Progress In Electromagnetics Research Letters, Vol. 81, 65–70, 2019

An Exact Expression for the Mutual Impedance between Coaxial Circular Loops on a Homogeneous Ground

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Abstract—This paper presents an exact expression for the mutual impedance of two coaxial loops located on the surface of a conductive ground. The semi-infinite complete integral representation for the impedance is first converted into a finite integral. Then the spherical Hankel function contained in the integrand is expanded according to Gegenbauer addition theorem. This makes it possible to perform analytical integration and express the mutual impedance as a sum of products of spherical Bessel functions. Since no assumptions are made in the mathematical derivation, the obtained formula is valid in quasi-static as well as non-quasi-static frequency ranges. Numerical examples show how, accuracy being equal, the proposed expression is less computationally expensive than standard Gauss-Kronrod numerical integration technique.

1. INTRODUCTION

It is well known that transmitter-receiver coil systems may be used to detect the presence of objects buried below the top surface of a terrestrial area [1–27]. At first, the voltage induced by the emitter in the receiver is measured at a discrete set of frequencies. Next, the presence of inhomogeneities is revealed by the mismatch between the recorded experimental data and the theoretical response curves associated with standard homogeneous earth models [4, 17, 21]. Despite the need to accurately calculate such theoretical response curves, at present only for special cases can we derive explicit expressions for the mutual impedance between two loops lying on a homogeneous soil. In fact, most published analytical formulations describing the mutual impedance either are subject to the assumption that the loops are electrically as well as physically small [2, 6, 7, 18], or treat the fields radiated by the emitter only, without dealing with the effect induced on the secondary loop [15, 16, 19, 24, 26]. Moreover, the validity of many contributions is limited to the quasi-static frequency range [6, 11, 12, 14], where the effects of the displacement currents in both the air and the ground are negligible. Thus, when the computation of the mutual impedance is required in a wide frequency range, the only possible solution is to resort to numerical integration of the field integrals, but this approach has the disadvantage of being time demanding, especially when both the source and the receiver lie on the conductive medium [6, 26].

The aim of the present paper is to derive an exact explicit expression for the mutual impedance of two coaxial loops placed on the surface of a conductive soil. The proposed formula is not subject to restrictions on frequency and the sizes of the loops, and, as fully analytical, is less computationally expensive than standard numerical integration techniques. The solution is obtained through an analytical procedure, which leads to convert the semi-infinite complete integral representation for the impedance into a finite integral. Next, application of the Gegenbauer addition theorem makes it possible to move the spherical Bessel functions contained in the integrand outside the integral sign, and, as a consequence, to express the impedance as a sum of simpler integrals whose analytical evaluation is

Received 9 November 2018, Accepted 3 January 2019, Scheduled 9 January 2019
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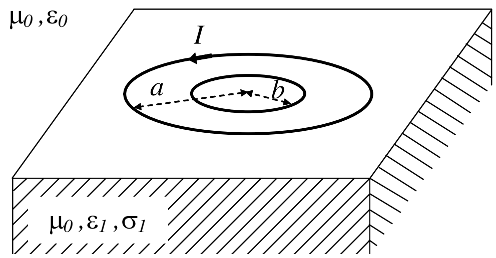

Figure 1. A couple of concentric loops on an earth structure.

straightforward. As a result, the mutual impedance is expressed as a linear combination of products of two spherical Bessel functions, with algebraic coefficients. Numerical simulations are performed to show that the proposed approach offers advantages in terms of time costs with respect to Gauss-Kronrod quadrature formula, while maintaining the same level of accuracy.

2. THEORY

The geometric configuration of the problem is shown in Fig. 1. Two concentric circular loop antennas, with radii a and b ($a>b$), are placed on the top surface of a homogeneous medium. Our scope is to determine the mutual impedance between the two loops, by evaluating its well-known integral representation, namely [6]

$$Z = -2\pi j\omega\mu_0 ab \int_0^\infty \frac{1}{u_0 + u_1} J_1(\lambda a)\, J_1(\lambda b)\, \lambda d\lambda, \tag{1}$$

where $J_\nu(\cdot)$ is the νth-order Bessel function, and with

$$u_n = \sqrt{\lambda^2 - k_n^2}, \qquad k_n^2 = \omega^2\mu_n\epsilon_n - j\omega\mu_n\sigma_n. \tag{2}$$

To this end, it is first convenient to apply the identity

$$\frac{1}{u_0{+}u_1} = \frac{u_0{-}u_1}{k_1^2{-}k_0^2} = \frac{1}{k_1^2{-}k_0^2}\left(\frac{\lambda^2 - k_0^2}{u_0} - \frac{\lambda^2 - k_1^2}{u_1}\right) \tag{3}$$

and rewrite Equation (1) as

$$Z = \frac{2\pi j\omega\mu_0 ab}{k_1^2{-}k_0^2}\left[\int_0^\infty \frac{\lambda^2 - k_n^2}{u_n} J_1(\lambda a)\, J_1(\lambda b)\, \lambda d\lambda\right]_{n=0}^{n=1}, \tag{4}$$

where $[A_n]_{n=0}^{n=1}$ indicates the quantity $A_1{-}A_0$. Next, use of the Bessel differential equation [28]

$$\lambda^2 J_1(\lambda a) = \left(-\frac{\partial^2}{\partial a^2} - \frac{1}{a}\frac{\partial}{\partial a} + \frac{1}{a^2}\right) J_1(\lambda a), \tag{5}$$

makes it possible to obtain the expression

$$Z = -\frac{2\pi j\omega\mu_0 b}{k_1^2{-}k_0^2}\left[k_n L_n \int_0^\infty \frac{1}{u_n} J_1(\lambda a)\, J_1(\lambda b)\, \lambda d\lambda\right]_{n=0}^{n=1}, \tag{6}$$

where L_n is the differential operator

$$L_n = \frac{a}{k_n}\left(\frac{\partial^2}{\partial a^2} + \frac{1}{a}\frac{\partial}{\partial a} - \frac{1}{a^2} + k_n^2\right). \tag{7}$$

The integral in Equation (6) may now be turned into a double integral by applying the identity [28, 11.41.17]

$$J_1(\lambda a)\, J_1(\lambda b) = \frac{1}{\pi}\int_0^\pi J_0\left(\lambda c\right) \cos\phi\, d\phi, \tag{8}$$

with

$$c = \sqrt{a^2 + b^2 - 2ab\cos\phi}. \tag{9}$$

It yields

$$Z = -\frac{2j\omega\mu_0 b}{k_1^2-k_0^2}\left[k_n L_n \int_0^{\pi} \cos\phi\, d\phi \int_0^{\infty} \frac{1}{u_n} J_0\left(\lambda c\right)\lambda d\lambda\right]_{n=0}^{n=1}, \tag{10}$$

where the inner semi-infinite integral is the well-known tabulated Sommerfeld Integral [29, 30]

$$\int_0^{\infty} \frac{1}{u_n} J_0(\lambda c)\lambda d\lambda = -jk_n h_0^{(2)}\left(k_n c\right), \tag{11}$$

$h_0^{(2)}(\xi)$ being the zeroth-order spherical Hankel function of the second kind. After substituting Equation (11) into Equation (10), so as to obtain

$$Z = -\frac{2\omega\mu_0 b}{k_1^2-k_0^2}\left[k_n^2 L_n \int_0^{\pi} h_0^{(2)}\left(k_n c\right)\cos\phi\, d\phi\right]_{n=0}^{n=1}, \tag{12}$$

we apply the addition formula [28, 11.41.4]

$$h_0^{(2)}\left(k_n c\right) = \sum_{m=0}^{\infty}\left(2m+1\right) h_m^{(2)}\left(k_n a\right) j_m\left(k_n b\right) C_m^{1/2}\left(\cos\phi\right), \tag{13}$$

where $j_m(\xi)$ is the mth-order spherical Bessel function, and $C_m^{1/2}(\cos\phi)$ denotes the coefficient of α^m in the expansion of $\left(1 - 2\alpha\cos\phi + \alpha^2\right)^{-1/2}$ in ascending powers of α. As a result, Z assumes the form

$$Z = -\frac{2\omega\mu_0 b}{k_1^2-k_0^2}\sum_{m=0}^{\infty}\left(2m+1\right)\left\{k_n^2 j_m\left(k_n b\right) L_n\left[h_m^{(2)}\left(k_n a\right)\right]\right\}_{n=0}^{n=1}\int_0^{\pi} C_m^{1/2}\left(\cos\phi\right)\cos\phi\, d\phi, \tag{14}$$

whose integral on the right-hand side is, for odd values of m, equal to

$$\int_0^{\pi} C_m^{1/2}\left(\cos\phi\right)\cos\phi\, d\phi = \frac{\pi(m!!)^2}{2^m m[(m-1)/2]![(m+1)/2]!}, \tag{15}$$

while it is null when m is even. Thus, upon setting $m = 2l - 1$ and performing the differentiations, expression (14) provides the exact series representation for Z, namely

$$Z = -\frac{2\pi\omega\mu_0 b}{k_1^2-k_0^2}\sum_{l=1}^{\infty}\frac{(4l-1)\left[(2l-1)!!\right]^2}{(2l-1)(2l-2)!!(2l)!!}\left\{k_n^2 j_{2l-1}\left(k_n b\right)\left[(4l^2-1)\frac{h_{2l-1}^{(2)}\left(k_n a\right)}{k_n a} - h_{2l-2}^{(2)}\left(k_n a\right)\right]\right\}_{n=0}^{n=1}. \tag{16}$$

It should be recalled that the integral in Equation (1) and its explicit counterpart in Equation (16) are valid only for homogeneous media. On the other hand, for material media made up of N layers, the complete integral expression of Z reads [6]

$$Z = -2\pi j\omega\mu_0 ab\int_0^{\infty}\frac{1}{u_0+\hat{u}_1} J_1(\lambda a)\, J_1(\lambda b)\lambda\, d\lambda, \tag{17}$$

where it is assumed that the Nth layer is the semi-infinite bottom-most layer of the medium, and with

$$\hat{u}_n = u_n\frac{\hat{u}_{n+1} + u_n\tanh(u_n d_n)}{u_n + \hat{u}_{n+1}\tanh(u_n d_n)}, \qquad n = N-1, \ldots, 1, \tag{18}$$

$d_1, d_2, \ldots, d_{N-1}$ being the thicknesses of the $N-1$ finite layers, and $\hat{u}_N = u_N$. The general integral representation in Equation (17) can be still evaluated, but at the price of introducing simplifying approximations in the integrand that make it possible analytical integration. For instance, use of the rational approximation

$$\frac{1}{u_0+\hat{u}_1} \cong \sum_{m=1}^{M}\frac{r_m}{j\lambda^2 - p_m}, \tag{19}$$

obtained through a least squares-based fitting procedure [25], allows to apply the identity [27, No. 14, p. 6] and obtain the explicit expression

$$Z \cong -2\pi\omega\mu_0 ab \sum_{m=1}^{M} r_m K_1\left(\sqrt{jp_m}a\right) I_1\left(\sqrt{jp_m}b\right), \tag{20}$$

where $I_1(\cdot)$ and $K_1(\cdot)$ are first-order modified Bessel functions of the first and second kind, respectively.

3. NUMERICAL RESULTS

To test the developed method, the proposed analytical formula (16) is applied to the computation of the amplitude- and phase-frequency spectra of the impedance Z between two concentric coils, with radii $a = 50\,\text{cm}$ and $b = 20\,\text{cm}$, located on the top surface of a clay soil. The electrical conductivity and dielectric permittivity of the ground are taken to be equal to $\sigma_1 = 10^{-2}\,\text{S/m}$ and $\epsilon_1 = 10\epsilon_0$ [14, 17], and the obtained results, illustrated in Figs. 2 and 3, are compared with those arising from numerically integrating Equation (1) through G7-K15 Gauss-Kronrod quadrature formula [22]. Figure 2 shows the amplitude-frequency spectrum of Z, and different curves originating from Equation (16) are depicted, each one corresponding to a particular value of the index L at which the infinite sum is truncated. As can be seen, for $L = 8$ the data generated by the proposed formula coincide with those provided by the Gauss-Kronrod scheme over the whole considered frequency range, and this confirms that Equation (16) converges to the exact solution. Furthermore, in the low-frequency range the outcomes from the two approaches are overlapping starting from $L = 2$, and this suggests that the quasi-static solution to the problem may be obtained from Equation (16) by retaining only the lower-order terms.

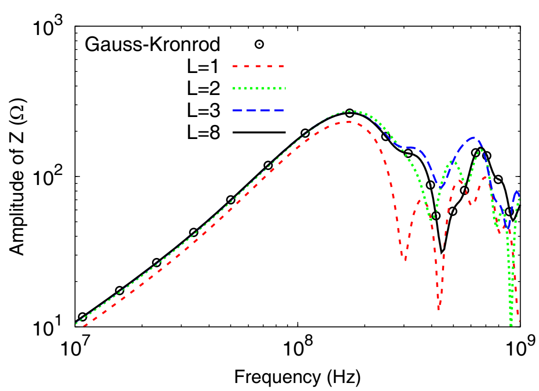

Figure 2. Amplitude of the impedance between two concentric loops.

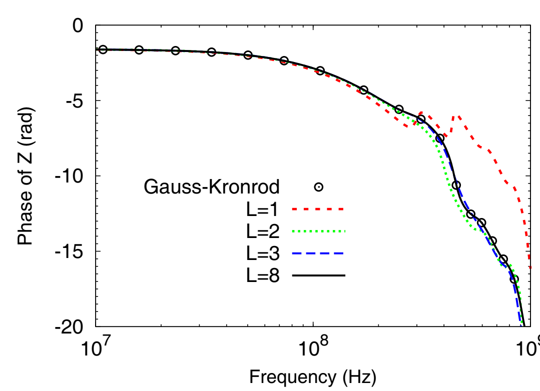

Figure 3. Phase of the impedance between two concentric loops.

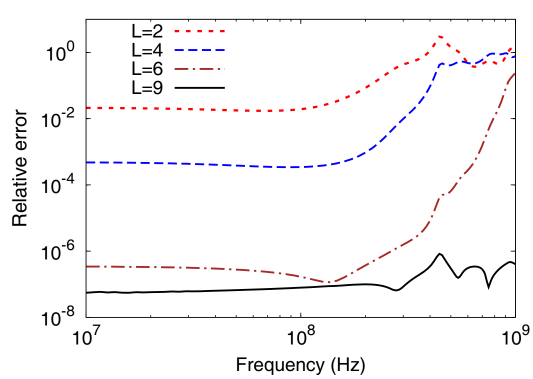

Figure 4. Relative error of the outcomes from (16), as compared to numerical integration data.

Convergence of the proposed solution is also confirmed by the curves plotted in Fig. 3, which shows the sequence of phase-frequency spectra originating from the partial sums in Equation (16), as L is increased. As is evident, it suffices to terminate Equation (16) at $L = 3$ to approximately reproduce the trend provided by Gauss-Kronrod numerical integration. The improvement in accuracy that follows from increasing L may be better understood by taking a glance at Fig. 3, which depicts the relative error resulting from using the proposed expression rather than the Gauss-Kronrod scheme. As can be observed, for small values of the truncation index L the relative error is large, especially at higher frequencies. However, as L is increased it rapidly decreases, down to less than 10^{-6} for $L = 9$, regardless of the operating frequency. Moreover, the error reduction is not achieved at the price of significantly increasing the computational cost of the derived expression. This aspect is illustrated by Table 1, which depicts the average CPU times taken by Equation (16) and G7-K15 Gauss-Kronrod quadrature formula to calculate the impedance. The values of Table 1 show that the developed approach offers excellent speed-up with respect to numerical integration.

Table 1. CPU time comparisons for the impedance calculation.

Approach	average CPU time [s]	Speed-Up
Gauss-Kronrod	$2.142 \cdot 10^2$	-
(16) with $L = 2$	$1.873 \cdot 10^{-3}$	$1.144 \cdot 10^5$
(16) with $L = 4$	$5.428 \cdot 10^{-3}$	$3.946 \cdot 10^4$
(16) with $L = 6$	$1.292 \cdot 10^{-2}$	$1.658 \cdot 10^4$
(16) with $L = 9$	$5.327 \cdot 10^{-2}$	$4.021 \cdot 10^3$

4. CONCLUSIONS

The aim of this work has been to derive an exact explicit expression for the mutual impedance of two concentric loops lying on a conductive soil. The solution has been obtained through a two-step procedure. First, the semi-infinite Sommerfeld integral representation for the impedance is cast into a simpler finite integral. Next, analytical integration is performed after expanding the spherical Hankel function contained in the integrand according to the Gegenbauer addition theorem. As a result, the mutual impedance is expressed as a series of products of two spherical Bessel functions. Numerical simulations have been performed to show that the proposed formula offers a good level of accuracy, and that it is significantly less time consuming than standard numerical integration procedures.

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