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Electromagnetic Field Solutions in an Isotropic Medium with Weakly-Random Fluctuations in Time and Some Applications in the Electrodynamics of the Ionosphere

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Abstract—Stochastic wave equations are derived to describe electromagnetic wave propagation in an isotropic medium in which the electric permittivity and the magnetic permeability are weakly-random functions of time. Approximate analytical solutions are obtained using separation of variables and the WKB method for some configurations that can be used to model the electromagnetic field in the ionosphere. The form of the initial and boundary conditions determines whether the solution takes a form representing a direct current electric field or continuous pulsation electromagnetic waves. The temporal variation of the calculated induced electromotive force (EMF) is in agreement with observations.

1. INTRODUCTION

A random medium is a nonhomogeneous medium whose properties are random functions of either time or position or both. The randomness may be due to the fluctuations in the thermodynamic properties of the medium or due to irregular scatterers in the medium [25]. There are a number of situations in nature where wave propagation occurs in a random nonhomogeneous medium, for example, acoustic-gravity waves in the ocean and atmosphere, seismic waves and regular pulsation electromagnetic waves in the ionosphere [19], and electromagnetic fluctuations in the human brain [2, 16]. When electromagnetic waves propagate in a random medium, there are complex interactions between the fields which change the spatial configuration, orientation or polarization of the electromagnetic field.

Electromagnetic wave propagation in a random medium can be simulated using a discrete model in which it is assumed that there are randomly-distributed discrete scatters in the medium or a continuous model in which the medium is considered to be a continuum whose properties are characterized by its electric permittivity and magnetic permeability, which are considered to be statistical quantities [25]. In the latter approach, the governing equations for the wave propagation are stochastic partial differential equations. In this paper, we follow this continuum approach to examine the case of electromagnetic wave propagation in an isotropic medium for which the electric permittivity and magnetic permeability are functions of time, independent of position, and there is weakly-random temporal variation about their temporal mean values. We derive stochastic wave equations for this configuration and consider some initial and boundary conditions that allow us to obtain analytical solutions for the electromagnetic field.

The configurations that we examine here can be considered as simple representations for electromagnetic wave propagation in the ionosphere. The ionosphere is the region in the upper atmosphere where the concentration of ions and electrons is high enough to affect the propagation of radio waves [9]. Solar winds, magnetic storms and other phenomena transport random scatterers and

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perturb the geomagnetic field and can generate electromagnetic waves and ionospheric disturbances [19]. Downward-propagating ionospheric disturbances sometimes reach the lower atmosphere where they interact with atmospheric waves and consequently affect the general circulation of the atmosphere and, hence, weather and climate [13, 15]. On the other hand, waves generated in the lower and middle atmosphere propagate upward and sometimes reach the ionosphere where they generate travelling ionospheric disturbances [10, 25] and perturb radio wave propagation over long distances [8]. These atmosphere-ionosphere interactions can be studied using electromagnetohydrodynamic (EMHD) equations [18], but in order to develop realistic models for the interactions based on these equations, knowledge of the structure and propagation characteristics of the electromagnetic waves is required.

The ionosphere is a medium that fluctuates randomly in both space and time. There have been analytical studies on electromagnetic wave propagation in a spatially-random ionospheric configuration, dating back over four decades to the classical text of Yeh and Liu [25]. Other examples of wave propagation in spatially-random media, besides the ionosphere, occur in medical imaging and radio wave propagation [3, 20]. In this paper, we focus on the case of electromagnetic wave propagation in a medium that varies randomly in time and we discuss the relevance of our solutions to the dynamics of the ionosphere.

The paper is organized as follows. In Section 2, we describe the equations of the electromagnetic fields in a time-dependent conducting or nonconducting medium with or without a charge source, and we derive a version of the Kramers-Kronig relations which is valid for such a medium. We also derive the equation describing the evolution of the charge density. In Section 3, we solve these equations in an isotropic medium with weakly-random fluctuations in time using separation of variables and the WKB (Wentzel, Kramers and Brillouin) method [4]. In Section 4, applications which are relevant to the ionosphere are given and the induced electromotive force (EMF) is computed. In Section 5, a brief summary of the results obtained in the paper is given.

2. ELECTROMAGNETIC WAVE EQUATIONS IN A TIME-DEPENDENT ISOTROPIC MEDIUM

We start with Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1}$$

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \tag{2}$$

$$\nabla \cdot \mathbf{D} = \rho \tag{3}$$

and

$$\nabla \cdot \mathbf{B} = 0, \tag{4}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field vectors; $\mathbf{D}$ and $\mathbf{B}$ are the electric and magnetic flux densities; $\mathbf{J}$ is the current density and ρ is the charge density. The current density and the charge density are related by the continuity equation

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \mathbf{J}. \tag{5}$$

Following Kormiltsev and Mesentsev [14], we write the relationships between $\mathbf{D}$ and $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{H}$, and $\mathbf{J}$ and $\mathbf{E}$ in a dispersive time-dependent medium as Eq. (6)

$$\begin{aligned}\mathbf{D}(\mathbf{r},t) &= \frac{d}{dt}(\epsilon(t) * \mathbf{E}(\mathbf{r},t)) = \frac{d}{dt}\int_0^t \epsilon(\tau)\mathbf{E}(\mathbf{r},t-\tau)d\tau \\ &= \epsilon(t)\mathbf{E}(\mathbf{r},0) + \int_0^t \epsilon(\tau)\frac{\partial \mathbf{E}}{\partial \tau}(\mathbf{r},t-\tau)d\tau = \epsilon(0)\mathbf{E}(\mathbf{r},t) + \int_0^t \frac{\partial \epsilon(\tau)}{\partial \tau}\mathbf{E}(\mathbf{r},t-\tau)d\tau,\end{aligned} \tag{6}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{d}{dt}(\mu(t) * \mathbf{H}(\mathbf{r},t)) = \frac{d}{dt}\int_0^t \mu(\tau)\mathbf{H}(\mathbf{r},t-\tau)d\tau$$
$$= \mu(t)\mathbf{H}(\mathbf{r},0) + \int_0^t \mu(\tau)\frac{\partial \mathbf{H}}{\partial \tau}(\mathbf{r},t-\tau)d\tau = \mu(0)\mathbf{H}(\mathbf{r},t) + \int_0^t \frac{\partial \mu(\tau)}{\partial \tau}\mathbf{H}(\mathbf{r},t-\tau)d\tau \quad (7)$$

and

$$\mathbf{J}(\mathbf{r},t) = \frac{d}{dt}(\sigma(t) * \mathbf{E}(\mathbf{r},t)) = \frac{d}{dt}\int_0^t \sigma(\tau)\mathbf{E}(\mathbf{r},t-\tau)d\tau$$
$$= \sigma(t)\mathbf{E}(\mathbf{r},0) + \int_0^t \sigma(\tau)\frac{\partial \mathbf{E}}{\partial \tau}(\mathbf{r},t-\tau)d\tau = \sigma(0)\mathbf{E}(\mathbf{r},t) + \int_0^t \frac{\partial \sigma(\tau)}{\partial \tau}\mathbf{E}(\mathbf{r},t-\tau)d\tau, \quad (8)$$

where $*$ represents the Laplace convolution integral, ϵ the permittivity of the medium, μ the permeability of the medium, and σ the conductivity of the medium.

We note that it is important that the convolution integrals reach some stationary states as $t \to \infty$ in order for the principle of causality to be satisfied [25]. In that case, we can take the limit of $t \to \infty$ and obtain the Fourier convolution integral. We also observe, for instance, that the convolution integral in (6),

$$\epsilon(t) * \mathbf{E}(\mathbf{r},t) = \int_0^t \epsilon(\tau)\mathbf{E}(\mathbf{r},t-\tau)d\tau,$$

has the unit of the electric flux density $\mathbf{D}$ times the unit of time. In order to obtain a correct unit for $\mathbf{D}$, it is intuitive to differentiate the convolution integral with respect to time. One can interpret the right hand side of Eq. (6) as the response of the medium to the electric field, while the left hand side of Eq. (6) can be interpreted as the response of the electric field to the medium. This indicates that the medium and the field affect each other. Differentiating the convolution integral with respect to time t, on one hand, is to take into consideration the variation of the field as a result of the effects of the medium on the field (the expressions on the left hand sides of Equations (6), (7) and (8)), while on another hand, it is to take into consideration the variation of the properties of the medium as a result of the effects of the field on the medium (the expressions on the right hand sides of Equations (6), (7) and (8)).

The Fourier transforms of $\mathbf{D}, \mathbf{B}$ and $\mathbf{J}$ are given respectively by

$$\hat{\mathbf{D}}(\mathbf{r},\omega) = i\omega\hat{\epsilon}(\omega)\hat{\mathbf{E}}(\mathbf{r},\omega), \hat{\mathbf{B}}(\mathbf{r},\omega) = i\omega\hat{\mu}(\omega)\hat{\mathbf{H}}(\mathbf{r},\omega) \text{ and } \hat{\mathbf{J}}(\mathbf{r},\omega) = i\omega\hat{\sigma}(\omega)\hat{\mathbf{E}}(\mathbf{r},\omega), \quad (9)$$

provided that the convolution integrals in Eqs. (6)–(8) reach some stationary states as $t \to \infty$. Taking the Fourier transform in time of the second Maxwell's Equation (2) and using Eq. (9) gives

$$\nabla \times \hat{\mathbf{H}}(\mathbf{r},\omega) = [\sigma(0) + i\omega\hat{\sigma}(\omega) + i\omega\epsilon(0) - \omega^2\hat{\epsilon}(\omega)]\hat{\mathbf{E}}(\mathbf{r},\omega). \quad (10)$$

We write $\hat{\epsilon}$ as $\hat{\epsilon} = \hat{\epsilon}_{\mathrm{re}} + i\hat{\epsilon}_{\mathrm{im}}$, and $\hat{\sigma}$ as $\hat{\sigma} = \hat{\sigma}_{\mathrm{re}} + i\hat{\sigma}_{\mathrm{im}}$, where the subscripts re and im stand for real and imaginary parts respectively. Substituting these in Eq. (10) gives

$$\nabla\times\hat{\mathbf{H}}(\mathbf{r},\omega)=\{[\sigma_{re}(0)-\omega\hat{\sigma}_{\mathrm{im}}(\omega)-\omega\epsilon_{\mathrm{im}}(0)-\omega^2\hat{\epsilon}_{\mathrm{re}}(\omega)]+i[\sigma_{\mathrm{im}}(0)+\omega\hat{\sigma}_{\mathrm{re}}(\omega)+\omega\epsilon_{\mathrm{re}}(0)-\omega^2\hat{\epsilon}_{\mathrm{im}}(\omega)]\}\hat{\mathbf{E}}(\mathbf{r},\omega). \quad (11)$$

Taking into consideration the fact that, initially, the permittivity is nonzero, $\epsilon(0) = \epsilon_{\mathrm{re}}(0) + i\epsilon_{\mathrm{im}}(0)$ as in Eq. (11), and using the Poisson Kernel and its pair conjuguate, the principle of causality leads to the following version of the Kramers-Kronig relations in Fourier space (the ω domain),

$$\hat{\epsilon}_{\mathrm{re}}(\omega) = \epsilon_{\mathrm{re}}(0)\delta(\omega-0) + \frac{1}{i\pi}\mathrm{PV}\int_{-i\infty}^{+i\infty} \frac{\xi\hat{\epsilon}_{\mathrm{im}}(\xi)}{\xi^2+\omega^2}d\xi \quad (12)$$

and

$$\hat{\epsilon}_{\rm im}(\omega) = \epsilon_{\rm im}(0)\delta(\omega - 0) - \frac{1}{i\pi\omega}{\rm PV}\int_{-i\infty}^{+i\infty} \frac{\xi^2\hat{\epsilon}_{\rm re}(\xi)}{\xi^2+\omega^2}d\xi, \tag{13}$$

where PV stands for the principal value of the integral.

Relations (12) and (13) are slightly different from those found in the classical books on electromagnetism, such as Van Bladel [5], since they are obtained using $\epsilon' = d\epsilon/dt$ as the causal function, rather than ϵ as in Van Bladel [5], in order to satisfy Eqs. (6) and (11).

Under certain conditions, it is possible to simplify Eqs. (6)–(8) and get rid of the convolution integrals. Here, we consider electromagnetic fields with small noncausal events as those in a weakly-dispersive medium. In a weakly-dispersive time-dependent medium, the medium has a weaker effect on the electromagnetic field, and so $\epsilon(\tau)$, $\mu(\tau)$ and $\sigma(\tau)$ can be approximated as $\epsilon(t)$, $\mu(t)$ and $\sigma(t)$ in the expressions on the left hand sides of Eqs. (6), (7), and (8), respectively. In doing so, we obtain the simplified relations

$$\mathbf{D}(\mathbf{r},t) = \epsilon(t)\mathbf{E}(\mathbf{r},t),\ \mathbf{B}(\mathbf{r},t) = \mu(t)\mathbf{H}(\mathbf{r},t) \text{ and } \mathbf{J}(\mathbf{r},t) = \sigma(t)\mathbf{E}(\mathbf{r},t). \tag{14}$$

On the other hand, in a weakly-dispersive time-dependent medium, the electromagnetic field also has a weaker effect on the medium. In that case, the electric field vector $\mathbf{E}(r,t-\tau)$ in the expressions on the right hand sides of Eqs. (6) and (8) can be approximated as $\mathbf{E}(r,t)$, while $\mathbf{H}(r,t-\tau)$ can be approximated as $\mathbf{H}(r,t)$ in the expression on the right hand side of Eq. (7). This, as expected, gives Eq. (14). We also note that Eqs. (6)–(8) are reduced to Eq. (14) in a non-dispersive medium with constant ϵ, μ and σ.

Next, taking the curl of Eq. (1) and using Eqsw. (2) and (14) gives

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\frac{\partial(\mu\mathbf{J})}{\partial t} - \mu\frac{\partial^2(\epsilon\mathbf{E})}{\partial t^2} - \frac{d\mu}{dt}\frac{\partial(\epsilon\mathbf{E})}{\partial t}. \tag{15}$$

Similarly, taking the curl of Eq. (2) and using Eqs. (1) and (14) gives

$$\nabla^2\mathbf{H} = \epsilon\frac{\partial^2(\mu\mathbf{H})}{\partial t^2} + \frac{d\epsilon}{dt}\frac{\partial(\mu\mathbf{H})}{\partial t} - \nabla\times\mathbf{J}. \tag{16}$$

In a non-conducting source-free medium, $\rho = 0$ and $\mathbf{J} = 0$; in that case, Equations (15) and (16) are simplified to

$$\nabla^2\mathbf{E} - \mu\frac{\partial^2(\epsilon\mathbf{E})}{\partial t^2} - \frac{d\mu}{dt}\frac{\partial(\epsilon\mathbf{E})}{\partial t} = 0 \tag{17}$$

and

$$\nabla^2\mathbf{H} - \epsilon\frac{\partial^2(\mu\mathbf{H})}{\partial t^2} - \frac{d\epsilon}{dt}\frac{\partial(\mu\mathbf{H})}{\partial t} = 0. \tag{18}$$

The case with constants ϵ and μ leads to the standard form of the equations for electromagnetic wave propagation in a steady homogeneous medium.

More generally, Equations (17) and (18) are obtained in any configuration where

$$\nabla(\nabla\cdot\mathbf{E}) = -\frac{\partial(\mu\mathbf{J})}{\partial t} \text{ and } \nabla\times\mathbf{J} = 0. \tag{19}$$

Applying the divergence operator yields

$$\nabla^2(\nabla\cdot\mathbf{E}) = -\frac{\partial(\mu\nabla\cdot\mathbf{J})}{\partial t}. \tag{20}$$

Combining this with the third Maxwell Equation (3) and the continuity Equation (5) and expanding the right hand side of Eq. (20) gives the equation for the electric charge density ρ,

$$\nabla^2\rho = \epsilon\frac{d\mu}{dt}\frac{\partial\rho}{\partial t} + \epsilon\mu\frac{\partial^2\rho}{\partial t^2}. \tag{21}$$

This equation is useful in the sense that it allows us to derive the expression for ρ even if that of the electric field vector $\mathbf{E}$ is not known. If ϵ and μ are constants, ρ is thus a solution for the wave equation

$$\nabla^2\rho = \epsilon\mu\frac{\partial^2\rho}{\partial t^2}, \tag{22}$$

as can be found in books on the basics of electromagnetism (e.g., [6]).

A configuration with time-dependent permittivity, but constant permeability, has been examined in the past (see for example Pedrosa et al. [21]); they considered some special profiles of the permittivity that allowed them to derive exact solutions. In our investigation, both the permittivity and the permeability are functions of time and fluctuate randomly.

In the present paper, we examine Equations (17) and (18) in a two-dimensional domain described by rectangular coordinates x and z. We consider a bounded domain in the x-direction ($x_1 \le x \le x_2$) and define the z coordinate from some fixed altitude, say $z = z_1$ (the wave source level) and extending up to infinity ($z_1 \le z < \infty$). We seek solutions of Eqs. (17) and (18) that are oscillatory and periodic in the x direction and remain finite as $z \to \infty$ and as the time $t \to \infty$.

In total, six (initial and boundary) conditions are needed in order to define completely the electric or magnetic field. Four of these conditions are: $E(x, z, t)$ is finite as $z \to \infty$, $E(x, z, t)$ is finite as $t \to \infty$ and either periodic boundary conditions $E(x = x_1, z, t) = E(x = x_2, z, t)$ or non-periodic boundary conditions $E(x = x_1, z, t) \neq E(x = x_2, z, t)$. The others are the boundary condition at $z = z_1$ ($E(x, z = z_1, t)$), and either the initial condition $E(x, z, t = 0)$ or $\frac{\partial E}{\partial t}(x, z, t = 0)$ which will be specified later in Section 4. These types of initial-boundary conditions are relevant to the study of ionospheric disturbances such as electrohydrodynamic disturbances (EHD), magnetohydrodynamic disturbances (MHD), electromagnetohydrodynamic disturbances (EMHD) and their associated travelling ionospheric disturbances (TIDs) [18].

3. APPROXIMATE ANALYTICAL SOLUTIONS IN AN ISOTROPIC MEDIUM WITH WEAKLY-RANDOM FLUCTUATIONS IN TIME

In an isotropic medium where the electric and magnetic fields satisfy Eqs. (17) and (18) respectively, each component E of the electric field vector $\mathbf{E}$ and each component H of the magnetic field vector $\mathbf{H}$ satisfy, respectively,

$$\nabla^2 E - \mu\frac{\partial^2(\epsilon E)}{\partial t^2} - \frac{d\mu}{dt}\frac{\partial(\epsilon E)}{\partial t} = 0 \tag{23}$$

and

$$\nabla^2 H - \epsilon\frac{\partial^2(\mu H)}{\partial t^2} - \frac{d\epsilon}{dt}\frac{\partial(\mu H)}{\partial t} = 0. \tag{24}$$

Equations (23) and (24) are symmetric in ϵ and μ. Once the expression for the electric field E is derived, that of the magnetic field H is obtained by simply interchanging ϵ and μ. We thus focus on the solution of Equation (23). Using the method of separation of variables, we seek a solution of the form $E(\mathbf{r}, t) = R(\mathbf{r})T(t)$, where $\mathbf{r}$ is the position vector in two-dimensional space and t represents time. Equation (23) then gives

$$\frac{\nabla^2 R}{R} = \frac{(\mu(\epsilon T)')'}{T} = \lambda, \quad \text{a constant}, \tag{25}$$

where the prime stands for differentiation with respect to time t. This means that

$$\nabla^2 R - \lambda R = 0 \tag{26}$$

and

$$(\mu(\epsilon T)')' - \lambda T = 0. \tag{27}$$

The constant λ may take different values depending on the problem that is being studied. In the configuration considered here, which is relevant to the ionosphere, some particular values of λ are considered.

3.1. Solutions Corresponding to $\lambda = 0$

If $\lambda = 0$, the solutions are oscillatory in space and vary randomly in time. In the two-dimensional ionospheric configuration with periodic boundary conditions and sinusoidal oscillations in the x-direction and exponential variation in the z-direction, the solution corresponding to $\lambda = 0$ which is finite as $z \to \infty$ is

$$E_{\lambda=0}(x,z,t) = \frac{e^{-\eta z}}{\epsilon(t)}\left[a_1 + a_2 \int \frac{1}{\mu(t)} dt\right](b_1 \cos\eta x + b_2 \sin\eta x), \tag{28}$$

where η, a_1, a_2, b_1 and b_2 are constants, and η can be determined from the boundary conditions specified in the x direction.

In a medium with weakly-random temporal variations, the permittivity ϵ and permeability μ are represented as

$$\epsilon(t) = \epsilon_0\epsilon_r(t) = \epsilon_0\langle\epsilon_r\rangle + \Delta\epsilon(t) \text{ and } \mu(t) = \mu_0\mu_r(t) = \mu_0\langle\mu_r\rangle + \Delta\mu(t), \tag{29}$$

where ϵ_0 is the permittivity of free space, $\epsilon_r(t)$ the relative permittivity of the medium, $\langle\epsilon_r\rangle$ the temporal average of the relative permittivity, μ_0 the permeability of free space, $\mu_r(t)$ the relative permeability of the medium, and $\langle\mu_r\rangle$ the temporal average of the relative permeability. The fluctuations $\Delta\epsilon(t)$ and $\Delta\mu(t)$ from the average values are defined as stochastic variables each with mean zero and some variance. Weak random variation of the medium means that $|\Delta\epsilon(t)| \ll |\epsilon_0\langle\epsilon_r\rangle|$ and $|\Delta\mu(t)| \ll |\mu_0\langle\mu_r\rangle|$. The fluctuations are defined so that $\Delta\epsilon(0) = 0$ and $\Delta\mu(0) = 0$ and $\Delta\epsilon(t)$ and $\Delta\mu(t)$ both have variance that equals t, then $\Delta\epsilon(t)$ and $\Delta\mu(t)$ can be represented as standard Wiener processes (see Appendix B).

Using the asymptotic approximation $(1+a)^\theta \sim 1 + \theta a$ which is valid when $a \ll 1$, in a weakly-random medium, we can write

$$\frac{1}{\epsilon(t)} = \frac{1}{\epsilon_0\langle\epsilon_r\rangle + \Delta\epsilon(t)} \sim \frac{1}{\epsilon_0\langle\epsilon_r\rangle}\left[1 - \frac{\Delta\epsilon(t)}{\epsilon_0\langle\epsilon_r\rangle}\right] \text{ and } \frac{1}{\mu(t)} = \frac{1}{\mu_0\langle\mu_r\rangle + \Delta\mu(t)} \sim \frac{1}{\mu_0\langle\mu_r\rangle}\left[1 - \frac{\Delta\mu(t)}{\mu_0\langle\mu_r\rangle}\right]. \tag{30}$$

Thus, $\int \frac{1}{\epsilon(\tau)} d\tau \sim O(t)$ and $\int \frac{1}{\mu(\tau)} d\tau \sim O(t)$.

In order for $E_{\lambda=0}$ to remain finite as $t \to \infty$, a_2 must be zero. Therefore,

$$\begin{aligned} E_{\lambda=0}(x,z,t) =& \frac{e^{-\eta z}}{\epsilon(t)}(b_1 \cos\eta x + b_2 \sin\eta x) = \frac{e^{-\eta z}}{\epsilon_r(t)}\left(\tilde{b}_1 \cos\eta x + \tilde{b}_2 \sin\eta x\right) \\ &\sim \frac{e^{-\eta z}}{\langle\epsilon_r\rangle}\left(\tilde{b}_1 \cos\eta x + \tilde{b}_2 \sin\eta x\right)\left[1 - \frac{\Delta\epsilon(t)}{\epsilon_0\langle\epsilon_r\rangle}\right], \end{aligned} \tag{31}$$

where the new constants $\tilde{b}_1 = b_1/\epsilon_0$ and $\tilde{b}_2 = b_2/\epsilon_0$. The temporal average of $E_{\lambda=0}$ can be approximated by

$$\langle E_{\lambda=0}\rangle \sim \frac{e^{-\eta z}}{\langle\epsilon_r\rangle}\left(\tilde{b}_1 \cos\eta x + \tilde{b}_2 \sin\eta x\right). \tag{32}$$

We observe that the rate of decay of the solution in the z direction is equal to the horizontal wavenumber $k_x = \eta$. We can then compute the electric flux density D as

$$D_{\lambda=0}(x,z,t) = \epsilon(t)E_{\lambda=0}(x,z,t) = e^{-\eta(z-z_1)}(b_1 \cos\eta x + b_2 \sin\eta x). \tag{33}$$

We note that in this configuration E fluctuates in time, but D is time-independent and satisfies $\nabla^2 D = 0$, according to Eq. (23).

Interchanging ϵ and μ in Eq. (31) gives the solution for the magnetic field vector H corresponding to $\lambda = 0$,

$$H_{\lambda=0}(x,z,t) = \frac{e^{-\eta z}}{\mu(t)}\left[\bar{a}_1 + \bar{a}_2 \int \frac{1}{\epsilon(t)} dt\right]\left(\bar{b}_1 \cos\eta x + \bar{b}_2 \sin\eta x\right), \tag{34}$$

where η, $\bar{a}_1$, $\bar{a}_2$, $\bar{b}_1$ and $\bar{b}_2$ are constants. In order for $H_{\lambda=0}$ also to remain finite as $t \to \infty$, we must have $\bar{a}_2 = 0$. We can then obtain an expression for the corresponding magnetic flux density

$$B_{\lambda=0}(x,z,t) = \mu(t)H_{\lambda=0}(x,z,t) = e^{-\eta(z-z_1)}\left(\bar{b}_1 \cos\eta x + \bar{b}_2 \sin\eta x\right), \tag{35}$$

which is time-independent and satisfies $\nabla^2 B = 0$, according to Eq. (18).

3.2. Solutions Corresponding to $\lambda \neq 0$

Nonzero values of λ, on the other hand, give solutions that represent waves oscillating in both space and time and varying randomly in time. With $\lambda \neq 0$, Equation (26) can be solved using the method of separation of variables. We seek a solution of the form $R(x, z) = X(x)Z(z)$, which is oscillatory and periodic in x and finite as $z \to \infty$ and find that

$$X(x) = b_1 e^{i\eta x} + b_2 e^{-i\eta x} \text{ and } Z(z) = c_1 e^{\sqrt{\eta^2+\lambda}\, z} + c_2 e^{-\sqrt{\eta^2+\lambda}\, z}, \tag{36}$$

where λ, b_1, b_2, c_1 and c_2 are complex constants. As before, $k_x = \eta$ is the horizontal wavenumber, which is a real non-zero constant that can be determined from the boundary conditions specified in the x direction.

Equation (27) can be written as

$$\tilde{T}'' + \frac{\Delta\mu'}{\mu}\tilde{T}' - \frac{c^2\lambda}{\epsilon_r\mu_r}\tilde{T} = 0, \tag{37}$$

where $\tilde{T}(t) = \epsilon(t)T(t)$ and $c = (\epsilon_0\mu_0)^{-1/2} \approx 3 \times 10^8\,\mathrm{ms}^{-1}$ is the speed of light in free space. We then introduce another new function $V(t) = \sqrt{\mu(t)}\tilde{T}(t)$ which eliminates the first derivative term in Eq. (37) and gives

$$V'' - \left[\frac{c^2\lambda}{\epsilon_r\mu_r} + \frac{1}{2}\left(\frac{\Delta\mu'}{\mu}\right)' + \frac{1}{4}\left(\frac{\Delta\mu'}{\mu}\right)^2\right] V = 0. \tag{38}$$

In a plasma such as the ionosphere, the nondimensional functions $\epsilon_r(t)$ and $\mu_r(t)$ are $O(1)$, and the constant $c^2|\lambda|$ which has dimension t^{-2} and units s^{-2}, can be considered to be "large" provided that $|\lambda|$ is larger than $1/9 \times 10^{-16}\,\mathrm{m}^{-2}$. In that case, the second and third terms in the coefficient of V in (38) is negligible compared with the first term, and V can thus be approximated by the function $\tilde{V}$ which satisfies

$$\tilde{V}'' - \left[\frac{c^2\lambda}{\epsilon_r\mu_r}\right]\tilde{V} = 0. \tag{39}$$

We are therefore justified to make use of the WKB method [4] (see Equation (C2) in Appendix C) to obtain an approximate asymptotic solution for Equation (39). Applying the WKB method gives

$$\begin{aligned}\tilde{V}(t) \sim\; & d_1\left[\frac{\epsilon_r(t)\mu_r(t)}{c^2\lambda}\right]^{\frac{1}{4}} \exp\left\{c\sqrt{\lambda}\int_0^t [\epsilon_r(\tau)\mu_r(\tau)]^{-1/2} d\tau\right\} \\ & + d_2\left[\frac{\epsilon_r(t)\mu_r(t)}{c^2\lambda}\right]^{\frac{1}{4}} \exp\left\{-c\sqrt{\lambda}\int_0^t [\epsilon_r(\tau)\mu_r(\tau)]^{-1/2} d\tau\right\} + \text{c.c.},\end{aligned} \tag{40}$$

where d_1 and d_2 are constants, and "c.c." denotes the complex conjugate of the preceding expression. Thus, T can be approximated as

$$T(t) = \frac{V(t)}{\epsilon(t)\mu^{1/2}(t)} \sim \tilde{d}_1 \frac{\exp\left\{c\sqrt{\lambda}\int_0^t [\epsilon_r(\tau)\mu_r(\tau)]^{-1/2} d\tau\right\}}{[\epsilon_r^3(t)\mu_r(t)]^{1/4}} + \tilde{d}_2 \frac{\exp\left\{-c\sqrt{\lambda}\int_0^t [\epsilon_r(\tau)\mu_r(\tau)]^{-1/2} d\tau\right\}}{[\epsilon_r^3(t)\mu_r(t)]^{1/4}} + \text{c.c.}, \tag{41}$$

where the new constants $\tilde{d}_1 = d_1 c^{1/2}/(\epsilon_0^{1/2}\lambda^{1/4})$ and $\tilde{d}_2 = d_2 c^{1/2}/(\epsilon_0^{1/2}\lambda^{1/4})$.

In general, λ is complex, so we write it as $\lambda_r + i\lambda_i$ and define

$$\alpha_r + i\alpha_i = \sqrt{\lambda} = \sqrt{\lambda_r + i\lambda_i} = \frac{1}{\sqrt{2}}\sqrt{\lambda_r + \sqrt{\lambda_r^2 + \lambda_i^2}} + \mathrm{sgn}(\lambda_i)\frac{i}{\sqrt{2}}\sqrt{-\lambda_r + \sqrt{\lambda_r^2 + \lambda_i^2}} \tag{42}$$

and

$$\beta_r + i\beta_i \;=\; \sqrt{\eta^2+\lambda} = \sqrt{\eta^2+\lambda_r+i\lambda_i} = \frac{1}{\sqrt{2}}\sqrt{\eta^2+\lambda_r+\sqrt{(\eta^2+\lambda_r)^2+\lambda_i^2}}$$
$$+\mathrm{sgn}(\lambda_i)\frac{i}{\sqrt{2}}\sqrt{-(\eta^2+\lambda_r)+\sqrt{(\eta^2+\lambda_r)^2+\lambda_i^2}}. \quad (43)$$

To obtain a solution that is finite as $z \to \infty$ and as $t \to \infty$, we set c_1 and $\tilde{d}_1$ to zero. Using the asymptotic approximation $(1+a)^\theta \sim 1+\theta a$, $a \ll 1$, as before, the solution corresponding to $\lambda \neq 0$ can thus be approximated by

$$E(x,z,t) \sim$$
$$[1-\vartheta_E(t)]\exp\{-\gamma\,[t-\theta_{\mathit{shift}}(t)/\omega]\}\exp\{-i\omega\,[t-\theta_{\mathit{shift}}(t)/\omega]\}e^{-\beta_r z}e^{-i\beta_i z}\left(b_1e^{i\eta x}+b_2e^{-i\eta x}\right)+\text{c.c.}, \quad (44)$$

where $\vartheta_E(t) = (1/4)\,[3\Delta\epsilon(t)/(\epsilon_0\langle\epsilon_r\rangle)+\Delta\mu(t)/(\mu_0\langle\mu_r\rangle)]$, γ is the decay rate of the wave, ω the frequency of the wave, and γ and ω are given respectively by

$$\gamma = \frac{c}{\sqrt{\langle\epsilon_r\rangle\langle\mu_r\rangle}}\alpha_r = \frac{c}{\langle n\rangle}\alpha_r \text{ and } \omega = \frac{c}{\sqrt{\langle\epsilon_r\rangle\langle\mu_r\rangle}}\alpha_i = \frac{c}{\langle n\rangle}\alpha_i, \quad (45)$$

with $\langle n\rangle = \sqrt{\langle\epsilon_r\rangle\langle\mu_r\rangle}$ being the mean value of the refractive index of the medium.

The wave phase shift induced by the randomness of the medium is

$$\theta_{\mathit{shift}}(t) \sim \frac{1}{2}\frac{\alpha_i c}{\langle n\rangle}\int_0^t\left[\frac{\Delta\epsilon(\tau)}{\epsilon_0\langle\epsilon_r\rangle}+\frac{\Delta\mu(\tau)}{\mu_0\langle\mu_r\rangle}\right]d\tau = \frac{\omega}{2}\int_0^t\left[\frac{\Delta\epsilon(\tau)}{\epsilon_0\langle\epsilon_r\rangle}+\frac{\Delta\mu(\tau)}{\mu_0\langle\mu_r\rangle}\right]d\tau. \quad (46)$$

The index of refraction of the random medium is thus given by

$$n(t) = \langle n\rangle + \Delta n(t) \sim \langle n\rangle + \frac{\langle n\rangle}{2}\left[\frac{\Delta\epsilon(t)}{\epsilon_0\langle\epsilon_r\rangle}+\frac{\Delta\mu(t)}{\mu_0\langle\mu_r\rangle}\right], \quad (47)$$

where $\Delta n(t)$ is the stochastic fluctuation induced by the randomness of the medium.

The horizontal and vertical wavenumbers are $k_x = \eta$ and $k_z = \beta_i$, $\alpha_r \geq 0$, $\beta_r \geq 0$ and $\mathrm{sgn}(\alpha_i) = \mathrm{sgn}(\beta_i) = \mathrm{sgn}(\lambda_i)$. If λ is real, then we obtain different possibilities depending on the relative magnitude of λ and η. The case $\lambda < -\eta^2$ gives $\alpha_r = 0$ and $\beta_r = 0$, so there are sinusoidal oscillations in t and z with constant amplitude. This is the standard configuration given in textbooks, but it is physically unrealistic on an unbounded vertical and temporal domain since the wave amplitude cannot remain constant for infinite time and at infinite distance from the source level. The other possibilities do not give vertically-propagating waves: $\lambda = -\eta^2$ gives oscillations in t which are independent of z; $-\eta^2 < \lambda < 0$ gives $\alpha_r = 0$ and $\beta_i = 0$ which means that there are sinusoidal oscillations in t with no exponential decay and exponential decay in z with no oscillations; $-\eta^2 < 0 < \lambda$ gives exponential decay in both z and t with no oscillations.

The mean of the time-dependent function in the solution (44) can be approximated as

$$\langle T(t)\rangle \sim e^{-\gamma t}e^{-i\omega t}+\text{c.c.}, \quad (48)$$

and from Eqs. (42) and (43), we obtain

$$(\alpha_r+i\alpha_i)^2 = \lambda = (\beta_r+i\beta_i)^2-\eta^2. \quad (49)$$

The real part of Eq. (49) gives

$$\alpha_i^2 = \alpha_r^2-\beta_r^2+\beta_i^2+\eta^2,$$

and so

$$\omega^2 = \frac{c^2}{\langle\epsilon_r\rangle\langle\mu_r\rangle}(k_x^2+k_z^2+\alpha_r^2-\beta_r^2). \quad (50)$$

This is the dispersion relation for the electromagnetic wave propagation; it is a generalization of the standard form $\omega^2 = c^2(k_x^2+k_z^2)/(\epsilon_r\mu_r)$ normally given in textbooks on electromagnetism for waves with constant amplitude in a two-dimensional configuration with constant permittivity and permeability. From Eq. (50), we note that the vertical component of the group velocity is

$$\frac{\partial\omega}{\partial k_z} = \frac{c}{(\langle\epsilon_r\rangle\langle\mu_r\rangle)^{1/2}} \frac{k_z}{(k_x^2 + k_z^2 + \alpha_r^2 - \beta_r^2)^{1/2}}, \tag{51}$$

which is positive since ω and k_z have the same sign. This indicates that the waves propagate upwards from their source. If we were to consider instead a configuration where the waves were generated in the upper levels of the ionosphere and propagated downwards, then we would instead set $c_2 = 0$ in (36) and the exponent of the function $Z(z)$ in (36) would then be $(-\beta_r - i\beta_i)z$. In that case, the vertical component of the group velocity would be negative according to Eq. (51).

From Eq. (44), we obtain the following approximation for the electric flux density,

$$\begin{aligned} &D(x,z,t) = \epsilon(t)E(x,z,t) \\ &\sim [1 - \vartheta_D(t)] \exp\{-\gamma\,[t - \theta_{shift}(t)/\omega]\} \exp\{-i\omega\,[t - \theta_{shift}(t)/\omega]\} e^{-\beta_r z} e^{-i\beta_i z} \left(b_1 e^{i\eta x} + b_2 e^{-i\eta x}\right) + \text{c.c.}, \end{aligned} \tag{52}$$

where $\vartheta_D(t) = (1/4)\,[\Delta\epsilon(t)/(\epsilon_0\langle\epsilon_r\rangle) - \Delta\mu(t)/(\mu_0\langle\mu_r\rangle)]$. Solutions for H and B can be obtained in a similar manner by interchanging ϵ and μ.

4. SOME CONFIGURATIONS RELEVANT TO THE ELECTRODYNAMICS OF THE IONOSPHERE

In this section, we describe some configurations that lead to solutions that can be used to describe certain phenomena observed in the ionosphere. Measurements of electric field variables indicate that, under certain circumstances, the upper region of the ionosphere, known as the F region, may behave like an electric dynamo with a direct current electric field [12, 22]. This situation can be represented by the solutions of Eqs. (28) and (34) in a domain with suitably chosen boundary and initial conditions, while solutions of the form (44) corresponding to $\lambda \neq 0$ can be used to represent continuous pulsation electromagnetic waves [11, 19].

In a two-dimensional domain given by $x_1 < x < x_2$ and $z_1 < z < \infty$ shown in Figure 1, we let

$$\mathbf{E} = \hat{\mathbf{x}}E_x + \hat{\mathbf{z}}E_z \text{ and } \mathbf{D} = \hat{\mathbf{x}}D_x + \hat{\mathbf{z}}D_z, \tag{53}$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ are, respectively, the unit vectors in the x and z directions. We solve Eq. (17) subject to the initial conditions

$$E_x(x,z,0) = \frac{\rho_s e^{-\eta_1 z}}{\epsilon_0\langle\epsilon_r\rangle} \sin\eta_1 x \text{ and } E_z(x,z,0) = \frac{\rho_s e^{-\eta_2 z}}{\epsilon_0\langle\epsilon_r\rangle} \cos\eta_2 x,\ x_1 \le x \le x_2,\ z_1 \le z < \infty \tag{54}$$

and the boundary conditions

$$E_x(x,z_1,t) = \frac{\rho_s}{\epsilon_0\epsilon_r(t)} \sin\eta_1 x \text{ and } E_z(x,z_1,t) = \frac{\rho_s}{\epsilon_0\epsilon_r(t)} \cos\eta_2 x,\ t \ge 0, \tag{55}$$

where ρ_s is the surface charge density. We also consider non-periodic boundary conditions $E(x = x_1, z, t) \neq E(x = x_2, z, t)$ in order to have a potential difference between the boundaries x_1 and x_2.

According to Section 3, these conditions give the solutions

$$E_x(x,z,t) = \frac{\rho_s}{\epsilon(t)} e^{-\eta_1(z-z_1)} \sin\eta_1 x \sim \frac{\rho_s}{\epsilon_0\langle\epsilon_r\rangle} e^{-\eta_1(z-z_1)} \sin\eta_1 x \left[1 - \frac{\Delta\epsilon(t)}{\epsilon_0\langle\epsilon_r\rangle}\right] \tag{56}$$

and

$$E_z(x,z,t) = \frac{\rho_s}{\epsilon(t)} e^{-\eta_2(z-z_1)} \cos\eta_2 x \sim \frac{\rho_s}{\epsilon_0\langle\epsilon_r\rangle} e^{-\eta_2(z-z_1)} \cos\eta_2 x \left[1 - \frac{\Delta\epsilon(t)}{\epsilon_0\langle\epsilon_r\rangle}\right]. \tag{57}$$

From Eq. (14), the components of the electric flux density are

$$D_x(x,z,t) = \epsilon(t)E_x(x,z,t) = \rho_s e^{-\eta_1(z-z_1)} \sin\eta_1 x \text{ and } D_z(x,z,t) = \epsilon(t)E_z(x,z,t) = \rho_s e^{-\eta_2(z-z_1)} \cos\eta_2 x. \tag{58}$$

Thus, the charge density ρ is

$$\rho(x,z) = \nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_z}{\partial z} = \eta_1 \rho_s e^{-\eta_1 (z-z_1)} \cos \eta_1 x - \eta_2 \rho_s e^{-\eta_2 (z-z_1)} \cos \eta_2 x. \quad (59)$$

Since ρ is time-independent, the solution represents a situation where the ionosphere acts like an electric dynamo with a direct current.

The magnetic flux density is obtained using Eq. (1),

$$\mathbf{B} = -\int \nabla \times \mathbf{E} dt = \int \hat{\mathbf{y}} \left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z} \right) dt = \hat{\mathbf{y}} \rho_s \left[\eta_1 e^{-\eta_1 (z-z_1)} \sin \eta_1 x - \eta_2 e^{-\eta_2 (z-z_1)} \sin \eta_2 x \right] \int \frac{1}{\epsilon(t)} dt. \quad (60)$$

This corresponds to the situation where $\bar{a}_1 = 0$ and $\bar{a}_2 \neq 0$ in the magnetic field solution (34). The induced electromotive force (EMF) is obtained using Faraday's law of induction,

$$\begin{aligned} \text{EMF} &= -\frac{d}{dt} \oiint_A \mathbf{B} \cdot \hat{\mathbf{y}} dA = \frac{\rho_s}{\epsilon(t)} \left[\frac{1}{\eta_1} (\cos(\eta_1 x_2) - \cos(\eta_1 x_1)) - \frac{1}{\eta_2} (\cos(\eta_2 x_2) - \cos(\eta_2 x_1)) \right] \\ &\sim \frac{\rho_s}{\epsilon_0 \langle \epsilon_r \rangle} \left[\frac{1}{\eta_1} (\cos(\eta_1 x_2) - \cos(\eta_1 x_1)) - \frac{1}{\eta_2} (\cos(\eta_2 x_2) - \cos(\eta_2 x_1)) \right] \left[1 - \frac{\Delta\epsilon(t)}{\epsilon_0 \langle \epsilon_r \rangle} \right], \quad (61) \end{aligned}$$

where A is the area of our rectangular domain shown in Figure 1. Thus, the solutions corresponding to $\lambda = 0$ obtained in Section 3 give a representation of an electromagnetic field corresponding to a direct current.

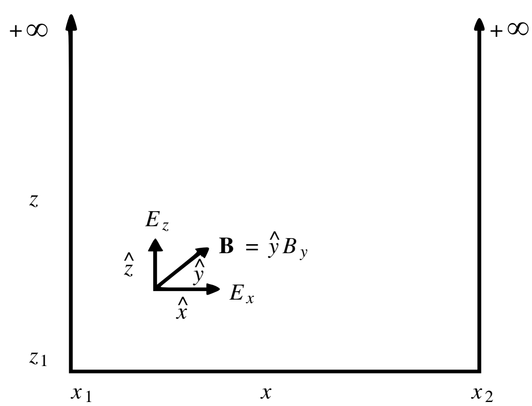

Figure 1. A schematic of the scattering configuration.

Figure 2 shows a plot of the horizontal component of the electric field E_x as a function of t at fixed x and z, as given by the solution (56). The fluctuation of the permittivity is represented as $\Delta\epsilon(t)/(\epsilon_0 \langle \epsilon_r \rangle) = \delta_E W(t)$, where $W(t)$ is a standard Wiener process (see Appendix B) and δ_E is a small constant. To ensure that $\delta_E W(t)$ is indeed small for all t, after generating the random vector of W values, we identify the maximum value $W_{\max}$ of W over the time interval and set $\delta_E = 0.05/W_{\max}$.

In the graph shown, four realizations corresponding to $\Delta\epsilon(t)/(\epsilon_0 \langle \epsilon_r \rangle) = \{0.21\, W_1(t), 0.19\, W_2(t), 0.27\, W_3(t), 0.31\, W_4(t)\}$ were performed. The time t is represented in milliseconds, and the amplitude of the electric field has been normalized to fluctuate around a mean value of 1, and the standard deviation is calculated as $\text{SD}\{E_x\} = [\delta_1^2 \,\text{Var}\{W_1(t)\} + \delta_2^2 \,\text{Var}\{W_2(t)\} + \delta_3^2 \,\text{Var}\{W_3(t)\} + \delta_4^2 \,\text{Var}\{W_4(t)\}]^{1/2}/4 = (\delta_1^2 + \delta_2^2 + \delta_3^2 + \delta_4^2)^{1/2} \sqrt{t}/4 = 0.12\sqrt{t} = 0.12\sqrt{250 \times 10^{-3}} = 0.02$. We observe that these graphs resemble the graphs of direct current electric field measurements in the ionosphere given in Figure 5 of [12].

The variance of the electric field E_x, $\text{Var}\{E_x\}$, as a function of time t in each realization is shown in Figure 3(a), while the variation of the variance, $\text{Var}\{E_x\}$, with respect to the nondimensional fluctuation of the permittivity $\Delta\epsilon(t)/(\epsilon_0 \langle \epsilon_r \rangle)$ in each realization is shown in Figure 3(b). We observe that the maximum value of the variance of the electric field in the four realizations is $O(10^{-3})$.

As a first example of the application of formula (61), we consider medium scale travelling ionospheric disturbances (MSTIDs) with horizontal wavelength $\lambda_x = 120\,\text{km}$ and vertical wavelength $\lambda_z = 90\,\text{km}$

observed in the ionospheric F region ($\rho_s \approx 10^{-10}\,\mathrm{C/m^2}$) generated by the electric field described by Eq. (53) and the magnetic field given by Eq. (60) as a result of a magnetic storm [24], then η_1 can be approximated as $\eta_1 \approx 2k_x = 2(2\pi/\lambda_x)$ while $\eta_2 \approx 2k_z = 2(2\pi/\lambda_z)$. This means that the MSTIDs

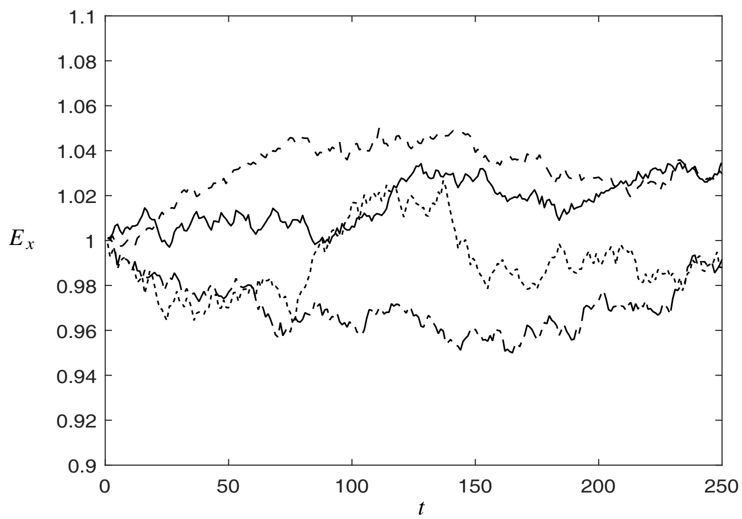

Figure 2. The horizontal component E_x of the electric field vector $\mathbf{E}$ (normalized to have a mean amplitude of 1) as a function of time t (in milliseconds), calculated according to the expression (56). In this configuration, the medium acts like an electric dynamo with a direct current [12]. The solid curve corresponds to the first realization, the dashed curve corresponds to the second realization, the dotted-dashed curve corresponds to the third realization and the dotted curve corresponds to the fourth realization.

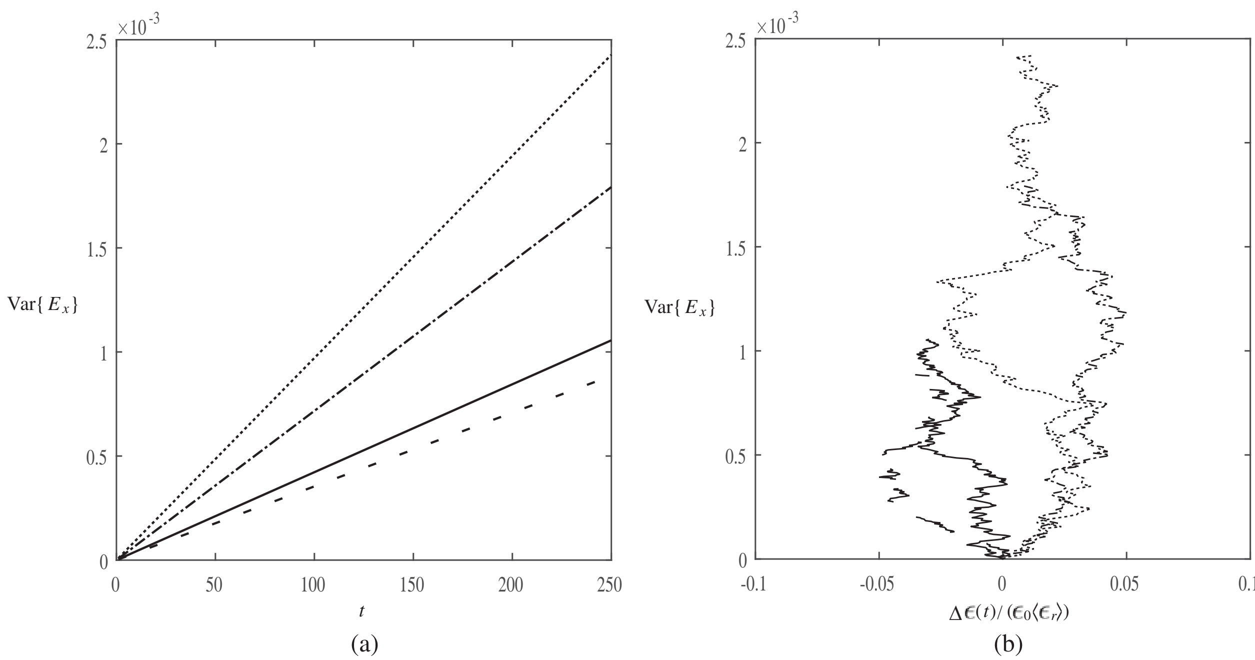

Figure 3. (a) The variance of the horizontal component E_x of the electric field vector $\mathbf{E}$, shown in Figure 2, as a function of time t (in milliseconds), calculated as $\mathrm{Var}\{E_x\} = \mathrm{Var}\{\delta_E W_E(t)\} = \delta_E^2\, t$. (b) The variation of the variance with respect to $\Delta\epsilon(t)/(\epsilon_0\langle\epsilon_r\rangle)$. The solid curve corresponds to the first realization, the dashed curve corresponds to the second realization, the dotted-dashed curve corresponds to the third realization and the dotted curve corresponds to the fourth realization.

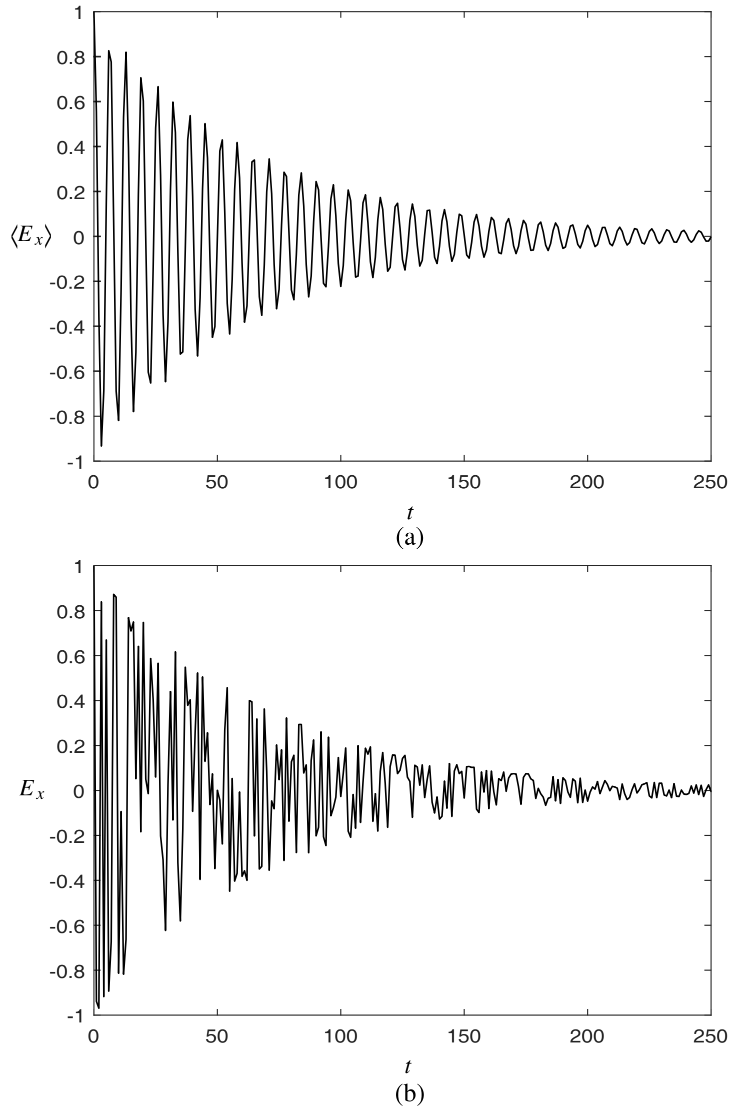

Figure 4. (a) The mean $\langle E_x \rangle$ of the horizontal component of the electric field vector $\mathbf{E}$ (normalized to have an initial amplitude of 1) as a function of time t (in milliseconds), calculated according to the expression (48), and (b) the corresponding stochastic function E_x calculated according to the expression (44). In this configuration, the solution resembles continuous pulsation waves in the ionosphere [11].

double their wavelengths as a result of the nonlinear interactions between the electromagnetic field and the medium [18]. Taking $x_1 = 0$, $x_2 = \lambda_x$ and $\langle \epsilon_r \rangle = -1.5$ [17], we obtain an induced average $\langle \text{EMF} \rangle = 45\,\text{kV}$ and a standard deviation $\text{SD}\{\text{EMF}\} = 1\,\text{kV}$, leading to a 95% confidence interval $\text{EMF} = \langle \text{EMF} \rangle \pm 2\text{SD}\{\text{EMF}\} = (45 \pm 2)\,\text{kV}$. And so, the overall average electric field vector is $\mathbf{E} = \hat{\mathbf{x}} E_x + \hat{\mathbf{z}} E_z = \hat{\mathbf{x}}\,\text{EMF}/\lambda_x + \hat{\mathbf{z}}\,\text{EMF}/\lambda_z = [(0.38 \pm 0.01)\hat{\mathbf{x}} + (0.50 \pm 0.02)\hat{\mathbf{y}}]$ mV/m as can be observed in the ionosphere [23].

As a second example of the application of formula (61), we consider large scale travelling ionospheric disturbances (LSTIDs) or electromagnetohydrodynamic disturturbances (EMHD) with horizontal wavelength $\lambda_x = 1200\,\text{km}$ and vertical wavelength $\lambda_z = 1000\,\text{km}$ observed in the ionospheric F region generated by the electric field (53) and the magnetic field (60) as a result of a magnetic storm [7, 24], make the approximations $\eta_1 \approx 2(2\pi/\lambda_x)$ and $\eta_2 \approx 2(2\pi/\lambda_z)$, and take $x_1 = 0$, $x_2 = \lambda_x$ and $\langle \epsilon_r \rangle = -1.5$ as before [17]. We then obtain the induced $\text{EMF} = (500 \pm 22)\,\text{kV}$ as might be observed in the ionosphere [23].

Solutions of the form (44), which correspond to $\lambda \neq 0$, resemble regular pulsation waves in the ionosphere [11, 19]. Figure 4(a) shows a plot of the mean of the horizontal component of the electric field $\langle E_x \rangle$ as a function of t at fixed x and z, as given by Eq. (48), while E_x given by Eq. (44) is shown in Figure 4(b). The fluctuations are represented as $\Delta\epsilon(t)/(\epsilon_0\langle\epsilon_r\rangle) = \delta_E\, W_E(t) = 0.21\, W_E(t)$ and $\Delta\mu(t)/(\mu_0\langle\mu_r\rangle) = \delta_H\, W_H(t) = 0.09\, W_H(t)$ as in Appendix B, where $W_E(t)$ and $W_H(t)$ are independent Gaussian random variables. We have set $\omega = 10^6\,\mathrm{s}^{-1}$ and $\gamma = 0.2 \times 10^6\,\mathrm{s}^{-1}$, t is represented in milliseconds, and the amplitude of the electric field has been normalized to start from an initial value of 1. The variance of E_x is asymptotically given by $\mathrm{Var}\{E_x\} \sim (0.5625\delta_E^2 + 0.0625\delta_H^2)t\exp(-2\gamma t)\exp(-2i\omega t)$, and thus quickly approaches zero as the time t increases. According to the dispersion relation, this value of ω corresponds to waves with vertical wavelengths of the order of 10^3 m, as might be observed in the ionosphere.

On the other hand, Figure 5 shows the horizontal component of the electric field E_x for a case

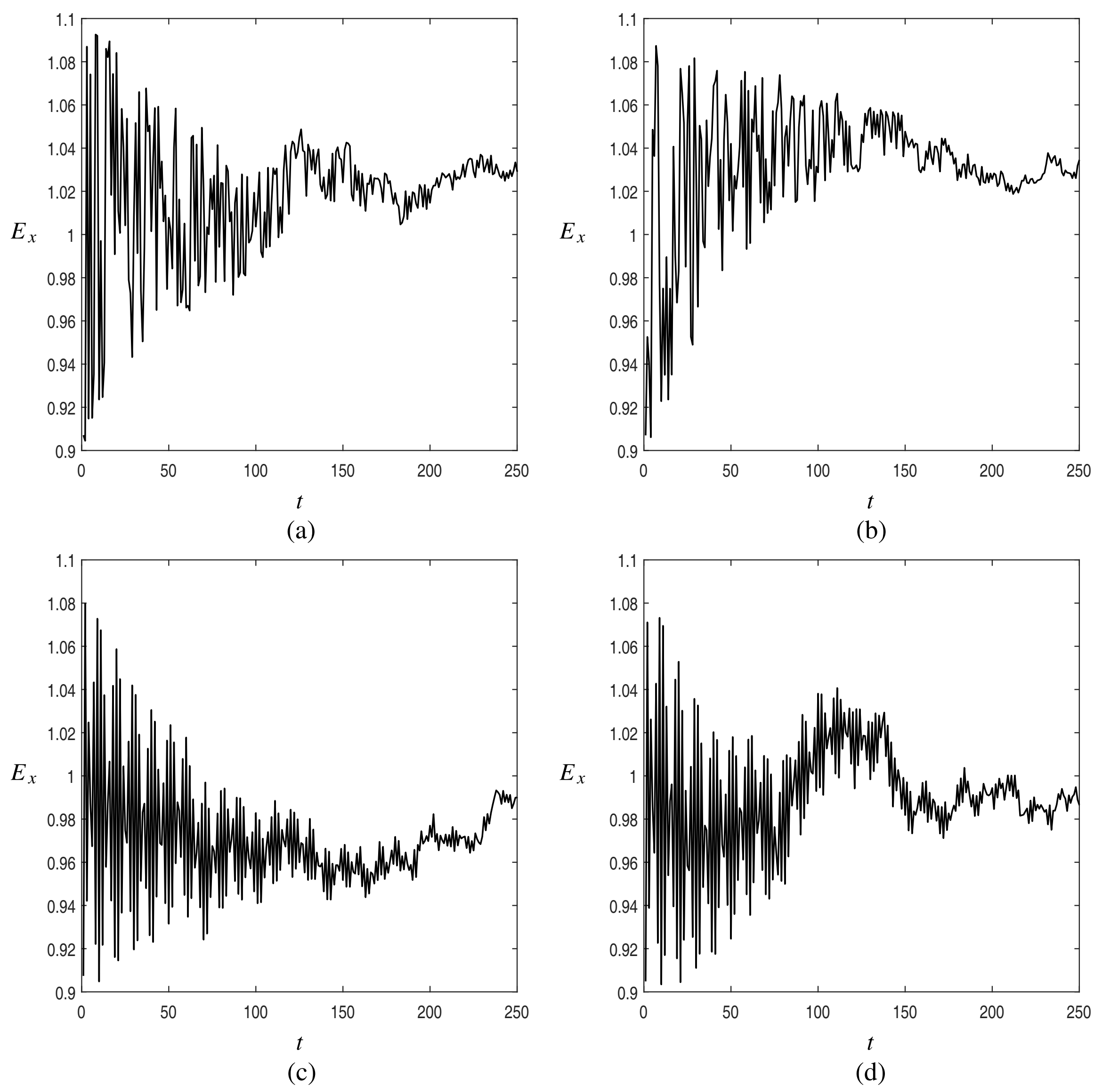

Figure 5. The horizontal component E_x of the electric field vector $\mathbf{E}$ (normalized to have a mean amplitude of 1) as a function of time t (in milliseconds), calculated as a linear combination of the expressions (44) and (56). (a) first realization, (b) second realization, (c) third realization and (d) fourth realization. In this configuration, the solution resembles continuous pulsation waves superimposed on a direct current electric field in the ionosphere [11].

where the initial conditions are such that the solution shown is a linear combination of the functions shown in Figures 2 and 4(b) and thus represents a single wave superimposed on a direct current electric field. The coefficient of the wave term in the linear combination has been set to 0.1, and that of the direct current term has been set to 1. We observe that the graphs shown in Figures 4(b) and 5 resemble that in Figure 2 of [11] which shows the intermittent time evolution of the electromagnetic field for continuous pulsation waves in the ionosphere. The variance of the electromagnetic wave mean amplitude is negligible compared to that of the direct current dynamo term shown in Figure 3. Therefore, the variance of E_x in Figure 5 is asymptotically given by $\mathrm{Var}\{E_x\} \sim \delta_E^2 t$, the variance of the direct current electric field shown in Figure 3.

5. CONCLUSIONS

In this paper, we obtained a version of Kramers-Kronig relations for a time-dependent dispersive medium taking into consideration the derivative of the convolution integrals in Eqs. (6)–(7). We also showed that for a weakly-dispersive medium with small noncausal events, the convolution integrals in Eqs. (6)–(10) become multiplicative relations as in dynamic time-dependent media, relations given by Eq. (14).

We derived solutions to electromagnetic wave equations in an isotropic medium where the permittivity and permeability are weakly-random functions of time. We described some configurations that lead to solutions that resemble electromagnetic fields in the ionosphere, in particular a situation where the ionosphere behaves like an electric dynamo with a direct current and a case where there are continuous pulsation waves. These special cases were illustrated in Figures 2–5. We also obtained a formula for the EMF in terms of the ionospheric disturbance characteristics (wavenumber or wavelength) by means of Faraday's law of induction, Equation (61).

We also note that the graphs shown in Figures 2–5 resemble electromagnetic field signals that could be obtained in the other contexts involving wave propagation in a medium with weakly-random variations in time, such as the bio-electromagnetic fluctuations from the human brain [2] that are measured using the EEG method [1, 16]. Configurations similar to those described here could be used to model these other types of waves.

APPENDIX A. NOTATIONS FOR THE PHYSICAL PARAMETERS AND ABBREVIATIONS

A list of the main notations for the physical parameters and abbreviations used throughout the paper:

$\langle\ \rangle$	:	mean
Δ	:	temporal fluctuations about the mean
Var	:	Variance
SD	:	standard deviation
ϵ	:	permittivity of the medium
ϵ_r	:	relative permittivity
ϵ_0	:	permittivity of the free space
ϵ_{re}	:	the real part of the permittivity
ϵ_{im}	:	the imaginary part of the permittivity
μ	:	permeability of the medium
μ_r	:	relative permeability
μ_0	:	permeability of the free space
μ_{re}	:	the real part of the permeability
μ_{im}	:	the imaginary part of the permeability
σ	:	conductivity of the medium
σ_{re}	:	the real part of the conductivity
σ_{im}	:	the imaginary part of the conductivity

ρ_s	:	surface charge density
n	:	refractive index of the medium
k_x	:	horizontal wavenumber
k_z	:	vertical wavenumber
λ_x	:	horizontal wavelength
λ_z	:	vertical wavelength
EMF	:	electromotive force
EHD	:	electrohydrodynamics disturbances
MHD	:	magnetohydrodynamics disturbances
EMHD	:	electromagnetohydrodynamics disturbances
TIDs	:	travelling ionospheric disturbances
MSTIDs	:	medium scale travelling ionospheric disturbances
LSTIDs	:	large scale travelling ionospheric disturbances
W	:	standard Wienner process
WKB	:	Wentzel-Kramers-Brillouin (methods)

APPENDIX B. STANDARD WIENER PROCESS REPRESENTATION OF THE PROPERTIES OF THE MEDIUM ϵ AND μ

A standard Wiener process $W(t)$, $0 \leq t \leq t_{\max}$ is a Gaussian random variable satisfying the properties [26]:

(i) $W(0) = 0$ with probability 1.

(ii) If $0 < s < t < t_{\max}$, then the random variable $\Delta W = W(t) - W(s)$ is normally distributed with mean 0 and variance $t - s$, and satisfies

$$\Delta W \sim \sqrt{t-s}N(0,1). \tag{B1}$$

(iii) If $0 < s < t < v < w < t_{\max}$, $\Delta W_1 = W(t) - W(s)$ and $\Delta W_2 = W(w) - W(v)$, then ΔW_1 and ΔW_2 are independent.

Note that if $s = 0$, then the variance of $\Delta W = W(t) - W(s) = W(t) - W(0)$ is simply t.

In a medium with weakly-random fluctuations in time as defined by Eq. (29), if $\Delta\epsilon(0) = 0$ and $\Delta\mu(0) = 0$ and we define $\Delta\epsilon(t)$ and $\Delta\mu(t)$ so that each has a variance of t, then $\Delta\epsilon(t)$ and $\Delta\mu(t)$ can be represented as standard Wiener processes by

$$\frac{\Delta\epsilon(t)}{\langle\epsilon\rangle} = \delta_E W_E(t) \quad \text{and} \quad \frac{\Delta\mu(t)}{\langle\mu\rangle} = \delta_H W_H(t), \tag{B2}$$

where δ_E and δ_H are small constants.

APPENDIX C. WENTZEL-KRAMERS-BRILLOUIN (WKB) METHOD

For reference, on the interval $0 \leq t < \infty$, the second order ordinary differential equation

$$y''(t) - c^2 P(t) y(t) = 0, \tag{C1}$$

where c is a constant and $P(t) \neq 0$, has the asymptotic WKB solution [4], valid for $c \gg 1$

$$y(t) \sim d_1 [P(t)]^{-1/4} e^{c\int_0^t \sqrt{P(u)}du} + d_2 [P(t)]^{-1/4} e^{-c\int_0^t \sqrt{P(u)}du}, \tag{C2}$$

where d_1 and d_2 are constants.

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