Unimodular Magnetoelectric Media

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Abstract—This article introduces a new class of electromagnetic materials: unimodular media. Unimodular media are magnetoelectric bi-isotropic media for which the determinant of the normalized four-parameter constitutive material matrix is unity. As special cases of such media are perfect electric conductor, perfect magnetic conductor, perfect electromagnetic conductor, simple skewon media, and simple isotropic media with unit refractive index. The essential parameters in the description of unimodular media (strength of impedance, degree of magnetoelectricity, angle of reciprocity) allow for illuminating visualizations of this class of materials.

1. INTRODUCTION

When an electromagnetics problem has to be solved in an environment that includes material media and boundaries, Maxwell equations need to be supplemented with the necessary constitutive relations that define the response of these materials. The effect of material on the electric and magnetic fields may sometimes be very simple: a plain isotropic, homogeneous, lossless, dielectric medium can be represented in electromagnetic equations by a single real number, the permittivity. In such special case the commonly-used term *dielectric constant* is in proper place [1].

However, in the real world, the electromagnetic responses of materials are varied. Already everyday-life natural materials, like snow, ice, soil, rocks, vegetation, can have a more complicated response. And in engineering applications, the designs purposely aim at composite structures whose macroscopic properties and material response functions surpass those of their constituent materials. Recent advances in complex media and metamaterials have exposed the extraordinary variety of reactions that a medium can display under excitation of electric and magnetic fields. The simple isotropic description has been generalized into a multitude of new directions: additional responses to be accounted for are anisotropy, magnetic activity, dissipation, gyrotropy, chirality, non-reciprocity, bi-anisotropy, non-linearity, coupling with non-electromagnetic excitations and responses, and others [2–4].

The present article presents a systematic manner to chart one subdomain of the wide space of complex materials. The focus is on *bi-isotropic* media which are special in the sense that an incident electric field creates (in addition to causing dielectric polarization) magnetic polarization in the medium, and vice versa. Still, the medium behaves isotropically, in other words, the strength of the polarization responses does not depend on the vector direction of the exciting fields [5]. The conventional way of quantifying bi-isotropic media is to assign four material parameters: two co- and two cross-polarization numbers. In the following, we will arrive at a new categorization in which the crucial point is to start from extreme materials (perfect electric conductor and perfect magnetic conductor). This leads to a taxonomy of bi-isotropic media in a manner that reveals very interesting physical characteristics and which finds instructive visualization possibilities. In particular, a new class of bi-isotropic media will be introduced: that of *unimodular media*.

To find proper tools in this approach towards unimodular materials, let us start from a general standpoint of fields in bi-anisotropic media and find a frame in which the material parameters are conveniently normalized.

2. RENORMALIZATION OF THE BI-ANISOTROPIC CONSTITUTIVE RELATIONS

The constitutive relations for bianisotropic media give the dyadic connection between the (absolute) electric and magnetic fields $\vec{E}_a, \vec{H}_a$ and the electric and magnetic flux densities $\vec{D}_a, \vec{B}_a$ [6,7]:

$$\vec{D}_a = \overline{\overline{\varepsilon}}_a \cdot \vec{E}_a + \overline{\overline{\xi}}_a \cdot \vec{H}_a \tag{1}$$

$$\vec{B}_a = \overline{\overline{\zeta}}_a \cdot \vec{E}_a + \overline{\overline{\mu}}_a \cdot \vec{H}_a \tag{2}$$

with the (absolute) permittivity and permeability dyadics $\overline{\overline{\varepsilon}}_a, \overline{\overline{\mu}}_a$ and the magnetoelectric dyadics $\overline{\overline{\xi}}_a, \overline{\overline{\zeta}}_a$: These come with dimensions As/Vm, Vs/Am, s/m, s/m, respectively.

With the following normalization which uses the free-space permittivity ε_0 and permeability μ_0,

$$\vec{E}_a = \sqrt{\varepsilon_0}\vec{E}, \quad \vec{H}_a = \sqrt{\mu_0}\vec{H}, \quad \vec{D}_a = \frac{\vec{D}}{\sqrt{\varepsilon_0}}, \quad \vec{B}_a = \frac{\vec{B}}{\sqrt{\mu_0}} \tag{3}$$

the constitutive relations look like

$$\vec{D} = \overline{\overline{\varepsilon}}_r \cdot \vec{E} + \overline{\overline{\xi}}_r \cdot \vec{H} \tag{4}$$

$$\vec{B} = \overline{\overline{\zeta}}_r \cdot \vec{E} + \overline{\overline{\mu}}_r \cdot \vec{H} \tag{5}$$

where now the four constitutive dyadics are dimensionless: the relative permittivity and permeability $\overline{\overline{\varepsilon}}_r, \overline{\overline{\mu}}_r$, and the relative magnetoelectric dyadics $\overline{\overline{\xi}}_r, \overline{\overline{\zeta}}_r$. The units for the four field/flux vectors are the same ($\sqrt{\text{VAs/m}^3}$); their dimension is square root of energy density.

Six-vector formalism [8] helps condense the constitutive relation into a matrix form:

$$\begin{pmatrix} \vec{D} \\ \vec{B} \end{pmatrix} = \begin{pmatrix} \overline{\overline{\varepsilon}}_r & \overline{\overline{\xi}}_r \\ \overline{\overline{\zeta}}_r & \overline{\overline{\mu}}_r \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \tag{6}$$

defining the material matrix $\mathcal{M}$. The magnetoeleric dyadics can be decomposed in the following manner:

$$\mathcal{M} = \begin{pmatrix} \overline{\overline{\varepsilon}}_r & \overline{\overline{\xi}}_r \\ \overline{\overline{\zeta}}_r & \overline{\overline{\mu}}_r \end{pmatrix} = \begin{pmatrix} \overline{\overline{\varepsilon}}_r & \overline{\overline{\chi}}^T - j\overline{\overline{\kappa}}^T \\ \overline{\overline{\chi}} + j\overline{\overline{\kappa}} & \overline{\overline{\mu}}_r \end{pmatrix} \tag{7}$$

where $\overline{\overline{\kappa}}$ is the chirality dyadic (responsible for reciprocal magnetoelectric effects), and $\overline{\overline{\chi}}$ is the non-reciprocal magnetoelectric dyadic [5].[†]

3. UNIMODULAR BI-ISOTROPIC MEDIA

In the following, let us focus the analysis on bi-isotropic[‡] media, in which case the four material matrix components are all multiples of the unit dyadic. Hence they can each be characterized by a single scalar, possibly complex. The material matrix in Eq. (7) reduces in this case into

$$\mathcal{M} = \begin{pmatrix} \varepsilon_r & \chi - j\kappa \\ \chi + j\kappa & \mu_r \end{pmatrix} \tag{8}$$

where the scalar κ is the chirality (Pasteur) parameter, and χ is the non-reciprocity (Tellegen) parameter.

In the present text, we look for materials for which the determinant of the material matrix $\mathcal{M}$ is unity. In terms of the scalar parameters, this means

$$\varepsilon_r\mu_r = 1 + \chi^2 + \kappa^2 \tag{9}$$

Let us give the label *unimodular (bi)-isotropic media* for media obeying Eq. (9), with a reference to matrix theory in which square matrices whose determinant is ± 1 are called unimodular [9, p. 60], [10, p. 294].

Due to this one condition, the number of free parameters for unimodular isotropic media is reduced to three.

[†] The operation $(\cdot)^T$ stands for the transpose of the dyadic, and j is the imaginary unit, refering to the time-harmonic behavior of the fields (here following the convention $\exp(j\omega t)$).

[‡] Bi-isotropic materials are non-anisotropic; in other words their response is independent on the vector direction of the exciting fields. However, they may display magnetoelectric coupling [5].

3.1. Non-Magnetoelectric Media

The special case for which the magnetoelectric coefficients vanish ($\chi = 0, \kappa = 0$) can be called a simple isotropic medium. Simple isotropic media (SIM) that also obey the condition (9) satisfy

$$\varepsilon_r = \frac{1}{\mu_r} \tag{10}$$

In other words, their refractive index which is also the square root of the determinant of the $\mathcal{M}$ matrix).[§] is unity, giving a reason to call non-magnetoelectric unimodular media as *unirefractive media*. Obviously there is only one free parameter remaining for the class of simple isotropic unirefractive materials: the ratio between the permittivity and permeability.

3.2. Non-Chiral Magnetoelectric Media

Allowing another dimension into the constitutive materials, let us consider non-chiral bi-isotropic media with condition (9). Then $\kappa = 0$ and the magnetoelectric coupling is measured by the Tellegen parameter χ. Instead of arbitrary permittivity, permeability, and Tellegen parameters, we now only have two degrees of freedom. This state can be conveniently written in the following form [12]

$$\mathcal{M} = \begin{pmatrix} \varepsilon_r & \chi \\ \chi & \mu_r \end{pmatrix} = \frac{1}{\cos\vartheta} \begin{pmatrix} 1/\eta & \sin\vartheta \\ \sin\vartheta & \eta \end{pmatrix} \tag{11}$$

Equation (11) shows directly that the determinant of $\mathcal{M}$ is unity.

The two new material parameters (η, ϑ) find clear physical interpretations. The parameter $\eta = \sqrt{\mu_r/\varepsilon_r}$ can be considered as the (relative) wave impedance of the medium. The other parameter $\vartheta = \arctan\chi$ measures the strength of the magnetoelectric coupling. For $\vartheta = 0$ the medium distills down to the simple isotropic medium. On the other hand, when $|\vartheta|$ increases and approaches the value $\pi/2$, all four primary material parameters approach infinity and the matrix description loses meaning.

Indeed, these limits $\vartheta \to \pm\pi/2$ lead to the PEMC medium. PEMC (perfect electromagnetic conductor) medium, introduced in 2005 [13], is defined by one of the most basic conditions in electromagnetics. It obeys the conditions

$$\vec{D} = M\vec{B}, \quad \vec{H} = -M\vec{E} \tag{12}$$

where M is (here unitless) PEMC parameter, having an admittance character. It generalizes the well-known perfect electric conductor $1/M = 0$ and perfect magnetic conductor $M = 0$ materials. In this limit $|\vartheta| = \pi/2$, the medium of Eq. (11) is a PEMC medium with $M = \mathrm{sgn}(\vartheta)/\eta$.

The PEMC medium is hence an extreme form of bi-isotropic Tellegen material. In theoretical physics, PEMC is also known as axion medium [14].

The equality of the PEMC descriptions between Eqs.— (11) and (12) is not immediately obvious. Nevertheless, writing the material matrix in form [13]

$$\begin{pmatrix} \vec{D} \\ \vec{B} \end{pmatrix} = \mathcal{M} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}, \quad \mathcal{M} = \lim_{q \to \infty} q \begin{pmatrix} M & 1 \\ 1 & 1/M \end{pmatrix} \tag{13}$$

the conditions (12) are seen to follow from Eq. (11).[||]

The two-parameter (η, ϑ) unimodular medium can be mapped into a plane as in Figure 1. There the vertical axis leads from PEC material ($\eta = 0, \log(\eta) = -\infty$) through free space ($\eta = 1, \log(\eta) = 0$) to PMC ($1/\eta = 0, \log(\eta) = +\infty$). The horizontal direction expands the medium into the magnetoelectric domain, and the limits for $|\vartheta| = \pi/2$ correspond to PEMC media, with positive PEMC admittance value M for $\vartheta = +\pi/2$ and negative M for $\vartheta = -\pi/2$. In [15], the medium spanned by the planes in Figure 1 is called *Minkowskian isotropic medium* and in [16, 17] as *Gibbsian isotropic medium*.

[§] This comes close to the concept of *isorefractivity* which was coined by P.L.E. Uslenghi for boundaries over which the square of the refractive index $\varepsilon_r\mu_r$ does not change [11].

[||] The matrix $\mathcal{M}$ can also be written in the form $q \begin{pmatrix} M(1 + q^{-2}) & 1 \\ 1 & 1/M \end{pmatrix}$ which preserves the unit determinant [15, 17]. This leads to the same result in the limit $q \to \infty$ as Eq. (13).
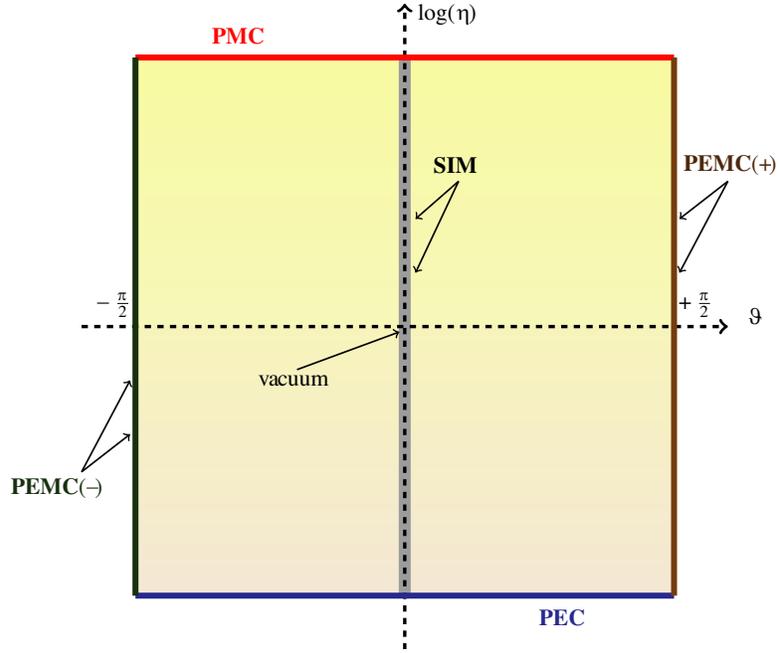
Figure 1. The unimodular Tellegen plane spanned by ϑ and η. The simple isotropic medium (SIM) goes along the line $\vartheta = 0$ while PEC and PMC occupy the lower and upper horizontal lines for "zero-impedance" and "very-large-impedance" limits, respectively.
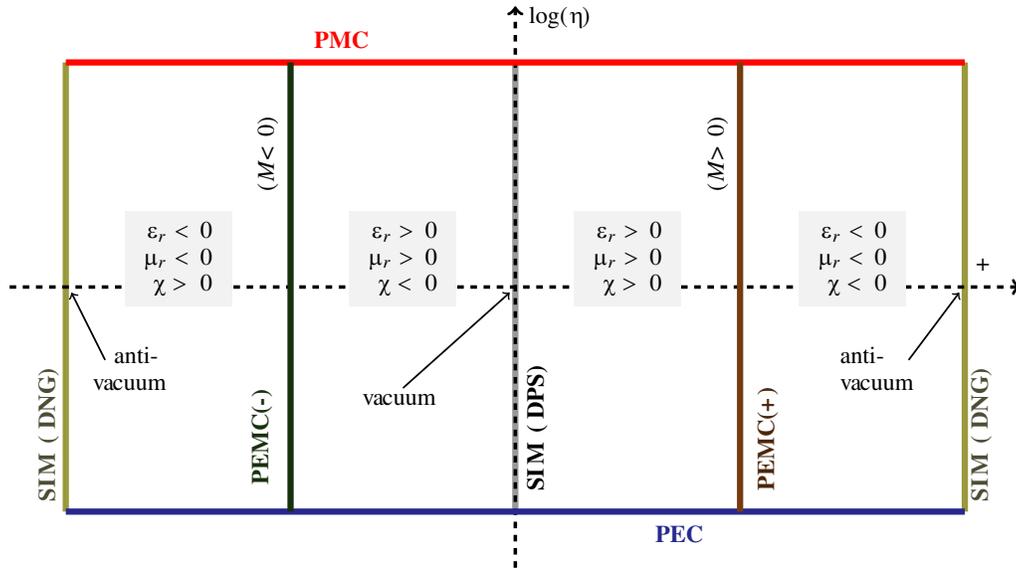
Figure 2. The expanded unimodular Tellegen plane spanned by ϑ and η. Note the 2π-periodicity on the ϑ axis.

Figure 2 extends the Tellegen plane by allowing all possible values for ϑ. When $|\vartheta| > \pi/2$, the permittivity and permeability parameters become negative (but η still remains positive). For values $\vartheta = \pm\pi$, the medium is again simple-isotropic and the Tellegen parameter χ vanishes. However, due to the negative character of ε_r and μ_r, the medium is "double-negative" (DNG) while along the SIM line through vacuum, the parameters are positive (DPS). Due to periodicity in ϑ, the DNG lines on the right and left correspond to the same medium. The DNG lines pass through the "anti-vacuum" point where $\varepsilon_r = -1$, $\mu_r = -1$, and $\log(\eta) = 0$.

3.3. Reciprocal Chiral Media

Another way to generalize the unirefractive simple isotropic medium into magnetoelectric domain is to keep it reciprocal but to allow chirality; in other words the material is a special case of Pasteur medium. Then the material matrix can be written in form

$$\mathcal{M} = \begin{pmatrix} \varepsilon_r & -\mathrm{j}\kappa \\ +\mathrm{j}\kappa & \mu_r \end{pmatrix} = \frac{1}{\cos\vartheta} \begin{pmatrix} 1/\eta & -\mathrm{j}\sin\vartheta \\ +\mathrm{j}\sin\vartheta & \eta \end{pmatrix} \tag{14}$$

Due to the property $\det\mathcal{M} = 1$, this medium is also unimodular.

Again, here the limit $\vartheta = 0$ leads to the simple isotropic medium with unit refractive index $(\varepsilon_r = 1/\mu_r)$. On the other hand, the limit $\vartheta = \pi/2$ corresponds to the so-called *simple skewon* medium [18]:

$$\vec{D} = -\mathrm{j}N\vec{B}, \quad \vec{H} = -\mathrm{j}N\vec{E} \tag{15}$$

where now N is the skewon parameter. Using four-dimensional analysis [14], the 36 constitutive parameters of a general bi-anisotropic medium can be naturally decomposed into three classes [14]: principal part (with 20 parameters), skewon part (15 parameters), and the axion part (1 parameter). While the single axion parameter corresponds to the PEMC admittance M, the skewon medium has many more degrees of freedom. The simple-skewon medium has only one degree of freedom (N), analogously to the PEMC medium.

The two parameters η and ϑ in Eq. (14) span the space of reciprocal unimodular bi-isotropic materials, similar to the non-reciprocal case illustrated in Figure 2.

4. THREE-PARAMETER UNIMODULAR MEDIUM

The unimodularity requirement subtracted one degree of freedom from the bi-isotropic medium description. Hence the most general unimodular bi-isotropic medium has three parameters. To visualize this manifold, volumetric space is needed.

4.1. Angle of Non-Reciprocity

To represent the three-parameter unimodular medium that obeys Eq. (9), a natural way is to follow the representations in Eqs. (11) and (14) and generalize these as

$$\mathcal{M} = \begin{pmatrix} \varepsilon_r & \chi - \mathrm{j}\kappa \\ \chi + \mathrm{j}\kappa & \mu_r \end{pmatrix} = \frac{1}{\cos\vartheta} \begin{pmatrix} 1/\eta & \mathrm{e}^{-\mathrm{j}\psi}\sin\vartheta \\ \mathrm{e}^{\mathrm{j}\psi}\sin\vartheta & \eta \end{pmatrix} \tag{16}$$

where the additional parameter ψ measures the degree of reciprocity: for $\psi = 0$, the magnetoelectric effect is purely non-reciprocal (Tellegen medium), and for $\psi = \pi/2$ it is reciprocal (Pasteur medium).

As ϑ increases from 0 to $\pi/2$, the medium transforms from simple isotropic medium to a simple-skewon–axion (SSA) medium with

$$\vec{D} = \frac{1}{\eta}\left(\cos\psi - \mathrm{j}\sin\psi\right)\vec{B}, \quad \vec{H} = \frac{1}{\eta}\left(-\cos\psi - \mathrm{j}\sin\psi\right)\vec{E} \tag{17}$$

Figure 3 maps general unimodular bi-isotropic media into the volume of a circular cylinder. The simple isotropic (non-magnetoelectric) medium is mapped on the symmetry axis ranging from PEC $(\eta = 0)$ to PMC $(1/\eta = 0)$. The radial coordinate is ϑ, and the side of the cylinder at $\vartheta = \pi/2$ leads to the SSA medium. There the special cases of PEMC (axion) medium $(\psi = 0, \pi)$ and simple skewon medium $\psi = \pi/2, 3\pi/2$ are mapped as the vertical lines ranging from PEC to PMC.

4.2. DH Versus DB Representation

A concise form to describe electromagnetic quantities calls for four-dimensional description [19] in which the material relations occur between quantities that combine $\vec{E}$ and $\vec{B}$ into one two-form and $\vec{D}$ and $\vec{H}$ into another. The constitutive relations between different notations are connected [20] but not always in a trivial manner, as the relation in Eq. (13) shows.
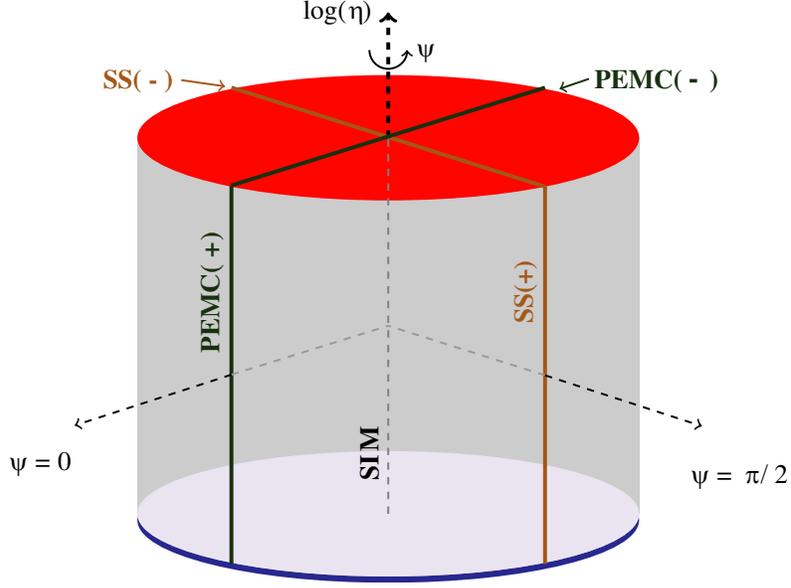
Figure 3. The three-parameter unimodular bi-isotropic medium illustrated within a circular cylinder (parameters η, ϑ, ψ). The radial axis is ϑ, and the side of the cylinder corresponds to $\vartheta = \pi/2$.

The constitutive relation for unimodular bi-isotropic media with matrix $\mathcal{M}$ in Eq. (16) changes its form in this transformation:

$$\begin{pmatrix} \vec{D} \\ \vec{H} \end{pmatrix} = \frac{1}{\eta} \begin{pmatrix} \cos\vartheta & +e^{-j\psi}\sin\vartheta \\ -e^{+j\psi}\sin\vartheta & \cos\vartheta \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} = \frac{1}{\eta}\mathcal{H} \begin{pmatrix} \vec{E} \\ \vec{B} \end{pmatrix} \tag{18}$$

Here the full matrix relation between $\vec{D}, \vec{H}$ and $\vec{E}, \vec{B}$ is no longer unimodular. However, the matrix $\mathcal{H}$ in Eq. (18) is unimodular, and it has also other interesting properties.

For real-valued ϑ and ψ, the matrix $\mathcal{H}$ is *unitary*; in other words its inverse is equal to its conjugate transpose. As a special case, for non-chiral materials ($\psi = 0$), $\mathcal{H}$ is a rotation matrix. A fundamental rotation is $\vartheta = \pi/2$ leading to the PEMC conditions (12) which "rotate" $\vec{B}$ to $\vec{D}$ and $\vec{E}$ to $-\vec{H}$.

5. PROPERTIES OF UNIMODULAR MEDIA

5.1. Refractive Indices

The refractive index for a plane wave propagating in bi-isotropic unimodular media can be computed from the constitutive parameters as [5]

$$n_{\pm} = \sqrt{\varepsilon_r \mu_r - \chi^2} \pm \kappa \tag{19}$$

For non-chiral materials ($\kappa = 0$), the refractive index is unity, $n_{\pm} = 1$, and they are called unirefractive. Chiral media are birefringent; in other words the two eigenwaves have different propagation factors ($n_+ \neq n_-$). Still, the condition $n_+ n_- = 1$ holds (cf. Eq. (9)).

Using the parameters ϑ, ψ, the refractive indices for unimodular media can be written in the form

$$n_{\pm} = \frac{1}{\cos\vartheta} \left(\sqrt{1 - \cos^2\psi \sin^2\vartheta} \pm \sin\psi \sin\vartheta \right) \tag{20}$$

An even more concise form comes, by using the inverse hyperbolic function:

$$n_{\pm} = e^{\pm\beta}, \quad \text{where} \quad \beta = \operatorname{arsinh}(\sin\psi \tan\vartheta) \tag{21}$$

Figure 4 shows the variation of the two refractive indices as functions of ϑ and ψ. The birefringence (which vanishes for $\kappa = 0$, $\psi = 0$) is strongest for reciprocal media ($\chi = 0$, $\psi = \pm\pi/2$) and it increases without limit when the medium approaches the SSA condition $\vartheta = \pi/2$. It is also worth noting that $n_{\pm}$ are independent of the parameter η.
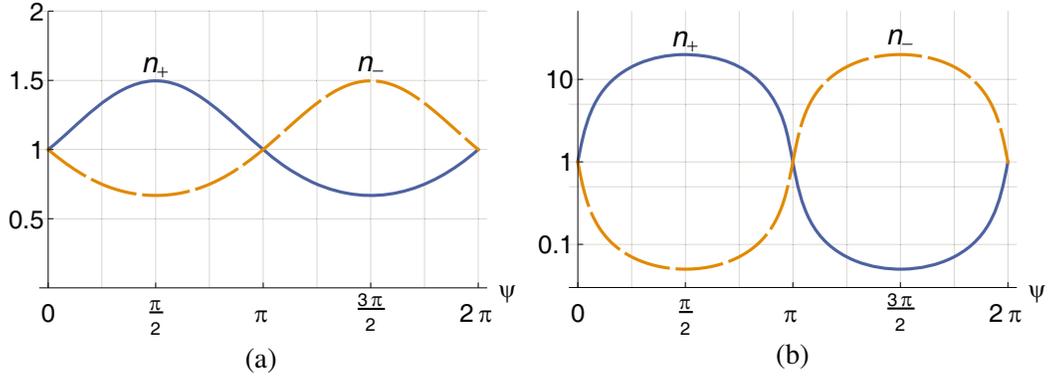
Figure 4. Refractive indices $n_\pm$ for unimodular bi-isotropic media as function of ψ. (a) $\vartheta = \pi/8$, (b) $\vartheta = \pi/2 - 0.1$.

5.2. Duality Transformation

The duality transformation is a very important operation in electromagnetics, and its usefulness arises from the symmetry between electric and magnetic quantities in Maxwell equations [6]. The duality transformation $\mathcal{D}$ between the pair of electric and magnetic fields that leaves free space invariant is the following:

$$\begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}_d = \mathcal{D} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \begin{pmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} \tag{22}$$

A natural question arises: how is a unimodular bi-isotropic medium affected by the duality transformation? Since the pair $(\vec{D}, \vec{B})$ also [6] transforms in duality with the same matrix $\mathcal{D}$, we can write the transformed constitutive relation as

$$\begin{pmatrix} \vec{D} \\ \vec{B} \end{pmatrix}_d = \mathcal{D} \begin{pmatrix} \vec{D} \\ \vec{B} \end{pmatrix} = \mathcal{D}\mathcal{M} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \underbrace{\mathcal{D}\mathcal{M}\mathcal{D}^{-1}}_{\mathcal{M}_d} \mathcal{D} \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix} = \mathcal{M}_d \begin{pmatrix} \vec{E} \\ \vec{H} \end{pmatrix}_d \tag{23}$$

The determinant of the transformed unimodular medium is

$$\det(\mathcal{M}_d) = \det(\mathcal{D}) \cdot \det(\mathcal{M}) \cdot \det(\mathcal{D}^{-1}) = \det(\mathcal{M}) = 1 \tag{24}$$

because the determinant of the duality matrix $\mathcal{D}$ is unity. This leads to the conclusion that a unimodular bi-isotropic medium remains unimodular in the duality transformation.

6. CONCLUSION

The new class of materials defined and introduced in the present article, unimodular media, forms a fundamental subclass of bi-isotropic magnetoelectric materials. The definition of unimodular media — the requirement of unit determinant of the normalized material matrix — reduced the number of degrees of freedom of fully bi-isotropic media from four to three. The approach in the article showed that the essential qualities of unimodular media are condensed in the following three material parameters: the strength of impedance η, the degree of magnetoelectricity ϑ, and the angle of reciprocity ψ.

The limiting cases of these parameters lead to fundamental electromagnetic material categories: $\eta = 0$ corresponds to perfect magnetic conductor (PMC), $1/\eta = 0$ to perfect electric conductor (PEC), $\vartheta = \pm\pi/2$, $\chi = 0$ to perfect electromagnetic conductor (PEMC) medium, $\vartheta = \pm\pi/2$, $\chi = \pm\pi/2$ to simple-skewon medium, and $\vartheta = 0$ leaves us with the simple isotropic medium with unit refractive index.

The various visualizations of unimodular materials show regions in two- or three-dimensional space in which the medium is located. The center of these regions correspond to free-space, and their boundaries reach to the limiting extreme materials, like PEC, PMC, PEMC, and simple-skewon media.

Admittedly, the dimension of the space of electromagnetic materials is far higher than three. Possibilities to generalize the concept of unimodular media is straightforward by the definition (requirement that the determinant of the material matrix remain unity), but finding the essential physical quantities that span the emerging new dimensions is not. Attempts towards this direction are to find unimodular classes of bi-anisotropic materials and tri-isotropic materials [21].

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