\nabla^2 \mathbf{E} - \mu(0)\epsilon(0)\frac{\partial^2 \mathbf{E}}{\partial t^2}(\mathbf{r},t) - \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) - \frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},t) - \frac{d\mu(t)}{dt}\left[\epsilon(0)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},0) + \frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},0)\right] - \left[\mu(0)\frac{d^2\epsilon(t)}{dt^2} + \frac{d\mu(0)}{dt}\frac{d\epsilon(t)}{dt}\right]\mathbf{E}(\mathbf{r},0) = 0. \tag{15}$$

and

$$\nabla^2 \mathbf{H} - \mu(0)\epsilon(0)\frac{\partial^2 \mathbf{H}}{\partial t^2}(\mathbf{r},t) - \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\partial \mathbf{H}}{\partial t}(\mathbf{r},t) - \frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}\mathbf{H}(\mathbf{r},t) - \frac{d\epsilon(t)}{dt}\left[\mu(0)\frac{\partial \mathbf{H}}{\partial t}(\mathbf{r},0) + \frac{d\mu(0)}{dt}\mathbf{H}(\mathbf{r},0)\right] - \left[\epsilon(0)\frac{d^2\mu(t)}{dt^2} + \frac{d\epsilon(0)}{dt}\frac{d\mu(t)}{dt}\right]\mathbf{H}(\mathbf{r},0) = 0. \tag{16}$$

We observe that Eqs. (15) and (16) are obtained in any configuration where

$$\nabla(\nabla \cdot \mathbf{E}) = -\mu(0)\frac{\partial \mathbf{J}}{\partial t}(\mathbf{r},t) + \frac{d\mu(0)}{dt}\mathbf{J}(\mathbf{r},t) - \frac{d\mu(t)}{dt}\mathbf{J}(\mathbf{r},0) \tag{17}$$

and

$$\nabla \times \mathbf{J} = 0. \tag{18}$$

The conductivity $\sigma(t)$ may be obtained using Eqs. (17) and (8).

We note that the general solutions for Eqs. (15) and (16) can be derived for any form of the time-dependent electric permittivity $\epsilon(t)$ and any form of the time-dependent magnetic permeability $\mu(t)$. Here, we rather consider a special case where $\epsilon(t)$ and $\mu(t)$ satisfy

$$\mu(0)\frac{d^2\epsilon(t)}{dt^2} + \frac{d\mu(0)}{dt}\frac{d\epsilon(t)}{dt} = 0 \text{ and } \epsilon(0)\frac{d^2\mu(t)}{dt^2} + \frac{d\epsilon(0)}{dt}\frac{d\mu(t)}{dt} = 0, \tag{19}$$

and discuss some configurations where this case may find applications in the next section (Section 3.1). In that case, Eqs. (15) and (16) are reduced respectively to

$$\nabla^2 \mathbf{E} - \mu(0)\epsilon(0)\frac{\partial^2 \mathbf{E}}{\partial t^2}(\mathbf{r},t) - \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) - \frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},t) - \frac{d\mu(t)}{dt}\left[\epsilon(0)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},0) + \frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},0)\right] = 0. \tag{20}$$

and

$$\nabla^2 \mathbf{H} - \mu(0)\epsilon(0)\frac{\partial^2 \mathbf{H}}{\partial t^2}(\mathbf{r},t) - \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\partial \mathbf{H}}{\partial t}(\mathbf{r},t) - \frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}\mathbf{H}(\mathbf{r},t) - \frac{d\epsilon(t)}{dt}\left[\mu(0)\frac{\partial \mathbf{H}}{\partial t}(\mathbf{r},0) + \frac{d\mu(0)}{dt}\mathbf{H}(\mathbf{r},0)\right] = 0. \tag{21}$$

We observe that if ϵ and μ are constants, then Eqs. (20) and (21) become the standard wave equations.

3. ANALYTICAL SOLUTIONS IN AN ISOTROPIC DISPERSIVE TIME-DEPENDENT MEDIUM

3.1. Temporal Evolution of the Properties of the Medium

The initial or boundary conditions determine the form of $\epsilon(t)$ and $\mu(t)$. Here, we solve Eq. (19) subject to the complex initial conditions

$$\epsilon(0) = \epsilon_1 + i\epsilon_2,\ \frac{d\epsilon(0)}{dt} = \dot{\epsilon}_1 + i\dot{\epsilon}_2,\ \mu(0) = \mu_1 + i\mu_2 \text{ and } \frac{d\mu(0)}{dt} = \dot{\mu}_1 + i\dot{\mu}_2, \tag{22}$$

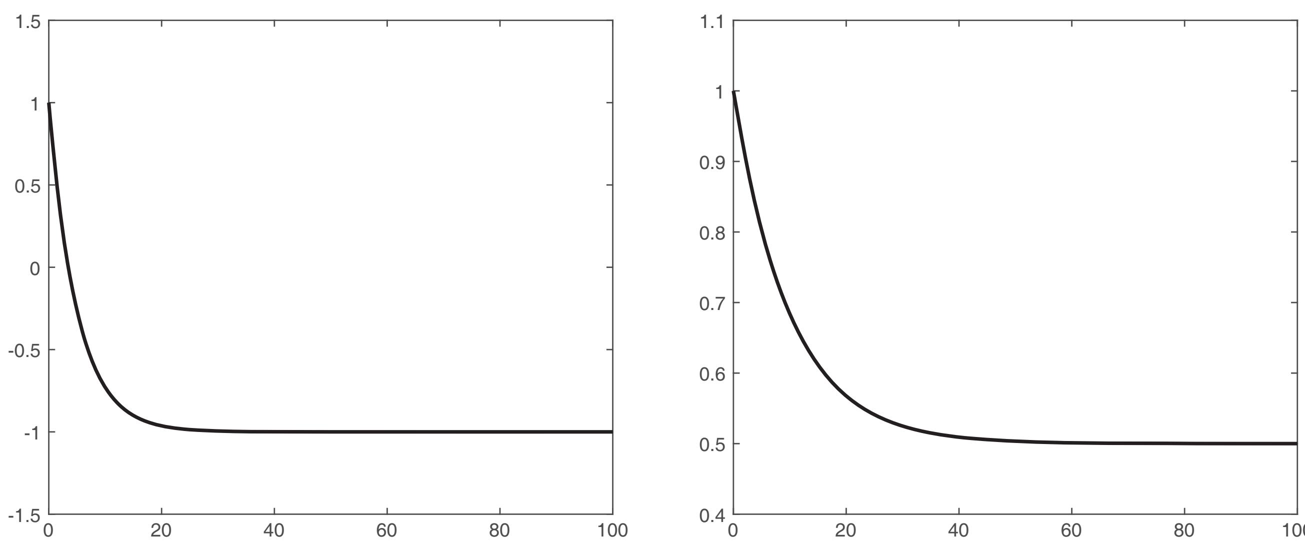
where the dots may be interpreted as time derivatives, and the value of $\dot{\epsilon}_1$ and that of $\dot{\mu}_1$ have to be positive in order for $\epsilon(t)$ and $\mu(t)$ to reach some steady states as the time becomes large ($t \to \infty$). Thus, solutions to Eq. (19) are

$$\begin{aligned}\epsilon(t) &= \epsilon(0) + \frac{d\epsilon(0)}{dt}\frac{\mu(0)}{\left[\dfrac{d\mu(0)}{dt}\right]}\left[\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right) - 1\right] \\ &= \epsilon_1 + i\epsilon_2 + (\dot{\epsilon}_1 + i\dot{\epsilon}_2)\frac{\mu_1 + i\mu_2}{\dot{\mu}_1 + i\dot{\mu}_2}\left[\exp\left(-\frac{\dot{\mu}_1\mu_1 + \mu_2\dot{\mu}_2 + i(\dot{\mu}_2\mu_1 - \dot{\mu}_1\mu_2)}{\mu_1^2 + \mu_2^2}t\right) - 1\right]\end{aligned} \tag{23}$$

and

$$\begin{aligned}\mu(t) &= \mu(0) + \frac{d\mu(0)}{dt}\frac{\epsilon(0)}{\left[\dfrac{d\epsilon(0)}{dt}\right]}\left[\exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right) - 1\right] \\ &= \mu_1 + i\mu_2 + (\dot{\mu}_1 + i\dot{\mu}_2)\frac{\epsilon_1 + i\epsilon_2}{\dot{\epsilon}_1 + i\dot{\epsilon}_2}\left[\exp\left(-\frac{\dot{\epsilon}_1\epsilon_1 + \epsilon_2\dot{\epsilon}_2 + i(\dot{\epsilon}_2\epsilon_1 - \dot{\epsilon}_1\epsilon_2)}{\epsilon_1^2 + \epsilon_2^2}t\right) - 1\right].\end{aligned} \tag{24}$$

In Figure 1, $(d\epsilon(0)/dt)/\epsilon(0)$ and $(d\mu(0)/dt)/\mu(0)$ are real and positive, while in Figures 3 and 4, $(d\epsilon(0)/dt)/\epsilon(0)$ and $(d\mu(0)/dt)/\mu(0)$ are complex and have positive real parts. The choice of initial



(a) (b)

Figure 1. (a) $\delta_\epsilon(t) = \epsilon(t)/\epsilon(0) = 1 + \alpha_\epsilon(e^{-0.2t} - 1)$ as a function of time t (in µs) and (b) $\delta_\mu(t) = \mu(t)/\mu(0) = 1 + \alpha_\mu(e^{-0.1t} - 1)$ as a function of time t (in µs). In this case, $\alpha_\epsilon = ((d\epsilon(0)/dt)/\epsilon(0))(\mu(0)/(d\mu(0)/dt)) = 2$ while $\alpha_\mu = 1/\alpha_\epsilon = 0.5$. The permittivity damping factor is $(-d\mu(0)/dt)/\mu(0) = -0.2 \cdot 10^6\,\mathrm{s}^{-1}$ while the permeability damping factor is $(-d\epsilon(0)/dt)/\epsilon(0) = -0.1 \cdot 10^6\,\mathrm{s}^{-1}$.

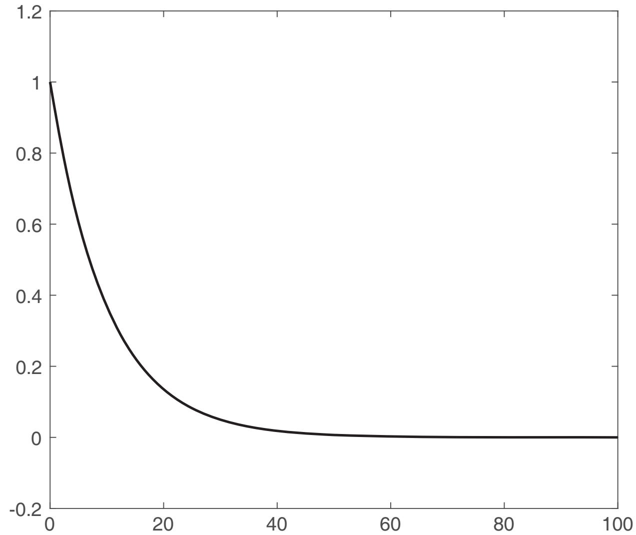


Figure 2. $\delta(t) = \epsilon(t)/\epsilon(0) = 1 + (e^{-0.1t} - 1) = \mu(t)/\mu(0)$ as a function of time t (in µs). In this case, the permittivity damping factor equals the permeability damping factor, $(-d\epsilon(0)/dt)/\epsilon(0) = (-d\mu(0)/dt)/\mu(0) = -0.1 \cdot 10^6\,\mathrm{s}^{-1}$.

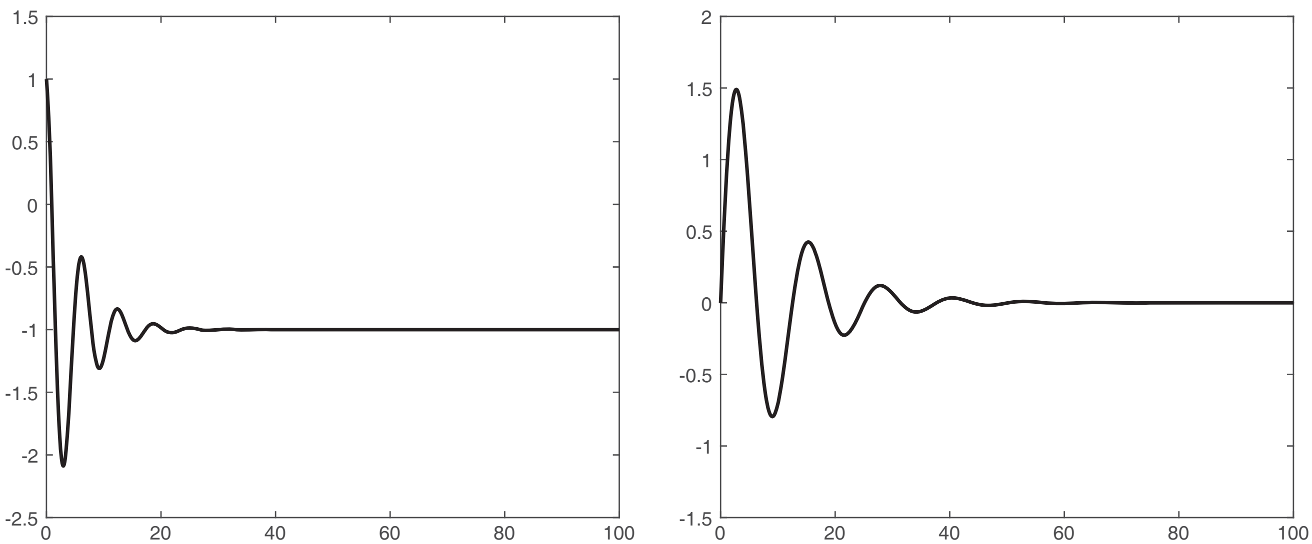


Figure 3. The permittivity and the permeability are complex. (a) The real part of $\delta_\epsilon = \epsilon(t)/\epsilon(0)$, $\mathrm{Re}\{\delta_\epsilon(t)\}$ as a function of time t (in µs), and (b) the imaginary part of $\delta_\epsilon = \epsilon(t)/\epsilon(0)$, $\mathrm{Im}\{\delta_\epsilon(t)\}$ as a function of time t (in µs). $(d\mu(0)/dt)/\mu(0) = (0.2+i)10^6\,\mathrm{s}^{-1}$ while $(d\epsilon(0)/dt)/\epsilon(0) = (0.1+0.5i)10^6\,\mathrm{s}^{-1}$.

conditions, which gives the results in Figures 1, 3 and 4, is such that the permittivity ϵ becomes negative as it evolves with time while the permeability μ remains positive. A material medium having either a negative permittivity or a negative permeability can be fabricated, and as mentioned before, it is called a metameterial [7, 9]. An application involving spontaneous photon production in a time-dependent material with a near-zero complex permittivity can be in found in [6]. The initial conditions that will make the permeability evolve to negative values while the permittivity remains positive may also be chosen. The refractive index $n(t) = (\epsilon_r(t)\mu_r(t))^{1/2}$ is thus complex. The temporal evolutions of real and imaginary parts of the refractive index, corresponding to the configuration in Figures 3 and 4, are shown in Figures 5(a) and 5(b), respectively.

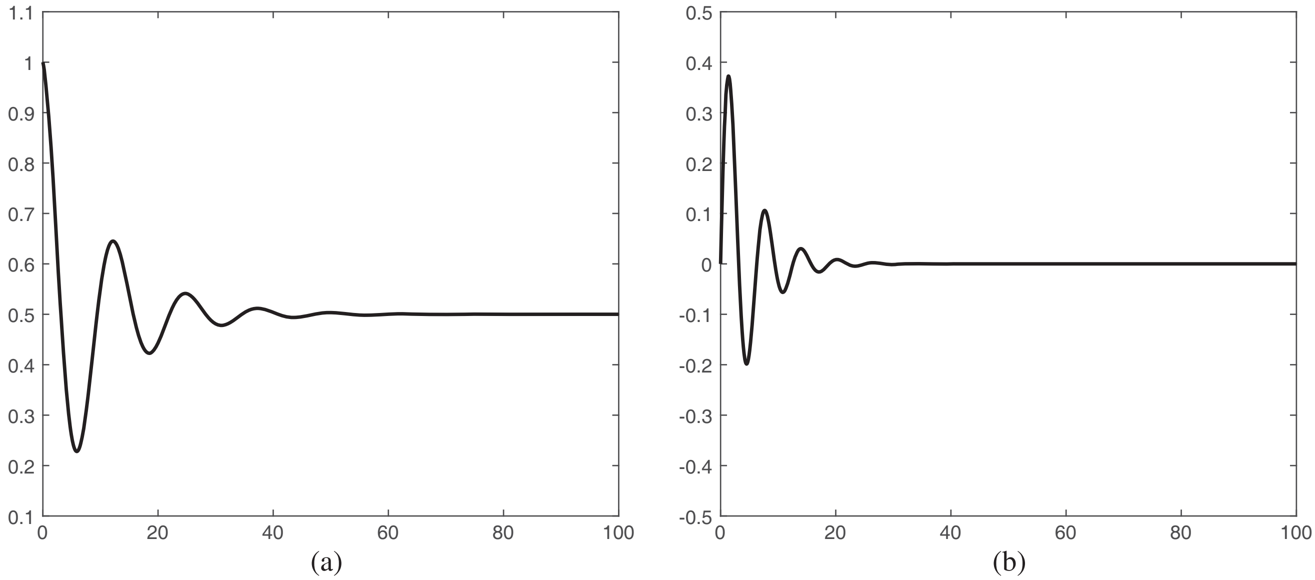


Figure 4. The permittivity and the permeability are complex. (a) The real part of $\delta_\mu = \mu(t)/\mu(0)$, $\mathrm{Re}\{\delta_\mu(t)\}$ as a function of time t (in µs), and (b) the imaginary part of $\delta_\mu = \mu(t)/\mu(0)$, $\mathrm{Im}\{\delta_\mu(t)\}$ as a function of time t (in µs). $(d\mu(0)/dt)/\mu(0) = (0.2+i)10^6\,\mathrm{s}^{-1}$ while $(d\epsilon(0)/dt)/\epsilon(0) = (0.1+0.5i)10^6\,\mathrm{s}^{-1}$.

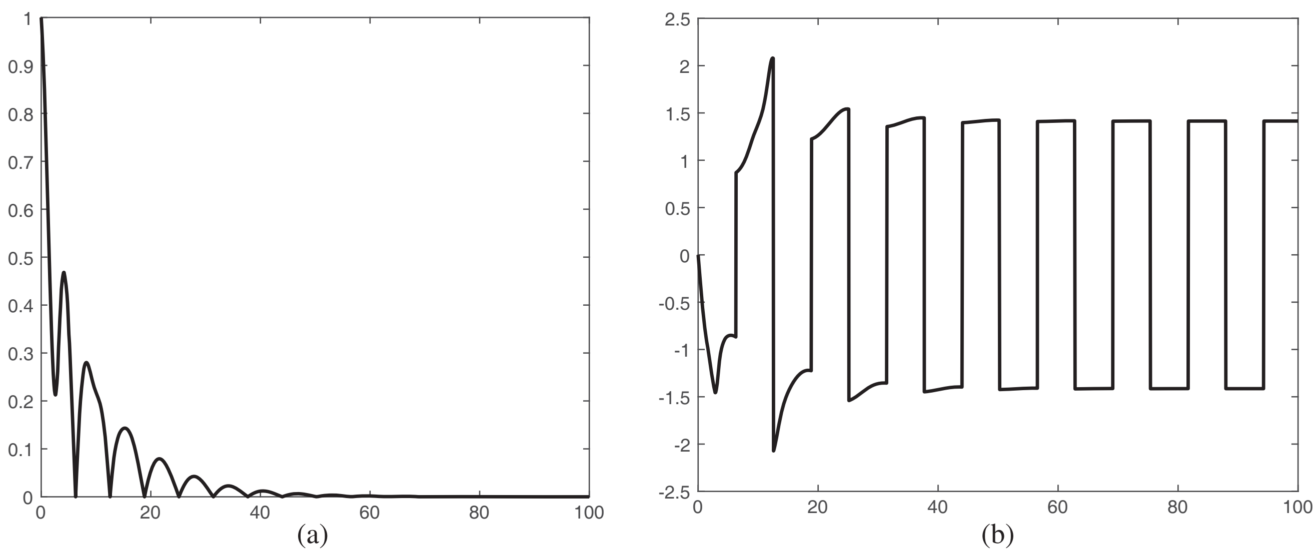


Figure 5. (a) The real part of the index of refraction $\mathrm{Re}\{n(t)\} = \mathrm{Re}\{c_0/c(t)\}$ as a function of time t (in µs), and (b) the imaginary part of the index of refraction $\mathrm{Im}\{n(t)\} = \mathrm{Im}\{c_0/c(t)\}$ as a function of time t (in µs). $(d\epsilon(0)/dt)/\epsilon(0) = (0.2+1i)10^6\,\mathrm{s}^{-1}$ while $(d\mu(0)/dt)/\mu(0) = (0.1+0.5i)10^6\,\mathrm{s}^{-1}$.

An important feature seen in Figures 3 and 4 is that although initially the permittivity and permeability are complex quantities, their corresponding imaginary parts vanish later in time. Thus, the permittivity and permeability become real quantities as in Figure 1 later in time.

Figure 2 illustrates that if $(d\epsilon(0)/dt)/\epsilon(0) = (d\mu(0)/dt)/\mu(0) > 0$, then the temporal evolution of ϵ and that of μ are identical. And both ϵ and μ tend to zero as t becomes large. As a result, the value of the refractive index tends to zero. This happens in the ionosphere when, for example, a gase is photoionized and turns into a plasma [10].

The other possibility is to have both permittivity and permeability evolving to negative values in time. This is possible if either $d\epsilon(0)/dt < 0$ while $d\mu(0)/dt > 0$ or $d\epsilon(0)/dt > 0$ while $d\mu(0)/dt < 0$. Material media which are designed to have this feature are called left-handed metamaterial (LHM) [8].

It is worth to point out that if $\epsilon(0) = \epsilon_0$ and $\mu(0) = \mu_0$ in Figures 1–4, then $\delta_\epsilon(t) = \epsilon_r(t)$ the relative permittivity while $\delta_\mu(t) = \mu_r(t)$ the relative permeability.

The speed of light $c(t) = 1/[\epsilon(t)\mu(t)]^{1/2} = c_0/n(t)$ where c_0 is the speed of light in free space, and the refractive index $n(t)$ is given by

$$n(t) = n(0)\left\{1 + \frac{\mu(0)}{\epsilon(0)}\frac{\left[\frac{d\epsilon(0)}{dt}\right]}{\left[\frac{d\mu(0)}{dt}\right]}\left[\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right) - 1\right] + \frac{\epsilon(0)}{\mu(0)}\frac{\left[\frac{d\mu(0)}{dt}\right]}{\left[\frac{d\epsilon(0)}{dt}\right]}\left[\exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right) - 1\right]\right.$$
$$\left. + \left[\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right) - 1\right]\left[\exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right) - 1\right]\right\}^{1/2}, \tag{25}$$

and $n(0)$ being the value of the index of refraction at $t = 0$.

3.2. Electric and Magnetic Field Solutions in an Isotropic Medium

From Eqs. (15) and (16), each component E of the electric field vector $\mathbf{E}$ and each component H of the magnetic field vector $\mathbf{H}$ satisfy, respectively,

$$\nabla^2 E - \mu(0)\epsilon(0)\frac{\partial^2 E}{\partial t^2}(\mathbf{r}, t) - \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\partial E}{\partial t}(\mathbf{r}, t)$$
$$-\frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}E(\mathbf{r}, t) - \frac{d\mu(t)}{dt}\left[\epsilon(0)\frac{\partial E}{\partial t}(\mathbf{r}, 0) + \frac{d\epsilon(0)}{dt}E(\mathbf{r}, 0)\right] = 0 \tag{26}$$

and

$$\nabla^2 H - \mu(0)\epsilon(0)\frac{\partial^2 H}{\partial t^2}(\mathbf{r}, t) - \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\partial H}{\partial t}(\mathbf{r}, t)$$
$$-\frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}H(\mathbf{r}, t) - \frac{d\epsilon(t)}{dt}\left[\mu(0)\frac{\partial H}{\partial t}(\mathbf{r}, 0) + \frac{d\mu(0)}{dt}H(\mathbf{r}, 0)\right] = 0. \tag{27}$$

Equations (26) and (27) are symmetric in ϵ and μ. Once Eq. (26) is solved, solutions to Eq. (27) may be obtained by interchanging ϵ and μ.

We use separation of variables to solve Eqs. (26) and (27). We write the solution to Eq. (26) as $E(\mathbf{r}, t) = R(\mathbf{r})T(t)$, where $\mathbf{r}$ is the position vector in two-dimensional space or three-dimensional space, and t represents time. Then, Eq. (26) gives

$$\frac{\nabla^2 R}{R} = \mu(0)\epsilon(0)\frac{\ddot{T}}{T} + \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\frac{\dot{T}}{T} + \frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt}$$
$$+\frac{d\mu(t)}{dt}\left[\frac{\epsilon(0)\dot{T}(0) + \frac{d\epsilon(0)}{dt}T(0)}{T}\right] = \lambda, \text{ a constant,} \tag{28}$$

where the superscript dot stands for differentiation with respect to time t. Thus, we obtain

$$\nabla^2 R - \lambda R = 0 \tag{29}$$

and

$$\mu(0)\epsilon(0)\ddot{T} + \left[\epsilon(0)\frac{d\mu(0)}{dt} + \mu(0)\frac{d\epsilon(0)}{dt}\right]\dot{T} + \left[\frac{d\mu(0)}{dt}\frac{d\epsilon(0)}{dt} - \lambda\right]T$$
$$= \frac{d\mu(0)}{dt}\left[\epsilon(0)\dot{T}(0) + \frac{d\epsilon(0)}{dt}T(0)\right]\exp\left[-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right]. \tag{30}$$

The general solution should be a series of products of the functions $R(\mathbf{r})$ and $T(t)$ involving all the possible values of λ. For simplification purposes, the two-dimensional configuration will be considered.

Equation (30) can be solved using the method of variation of parameter, while Eq. (29) can be solved using separation of variables. Thus, we obtain

$$\begin{aligned} E_{\lambda=0}(x,z,t) &\simeq R_{\lambda=0}(x,z)T_{\lambda=0}(t) \\ &= \left\{ a_1 \exp\left[-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right] + a_2 \exp\left[-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right] + \frac{d\mu(0)}{dt}\frac{\dot{T}_{\lambda=0}(0)+\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}T_{\lambda=0}(0)}{\mu(0)\vartheta_d} \right. \\ &\quad \left. \times t\exp\left[-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right]\right\} \left(a_3 e^{\eta z} + a_4 e^{-\eta z}\right)\left(a_5 e^{i\eta x} + a_6 e^{-i\eta x}\right) + \text{c.c.} \end{aligned} \tag{31}$$

and

$$\begin{aligned} E_{\lambda\neq0}(x,z,t) &\simeq R_{\lambda\neq0}(x,z)T_{\lambda\neq0}(t) \\ &= \left\{ b_1 \exp\left[-\frac{1}{2}\left(\vartheta_u - \frac{1}{\mu(0)\epsilon(0)}\sqrt{\vartheta_d^2 + 4\mu(0)\epsilon(0)\lambda}\right)t\right] \right. \\ &\quad + b_2 \exp\left[-\frac{1}{2}\left(\vartheta_u + \frac{1}{\mu(0)\epsilon(0)}\sqrt{\vartheta_d^2 + 4\mu(0)\epsilon(0)\lambda}\right)t\right] \\ &\quad \left. - \frac{1}{\lambda}\frac{d\mu(0)}{dt}\left[\epsilon(0)\dot{T}(0) + \frac{d\epsilon(0)}{dt}T(0)\right]\exp\left[-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right]\right\} \\ &\quad \left(b_3 e^{\sqrt{\eta^2+\lambda}\,z} + b_4 e^{-\sqrt{\eta^2+\lambda}\,z}\right)\left(b_5 e^{i\eta x} + b_6 e^{-i\eta x}\right) + \text{c.c.}, \end{aligned} \tag{32}$$

where $\vartheta_d = \frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt} - \frac{1}{\mu(0)}\frac{d\mu(0)}{dt}$, $\vartheta_u = \frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt} + \frac{1}{\mu(0)}\frac{d\mu(0)}{dt}$, c.c. represents the complex conjugate, and η, λ, a_i, and $b_i, i = 1, \ldots, 6$ are constants.

Using the fact that Eqs. (26) and (27) are symmetric in ϵ and μ, we interchange ϵ and μ in Eqs. (31) and (32) and obtain the magnetic field solutions

$$\begin{aligned} H_{\lambda=0}(x,z,t) &\simeq \left\{ \bar{a}_1 \exp\left[-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right] + \bar{a}_2 \exp\left[-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right] + \frac{d\epsilon(0)}{dt}\frac{\dot{T}_{\lambda=0}(0)+\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}T_{\lambda=0}(0)}{\epsilon(0)\vartheta_d} \right. \\ &\quad \left. \times t\exp\left[-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right]\right\} \left(\bar{a}_3 e^{\eta z} + \bar{a}_4 e^{-\eta z}\right)\left(\bar{a}_5 e^{i\eta x} + \bar{a}_6 e^{-i\eta x}\right) + \text{c.c.} \end{aligned} \tag{33}$$

and

$$\begin{aligned} H_{\lambda\neq0}(x,z,t) &\simeq \left\{ \bar{b}_1 \exp\left[-\frac{1}{2}\left(\vartheta_u - \frac{1}{\mu(0)\epsilon(0)}\sqrt{\vartheta_d^2 + 4\mu(0)\epsilon(0)\lambda}\right)t\right] \right. \\ &\quad + \bar{b}_2 \exp\left[-\frac{1}{2}\left(\vartheta_u + \frac{1}{\mu(0)\epsilon(0)}\sqrt{\vartheta_d^2 + 4\mu(0)\epsilon(0)\lambda}\right)t\right] \\ &\quad \left. - \frac{1}{\lambda}\frac{d\epsilon(0)}{dt}\left[\mu(0)\dot{T}(0) + \frac{d\mu(0)}{dt}T(0)\right]\exp\left[-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right]\right\} \\ &\quad \left(\bar{b}_3 e^{\sqrt{\eta^2+\lambda}\,z} + \bar{b}_4 e^{-\sqrt{\eta^2+\lambda}\,z}\right)\left(\bar{b}_5 e^{i\eta x} + \bar{b}_6 e^{-i\eta x}\right) + \text{c.c.}, \end{aligned} \tag{34}$$

where $\bar{a}_i$ and $\bar{b}_i, i = 1, \ldots, 6$ are constants.

3.3. Waves and the Group Velocity

If the permittivity $\epsilon(t)$ and permeability $\mu(t)$ are real-valued functions of time, then the eigenvalue $\lambda = 0$ gives periodic solutions in x which decay exponentially in both time t and z. If, on the other hand, the permittivity $\epsilon(t)$ and permeability $\mu(t)$ are complex-valued functions, then the eigenvalue $\lambda = 0$ gives superpositions of waves with frequency

$$\omega_1 = \frac{\dot{\mu}_2\mu_1 - \dot{\mu}_1\mu_2}{\mu_1^2 + \mu_2^2} \quad \text{and} \quad \omega_2 = \frac{\dot{\epsilon}_2\epsilon_1 - \dot{\epsilon}_1\epsilon_2}{\epsilon_1^2 + \epsilon_2^2}$$

respectively, propagating in the x direction and decaying exponentially in both time t and z. Those waves should have one of the following finite phase velocities

$$c_1 = \frac{\dot{\mu}_2\mu_1 - \dot{\mu}_1\mu_2}{(\mu_1^2 + \mu_2^2)\eta} \quad \text{and} \quad c_2 = \frac{\dot{\epsilon}_2\epsilon_1 - \dot{\epsilon}_1\epsilon_2}{(\epsilon_1^2 + \epsilon_2^2)\eta}$$

according to the problem configuration (the initial and boundary conditions). Indeed, the characteristics of such waves depend upon initial properties of the time-dependent medium under consideration. Solutions corresponding to $\lambda = 0$ may give a representation of an electric wave propagating in a conducting material medium.

The other possibility is to have a nonzero complex λ, $\lambda = \lambda_{\rm re} + i\lambda_{\rm im}$, as in [4], and write

$$\begin{aligned}\beta_{\rm re} + i\beta_{\rm im} &= \sqrt{\eta^2 + \lambda} = \sqrt{\eta^2 + \lambda_{\rm re} + i\lambda_{\rm im}} \\ &= \frac{1}{\sqrt{2}}\sqrt{\eta^2+\lambda_{\rm re}+\sqrt{(\eta^2+\lambda_{\rm re})^2+\lambda_{\rm im}^2}}+{\rm sgn}(\lambda_{\rm im})\frac{i}{\sqrt{2}}\sqrt{-(\eta^2+\lambda_{\rm re})+\sqrt{(\eta^2+\lambda_{\rm re})^2+\lambda_{\rm im}^2}}. \quad (35)\end{aligned}$$

For simplification purpose, let us choose a configuration in which both the permittivity and permeability are real functions of time as, for example, in Figure 2, and set

$$\Omega = \Omega_{\rm re} + i\Omega_{\rm im} = \sqrt{\vartheta_d^2 + 4\mu(0)\epsilon(0)(\lambda_{\rm re} + i\lambda_{\rm im})},$$

with

$$\Omega_{\rm re} = \left\{\vartheta_d^2 + 4\mu(0)\epsilon(0)\lambda_{\rm re} + \left[\left(\vartheta_d^2 + 4\mu(0)\epsilon(0)\lambda_{\rm re}\right)^2 + 16\mu^2(0)\epsilon^2(0)\lambda_{\rm im}^2\right]^{1/2}\right\}^{1/2}$$

and

$$\Omega_{\rm im} = \left\{-\vartheta_d^2 - 4\mu(0)\epsilon(0)\lambda_{\rm re} + \left[\left(\vartheta_d^2 + 4\mu(0)\epsilon(0)\lambda_{\rm re}\right)^2 + 16\mu^2(0)\epsilon^2(0)\lambda_{\rm im}^2\right]^{1/2}\right\}^{1/2}.$$

A solution corresponding to $\lambda \neq 0$ which is periodic in x should, for instance, be given by

$$\begin{aligned}E(x,z,t) \simeq &\left\{\exp\left[-\frac{1}{2}\left(\vartheta_u + \frac{\Omega_{\rm re} + i\Omega_{\rm im}}{\mu(0)\epsilon(0)}\right)t\right] - \frac{1}{\lambda}\frac{d\mu(0)}{dt}\left[\epsilon(0)\dot{T}(0) + \frac{d\epsilon(0)}{dt}T(0)\right]\exp\left[-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right]\right\} \\ &\times\left[b_3\exp(\beta_{\rm re}z)\exp(i\beta_{\rm im}z) + b_4\exp(-\beta_{\rm re}z)\exp(-i\beta_{\rm im}z)\right]\left(b_5e^{i\eta x} + b_6e^{-i\eta x}\right) + \text{c.c.}, \quad (36)\end{aligned}$$

where $\beta_{\rm re} \geq 0$ and ${\rm sgn}(\beta_{\rm im}) = {\rm sgn}(\lambda_{\rm im})$, the horizontal wavenumber is $k_x = \eta$ while the veridical wavenumber is $k_z = \beta_{\rm im}$. We note that this solution decays exponentially with time, so it will quickly vanish as the time becomes long. Therefore, we should expect that the photon rapidly vanishes prior its speed $c(t)$ reaches infinity ($c = \infty$).

If $\lambda < -\eta^2$, then $\beta = \beta_{\rm im}$ is purely imaginary. In that case, the fields are sinusoidal oscillations in t and z but have constant amplitudes (waves with constant amplitudes). This configuration is useful on a bounded domain. The case $\lambda = -\eta^2$ gives oscillations in t which they are independent of z describing waves propagating in the x direction. If λ is real on the other hand, then $\beta = \beta_{\rm re}$ is real. In that case, if $\vartheta_d^2 < -4\mu(0)\epsilon(0)\lambda_{\rm re}$, then the fields oscillate in t and decay exponentially in z, describing waves propagating in the x direction which are vertically damped.

The wave angular frequency is given by

$$\omega = \frac{\Omega_{\rm im}}{\sqrt{2}\epsilon(0)\mu(0)} = \frac{1}{\sqrt{2}\epsilon(0)\mu(0)}\left\{-\vartheta_d^2 - 4\mu(0)\epsilon(0)\lambda_{\rm re} + \left[\left(\vartheta_d^2 + 4\mu(0)\epsilon(0)\lambda_{\rm re}\right)^2 + 16\mu^2(0)\epsilon^2(0)\lambda_{\rm im}^2\right]^{1/2}\right\}^{1/2}.$$

The real and imaginary parts of Eq. (35) respectively give

$$\lambda_{\rm re} = \beta_{\rm re}^2 - \beta_{\rm im}^2 - \eta^2 = \beta_{\rm re}^2 - k_x^2 - k_z^2 \text{ and } \lambda_{\rm im} = 2\beta_{\rm re}\beta_{\rm im} = 2\beta_{\rm re}k_z.$$

Thus, we obtain the dispersion relation

$$\omega(k_x, k_z) = \frac{c(0)}{\sqrt{2}} \left\{ k_x^2 + k_z^2 - \beta_{\rm re}^2 - \frac{c^2(0)\vartheta_d^2}{4} + \left[\left(\frac{c^2(0)\vartheta_d^2}{4} + k_x^2 + k_z^2 - \beta_{\rm re}^2 \right)^2 + 4\beta_{\rm re}^2 k_z^2 \right]^{1/2} \right\}^{1/2}, \quad (37)$$

where $c(0) = 1/\sqrt{\epsilon(0)\mu(0)}$. From Eq. (37), we obtain that the vertical component of the group velocity $\partial\omega/\partial k_z$ is

$$\frac{\partial \omega}{\partial k_z} = \frac{c(0)}{\sqrt{2}} \frac{k_z \left\{ 1 + \frac{\frac{c^2(0)\vartheta_d^2}{4} + k_x^2 + k_z^2 + \beta_{\rm re}^2}{\left[\left(\frac{c^2(0)\vartheta_d^2}{4} + k_x^2 + k_z^2 - \beta_{\rm re}^2 \right)^2 + 4\beta_{\rm re}^2 k_z^2 \right]^{1/2}} \right\}}{\left\{ k_x^2 + k_z^2 - \beta_{\rm re}^2 - \frac{c^2(0)\vartheta_d^2}{4} + \left[\left(\frac{c^2(0)\vartheta_d^2}{4} + k_x^2 + k_z^2 - \beta_{\rm re}^2 \right)^2 + 4\beta_{\rm re}^2 k_z^2 \right]^{1/2} \right\}^{1/2}},$$

and has the sign of the vertical wavenumber k_z. Therefore, the waves will vertically propagate if and only if $k_z = \beta_{\rm im} > 0$. In that case, we set $b_3 = 0$ in Eq. (36). To obtain downward propagating waves, on the other hand, we set $b_4 = 0$ in Eq. (36).

3.4. Evaluation of the Electric Flux Density and the Magnetic Flux Density

In this section, we evaluate the electric flux density D and magnetic flux density B using formulas (6) and (7), respectively. We also describe a procedure to evaluate the current density J.

For $\lambda = 0$, we evaluate the electric flux density by substituting Eq. (31) in Eq. (6), and we obtain

$$\begin{aligned} D_{\lambda=0}(x,z,t) &\simeq \epsilon(0)E_{\lambda=0}(x,z,t) + \int_0^t \frac{d\epsilon(\tau)}{d\tau} E_{\lambda=0}(x,z,t-\tau)d\tau \\ &= \epsilon(0)E_{\lambda=0}(x,z,t) - \frac{d\epsilon(0)}{dt} \Bigg\{ \frac{a_1}{\vartheta_d} \left[\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t \right) - \exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t \right) \right] \\ &\quad + a_2 t \exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t \right) - \frac{d\mu(0)}{dt} \frac{\dot{T}_{\lambda=0}(0) + \frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}T_{\lambda=0}(0)}{\mu(0)\vartheta_d^2} \\ &\quad \times \left[t\exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t \right) + \frac{1}{\vartheta_d} \left[\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t \right) \right.\right. \\ &\quad \left.\left. - \exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t \right) \right] \right] \Bigg\} \left(a_3 e^{\eta z} + a_4 e^{-\eta z} \right) \left(a_5 e^{i\eta x} + a_6 e^{-i\eta x} \right) + \text{c.c.}, \end{aligned} \quad (38)$$

where, as before, $\vartheta_d = \frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt} - \frac{1}{\mu(0)}\frac{d\mu(0)}{dt}$. Substituting Eq. (33) in Eq. (7) gives the magnetic flux density,

$$\begin{aligned} B_{\lambda=0}(x,z,t) &\simeq \epsilon(0)E_{\lambda=0}(x,z,t) + \int_0^t \frac{d\epsilon(\tau)}{d\tau} E_{\lambda=0}(x,z,t-\tau)d\tau \\ &= \epsilon(0)E_{\lambda=0}(x,z,t) - \frac{d\mu(0)}{dt} \left\{ \frac{\bar{a}_1}{\vartheta_d} \left[\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t \right) - \exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t \right) \right] \right. \\ &\quad + \bar{a}_2 t \exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t \right) \end{aligned}$$

$$-\frac{d\epsilon(0)}{dt}\frac{\dot{T}_{\lambda=0}(0)+\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}T_{\lambda=0}(0)}{\epsilon(0)\vartheta_d^2}\Big[t\exp\Big(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\Big)+\frac{1}{\vartheta_d}\Big[\exp\Big(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\Big)$$
$$-\exp\Big(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\Big)\Big]\Big]\Big\}\left(\bar{a}_3e^{\eta z}+\bar{a}_4e^{-\eta z}\right)\left(\bar{a}_5e^{i\eta x}+\bar{a}_6e^{-i\eta x}\right)+\text{c.c.}. \quad (39)$$

For $\lambda \neq 0$, we substitute Eq. (32) in Eq. (6), and we obtain the electric flux density

$$\begin{aligned}D_{\lambda\neq 0}(x,z,t) &\simeq \epsilon(0)E_{\lambda\neq 0}(x,z,t)+\int_0^t\frac{d\epsilon(\tau)}{d\tau}E_{\lambda\neq 0}(x,z,t-\tau)d\tau\\ &= \epsilon(0)E_{\lambda\neq 0}(x,z,t)+\Bigg\{b_1\left[\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right)\right.\\ &\quad\left.-\exp\left(-\frac{1}{2}\left(\vartheta_u+\frac{1}{\mu(0)\epsilon(0)}\sqrt{\vartheta_d^2+4\mu(0)\epsilon(0)\lambda}\right)t\right)\right]\\ &\quad+b_2\left[\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right)-\exp\left(-\frac{1}{2}\left(\vartheta_u-\frac{1}{\mu(0)\epsilon(0)}\sqrt{\vartheta_d^2+4\mu(0)\epsilon(0)\lambda}\right)t\right)\right]\\ &\quad+\frac{1}{\lambda\vartheta_d}\frac{d\mu(0)}{dt}\left[\epsilon(0)\dot{T}(0)+\frac{d\epsilon(0)}{dt}T(0)\right]\left[\exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right)-\exp\left(-\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right)\right]\Bigg\}\\ &\quad\times\left(b_3e^{\sqrt{\eta^2+\lambda}z}+b_4e^{-\sqrt{\eta^2+\lambda}z}\right)\left(b_5e^{i\eta x}+b_6e^{-i\eta x}\right)+\text{c.c.},\end{aligned}\quad (40)$$

while we substitute Eq. (34) in Eq. (7) and obtain the magnetic flux density

$$\begin{aligned}B_{\lambda\neq 0}(x,z,t) &\simeq \mu(0)H_{\lambda\neq 0}(x,z,t)+\int_0^t\frac{d\mu(\tau)}{d\tau}H_{\lambda\neq 0}(x,z,t-\tau)d\tau\\ &= \mu(0)H_{\lambda\neq 0}(x,z,t)+\Bigg\{\bar{b}_1\left[\exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right)\right.\\ &\quad\left.-\exp\left(-\frac{1}{2}\left(\vartheta_u+\frac{1}{\mu(0)\epsilon(0)}\sqrt{\vartheta_d^2+4\mu(0)\epsilon(0)\lambda}\right)t\right)\right]\\ &\quad+\bar{b}_2\left[\exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right)-\exp\left(-\frac{1}{2}\left(\vartheta_u-\frac{1}{\mu(0)\epsilon(0)}\sqrt{\vartheta_d^2+4\mu(0)\epsilon(0)\lambda}\right)t\right)\right]\\ &\quad+\frac{1}{\lambda\vartheta_d}\frac{d\epsilon(0)}{dt}\left[\mu(0)\dot{T}(0)+\frac{d\mu(0)}{dt}T(0)\right]\left[\exp\left(-\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\right)-\exp\left(\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right)\right]\Bigg\}\\ &\quad\times\left(\bar{b}_3e^{\sqrt{\eta^2+\lambda}z}+\bar{b}_4e^{-\sqrt{\eta^2+\lambda}z}\right)\left(\bar{b}_5e^{i\eta x}+\bar{b}_6e^{-i\eta x}\right)+\text{c.c.},\end{aligned}\quad (41)$$

where, as before, $\vartheta_u=\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}+\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}$.

In a nondielectric material medium, we can solve Eq. (17) for the current density $\mathbf{J}$ using the method of variation of parameter and obtain

$$\begin{aligned}\mathbf{J}(\mathbf{r},t)=&-\exp\left(\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\right)\int_0^t\exp\left(\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}\tau\right)\nabla(\nabla\cdot\mathbf{E})(\mathbf{r},\tau)d\tau\\ &-\sigma(0)\mathbf{E}(\mathbf{r},0)\Big\{\frac{1}{\vartheta_d\mu(0)}\frac{d\mu(0)}{dt}\Big[\exp\Big(\frac{1}{\epsilon(0)}\frac{d\epsilon(0)}{dt}t\Big)-\exp\Big(\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\Big)\Big]+\exp\Big(\frac{1}{\mu(0)}\frac{d\mu(0)}{dt}t\Big)\Big\}.\end{aligned}\quad (42)$$

Once the integral in Eq. (42) is evaluated, the obtained expression has to be equated with the right hand side of Eq. (8). In that case, the part depending on the space variables x and z can readily be isolated, and the resulting equation can be solved for $\frac{d\sigma}{dt}$ by means of Laplace transform.

4. CONCLUSIONS

We found that the properties of the time-dependent dispersive medium such as the permittivity and the permeability can take negative and positive values, and can vanish as well. Therefore, the refractive index can be complex, negative and can vanish as well. This indicates that the electromagnetic wave can propagate at an infinite speed [1] in a time-dependent medium. Some applications where, for example, the refractive index vanishes can be found in [1, 10].

Beside, we have discussed the equations describing the electromagnetic field interactions in an isotropic time-dependent dispersive medium (Section 2), and we carefully worked out their solutions (Section 3). With the initial-boundary conditions suitable for the experimental setup, the later solutions can be used to validate experimental results.

APPENDIX A. IMPORTANT FORMULAS

Approximate expressions (without convolution integrals) for $\frac{\partial \mathbf{E}}{\partial t}$, $\frac{\partial^2 \mathbf{E}}{\partial t^2}$, $\frac{\partial \mathbf{B}}{\partial t}$ and $\frac{\partial^2 \mathbf{B}}{\partial t^2}$ can be obtained using the formula

$$\frac{\partial}{\partial t} f(t) * g(t) = f(0)g(t) + \int_0^t \frac{df(\tau)}{d\tau} g(t-\tau)d\tau = f(t)g(0) + \int_0^t f(\tau)\frac{d}{d\tau} g(t-\tau)d\tau,$$

and taking into consideration that the principle of causality has to be satisfied.

Now, differentiating Eq. (6) with respect to time t gives

$$\begin{aligned}\frac{\partial \mathbf{D}}{\partial t}(\mathbf{r},t) &= \epsilon(0)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) + \frac{d}{dt}\int_0^t \frac{d\epsilon(\tau)}{d\tau}\mathbf{E}(\mathbf{r},t-\tau)d\tau \\ &= \epsilon(0)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) + \frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},t) + \int_0^t \frac{d^2\epsilon(\tau)}{d\tau^2}\mathbf{E}(\mathbf{r},t-\tau)d\tau. \end{aligned} \tag{A1}$$

Next, differentiating Eq. (A1) with respect to t, we obtain

$$\begin{aligned}\frac{\partial^2 \mathbf{D}}{\partial t^2}(\mathbf{r},t) &= \epsilon(0)\frac{\partial^2 \mathbf{E}}{\partial t^2}(\mathbf{r},t) + \frac{d\epsilon(0)}{dt}\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) + \frac{d}{dt}\int_0^t \frac{d^2\epsilon(\tau)}{d\tau^2}\mathbf{E}(\mathbf{r},t-\tau)d\tau \\ &= \epsilon(0)\frac{\partial^2 \mathbf{E}}{\partial t^2}(\mathbf{r},t) + \frac{d\epsilon(0)}{dt}\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) + \frac{d^2\epsilon(t)}{dt^2}\mathbf{E}(\mathbf{r},0) + \int_0^t \frac{d^2\epsilon(\tau)}{d\tau^2}\frac{\partial \mathbf{E}}{\partial \tau}(\mathbf{r},t-\tau)d\tau. \end{aligned} \tag{A2}$$

Using the fact that the principle of causality has to be satisfied, the convolution integral on the right hand side of Eq. (A2) has to attain some stationary state $\mathbf{f}(\mathbf{r})$ for some $t > 0$. Thus, we obtain the approximation

$$\frac{\partial^2 \mathbf{D}}{\partial t^2}(\mathbf{r},t) \simeq \epsilon(0)\frac{\partial^2 \mathbf{E}}{\partial t^2}(\mathbf{r},t) + \frac{d\epsilon(0)}{dt}\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) + \frac{d^2\epsilon(t)}{dt^2}\mathbf{E}(\mathbf{r},0) + \mathbf{f}(\mathbf{r}). \tag{A3}$$

Integrating Eq. (A3) gives

$$\frac{\partial \mathbf{D}}{\partial t}(\mathbf{r},t) \simeq \epsilon(0)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) + \frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},t) + \frac{d\epsilon(t)}{dt}\mathbf{E}(\mathbf{r},0) + \mathbf{f}(\mathbf{r})t + \mathbf{g}(\mathbf{r}), \tag{A4}$$

where $\mathbf{g}(\mathbf{r})$ is some vector function.

The integrals in the right hand sides of Eqs. (6) and (A1) have to satisfy the principle of causality (attain stationary states for some $t > 0$) as well. Therefore, we have to have $\mathbf{f}(\mathbf{r}) = 0$ and $\mathbf{g}(\mathbf{r}) = 0$. Hence,

$$\frac{\partial \mathbf{D}}{\partial t}(\mathbf{r},t) \simeq \epsilon(0)\frac{\partial \mathbf{E}}{\partial t}(\mathbf{r},t) + \frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},t) + \frac{d\epsilon(t)}{dt}\mathbf{E}(\mathbf{r},0) \tag{A5}$$

and

$$\frac{\partial^2\mathbf{D}}{\partial t^2}(\mathbf{r},t) \simeq \epsilon(0)\frac{\partial^2\mathbf{E}}{\partial t^2}(\mathbf{r},t) + \frac{d\epsilon(0)}{dt}\frac{\partial\mathbf{E}}{\partial t}(\mathbf{r},t) + \frac{d^2\epsilon(t)}{dt^2}\mathbf{E}(\mathbf{r},0). \tag{A6}$$

It is important to point out that Eq. (A6) satisfies Eq. (A2) at $t = 0$, but however, Eq. (A5) does not satisfy Eq. (A1) at $t = 0$. The exact form, for $\frac{\partial\mathbf{D}}{\partial t}(\mathbf{r},t)$ at $t = 0$, has to be obtained from Eq. (A1). It is given by

$$\frac{\partial\mathbf{D}}{\partial t}(\mathbf{r},0) = \epsilon(0)\frac{\partial\mathbf{E}}{\partial t}(\mathbf{r},0) + \frac{d\epsilon(0)}{dt}\mathbf{E}(\mathbf{r},0),$$

and shall therefore be used in the derivation of the wave equations to improve the approximations (see Equation (10)).

Starting with Eq. (7) and following the same procedure, it can be shown that

$$\frac{\partial\mathbf{B}}{\partial t}(\mathbf{r},t) \simeq \mu(0)\frac{\partial\mathbf{H}}{\partial t}(\mathbf{r},t) + \frac{d\mu(0)}{dt}\mathbf{H}(\mathbf{r},t) + \frac{d\mu(t)}{dt}\mathbf{H}(\mathbf{r},0) \tag{A7}$$

and

$$\frac{\partial^2\mathbf{B}}{\partial t^2}(\mathbf{r},t) \simeq \mu(0)\frac{\partial^2\mathbf{H}}{\partial t^2}(\mathbf{r},t) + \frac{d\mu(0)}{dt}\frac{\partial\mathbf{H}}{\partial t}(\mathbf{r},t) + \frac{d^2\mu(t)}{dt^2}\mathbf{H}(\mathbf{r},0). \tag{A8}$$

The exact expression for $\frac{\partial\mathbf{B}}{\partial t}(\mathbf{r},0)$ is thus given by

$$\frac{\partial\mathbf{B}}{\partial t}(\mathbf{r},0) = \mu(0)\frac{\partial\mathbf{H}}{\partial t}(\mathbf{r},0) + \frac{d\mu(0)}{dt}\mathbf{H}(\mathbf{r},0),$$

and shall be used in the derivation of the wave equations to improve the approximations (see Equation (13)).

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