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Scattering Matrix of 2*N*-Port Hybrid Directional Couplers

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Abstract—The derivation of the scattering matrix of hybrid directional couplers with more than four ports is rather difficult to find in the literature. Some particular cases can be found, but a general form is not yet discussed. The aim of this contribution is to develop a simple procedure to write the $2N \times 2N$ S-matrix for hybrid directional couplers with N input and N output ports. This procedure is based on the separation of the phase of the scattering coefficients in two terms. The first is related to the presence of transmission lines, or phase shifters, connected to the coupler ports and the second to the intrinsic nature of the coupler that imposes particular phase relationships to the scattering coefficients to ensure that the S-matrix is unitary. These relationships are due to the presence of one polyphase systems of order N or to m polyphase subsystems of order N/m, if N is multiple of m. Finally, it will be shown that $2N$ port hybrid directional couplers with phase shift equal to 0 or π are possible only if N is an integer power of 2.

1. INTRODUCTION

Hybrid directional couplers with N input ports and N output ports have been used since '50 s to obtain microwave devices able to feed output ports with proper amplitude and phases. Typical applications can be found in the Beam Forming Network of antenna arrays [1–5], or in interferometric applications [6, 7] and measurement techniques. The main requirement is that all ports are matched, and the output ports are excited with the same power. The phase can be adjusted with phase shifters connected to the output ports to obtain the desired values. While the request on the amplitudes can be "translated" in the scattering coefficients of the $2N \times 2N$ S-matrix in a very simple way, i.e., $|S_{i,j}| = |S_{j,i}| = \frac{1}{\sqrt{N}}$, $i = 1, \dots, N$, $j = N+1, \dots, 2N$, the expressions for the scattering coefficient phases are more involved. Hence, is there a simple way to write the phase of the S-matrix of a hybrid directional coupler with N input and N output ports? Many researchers have evaluated the S-matrix of hybrid couplers in some cases, for example for 3×3 or 4×4 cases [2,8,9], but, to the Author's knowledge, a general formulation for the generic case has not been proposed. Actually, the S-matrix representing a Butler matrix, used to design BFN, is strictly related to the scattering matrix of hybrid couplers [10], but this S-matrix does not yet represent the general case. In fact, typical limitation of the Butler matrix is that N is an integer power of two.

The aim of this contribution is to define a simple procedure that permits to quickly write the $2N \times 2N$ S-matrix (in amplitude and phase) of a hybrid directional coupler with N input and N output ports, as will be shown in some examples for $N = 5$ and $N = 6$.

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2. THEORY

The S-matrix of an ideal hybrid directional coupler with N input and N output matched ports, excited with the same power, is ($^{\intercal}$ stands for transpose)

$$S=\begin{bmatrix} 0_{N\times N} & S^{\text{io}}_{N\times N} \\ \left(S^{\text{io}}\right)^{\intercal}_{N\times N} & 0_{N\times N} \end{bmatrix}_{2N\times 2N} \tag{1}$$

where S^{io} contains the input-to-output scattering coefficients

$$S^{\text{io}}=\frac{1}{\sqrt{N}}\begin{bmatrix} e^{j\varphi_{1,N+1}} & \dots & e^{j\varphi_{1,2N}} \\ \vdots & e^{j\varphi_{i,N+k}} & \vdots \\ e^{j\varphi_{N,N+1}} & \dots & e^{j\varphi_{N,2N}} \end{bmatrix}_{N\times N} \tag{2}$$

It should be noted that matrix S^{io} could be not symmetric. In fact, reciprocity of the coupler requires only that the block 2, 1 of the S-matrix in Eq. (1) is the transpose of the block 1, 2.

The condition that Eq. (1) must satisfy to be well defined is that the matrix representing the complex power, CP, is unitary, i.e., $CP = S \cdot S^{\dagger} = I$, where $\dagger$ stands for transpose and conjugate. This condition implies a number of relationships between the scattering coefficient phases, which could be very complex. In fact, elements i, k of CP must satisfy

$$CP_{ik}=\frac{1}{N}\sum_{m=1}^{N} e^{j\left(\varphi_{i,N+m}-\varphi_{k,N+m}\right)}=0 \qquad \text{if} \quad \begin{cases} 1\le i\le N-1 \\ i+1\le k\le N \end{cases} \tag{3}$$

$$CP_{N+i,N+k}=\frac{1}{N}\sum_{m=1}^{N} e^{j\left(\varphi_{m,N+i}-\varphi_{m,N+k}\right)}=0 \quad \text{if} \quad \begin{cases} 1\le i\le N-1 \\ i+1\le k\le N \end{cases} \tag{4}$$

$$CP_{ii}=1 \tag{5}$$

Moreover, $CP_{ki} = CP^{*}_{ik}$ and the elements i, k of CP not included in Eqs. (3)–(5) are null by construction of the S-matrix in Eq. (1). Eq. (5) is always satisfied for any i, while Eqs.(3) and (4) can be simplified if we set

$$\varphi_{i,N+k}=\varphi_{i,2N}+\varphi_{N,N+k}-\varphi_{N,2N}+\psi_{i,N+k} \quad \text{if} \quad \begin{cases} 1\le i\le N-1 \\ 1\le k\le N-1 \end{cases} \tag{6}$$

Equation (6) states that the phase of the scattering coefficient $S^{\text{io}}_{i,N+k}$ of matrix in Eq. (2) is obtained summing the phases of the scattering coefficients of the i-th element of the last column $(i, 2N)$ to the k-th element of the last row $(N, N+k)$ minus the last element of the N-th row $(N, 2N)$ and adding an unknown phase $\psi_{i,N+k}$. From Eq. (6), matrix S^{io} in Eq. (2) becomes

$$S^{\text{io}}=\begin{bmatrix} \left[S^{\text{io}}_{ik}\right]_{(N-1)\times(N-1)} & \dfrac{e^{j\varphi_{1,2N}}}{\sqrt{N}} \\ & \vdots \\ \dfrac{e^{j\varphi_{N,N+1}}}{\sqrt{N}} \quad \dots & \dfrac{e^{j\varphi_{N,2N}}}{\sqrt{N}} \end{bmatrix}_{N\times N} \tag{7}$$

where $\left[S^{\text{io}}_{ik}\right]$ is $(N-1)\times(N-1)$ matrix whose generic element i, k is equal to

$$S^{\text{io}}_{ik}=\frac{e^{j\varphi_{i,N+k}}}{\sqrt{N}}=\frac{e^{j\left(\varphi_{i,2N}+\varphi_{N,N+k}-\varphi_{2N,2N}+\psi_{i,N+k}\right)}}{\sqrt{N}} \quad \text{with} \quad \begin{cases} 1\le i\le N-1 \\ 1\le k\le N-1 \end{cases} \tag{8}$$

The phase of Eq. (8) can be obtained from the knowledge of the phase of the last row and column of matrix S^{io}. Hence, the last row and column of S^{io} contain $2N-1$ independent phases that can be properly chosen in order to impose desired values to the phases of the scattering coefficients. These independent phases are related to transmission lines, or to phase shifters, connected to the coupler ports, as will be discussed further on. From Eq. (6), Eqs. (3) and (4) split in:

$$
\begin{aligned}
CP_{ik} &= \sum_{m=1}^{N} e^{j\left(\varphi_{i,N+m}-\varphi_{k,N+m}\right)} = \\
&= e^{j\left(\varphi_{i,2N}-\varphi_{k,2N}\right)} + \sum_{m=1}^{N-1} e^{j\left[\varphi_{i,2N}+\varphi_{N,N+m}-\varphi_{N,2N}+\psi_{i,N+m}-\left(\varphi_{k,2N}+\varphi_{N,N+m}-\varphi_{N,2N}+\psi_{k,N+m}\right)\right]} \\
&= e^{j\left(\varphi_{i,2N}-\varphi_{k,2N}\right)} \left\{1+\sum_{m=1}^{N-1} e^{j\left(\psi_{i,N+m}-\psi_{k,N+m}\right)}\right\} = 0 \qquad \text{if} \quad \begin{cases} 1 \le i \le N-1 \\ i+1 \le k \le N-1 \end{cases}
\end{aligned} \tag{9}
$$

$$
\begin{aligned}
CP_{iN} &= \sum_{m=1}^{N} e^{j\left(\varphi_{i,N+m}-\varphi_{N,N+m}\right)} \\
&= e^{j\left(\varphi_{i,2N}-\varphi_{N,2N}\right)} + \sum_{m=1}^{N-1} e^{j\left(\varphi_{i,2N}+\varphi_{N,N+m}-\varphi_{N,2N}+\psi_{i,N+m}-\varphi_{N,N+m}\right)} \\
&= e^{j\left(\varphi_{i,2N}-\varphi_{N,2N}\right)} \left\{1+\sum_{m=1}^{N-1} e^{j\psi_{i,N+m}}\right\} = 0 \qquad \text{if} \quad \begin{cases} 1 \le i \le N-1 \\ k = N \end{cases}
\end{aligned} \tag{10}
$$

$$
\begin{aligned}
CP_{N+i,N+k} &= \sum_{m=1}^{N} e^{j\left(\varphi_{m,N+i}-\varphi_{m,N+k}\right)} = \\
&= e^{j\left(\varphi_{N,N+i}-\varphi_{N,N+k}\right)} \left\{1+\sum_{m=1}^{N-1} e^{j\left(\psi_{m,N+i}-\psi_{m,N+k}\right)}\right\} = 0 \quad \text{if} \quad \begin{cases} 1 \le i \le N-1 \\ i+1 \le k \le N-1 \end{cases}
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
CP_{N+i,2N} &= \sum_{m=1}^{N} e^{j\left(\varphi_{m,N+i}-\varphi_{m,2N}\right)} = \\
&= e^{j\left(\varphi_{N,N+i}-\varphi_{N,2N}\right)} \left\{1+\sum_{m=1}^{N-1} e^{j\psi_{m,N+i}}\right\} = 0 \qquad \text{if} \quad \begin{cases} 1 \le i \le N-1 \\ k = N \end{cases}
\end{aligned} \tag{12}
$$

Hence, from Eqs. (10) and (12), the conditions that the unknown phases ψ must satisfy are:

$$\sum_{m=1}^{N} e^{j\psi_{i,N+m}} = 0 \ \forall i \ \text{ with } \ \psi_{i,2N} = 0 \tag{13}$$

$$\sum_{m=1}^{N} e^{j\psi_{m,N+i}} = 0 \ \forall i \ \text{ with } \ \psi_{N,N+i} = 0 \tag{14}$$

Equations (13) and (14) assert that the two phase systems constituted by $\psi_{i,N+m}$, $m = 1, 2, \ldots, N$ and $\psi_{m,N+i}$, $m = 1, 2, \ldots, N$ form two polyphase systems of order N. A polyphase system of order N is a set of complex number with unit amplitude and equispaced phases with interval equal to $2\pi/N$.

If we define the $N \times N$ matrix Ψ containing the phases $\psi_{i,N+k}$ of Eqs. (13) and (14)

$$\Psi = \begin{bmatrix} \psi_{1,N+1} & \ldots & \psi_{1,2N} \\ \vdots & \vdots & \vdots \\ \ldots & \psi_{i,N+k} & \ldots \\ \vdots & \vdots & \vdots \\ \psi_{N,N+1} & \ldots & \psi_{N,2N} \end{bmatrix}_{N \times N} \tag{15}$$

and put $\psi_{i,2N} = 0$ and $\psi_{N,N+i} = 0$ in Eq. (15) as defined in Eqs. (13) and (14), matrix Ψ becomes

$$\Psi = \begin{bmatrix} \psi_{1,N+1} & \cdots & \psi_{1,2N-1} & 0 \\ \vdots & \psi_{i,N+k} & \vdots & 0 \\ \psi_{N-1,N+1} & \cdots & \psi_{N-1,2N-1} & 0 \\ 0 & \cdots & 0 & 0 \end{bmatrix}_{N\times N} \tag{16}$$

From Eq. (16), conditions Eqs. (13) and (14) imply that any row or column of Ψ represents the phases of a polyphase system of order N. With similar considerations on Eqs. (9) and (11), we can state that:

- from Eqs. (13) and (14), any column or any row of Ψ represents the phases of N-order polyphase system
- from Eq. (9), the difference between two rows of Ψ represents the phases of N-order polyphase system
- from Eq. (11), the difference between two columns of Ψ represents the phases of N-order polyphase system

Application of these conditions gives the correct numerical values of Ψ for the coupler under investigation. Moreover, we can define a matrix Φ of order $N \times N$ related to Eq. (6)

$$\Phi = \begin{bmatrix} [\Phi_{i,k}]_{(N-1)\times(N-1)} & \begin{matrix} \varphi_{1,2N} \\ \varphi_{i,2N} \end{matrix} \\ \varphi_{N,N+1} \cdots \varphi_{N,N+k} \cdots & \varphi_{N,2N} \end{bmatrix}_{N\times N} \tag{17}$$

where $[\Phi_{i,k}]$ is $(N-1) \times (N-1)$ matrix with the generic element i, k equal to

$$\Phi_{i,k} = \varphi_{i,2N} + \varphi_{N,N+k} - \varphi_{N,2N} \quad \text{with} \quad \begin{cases} 1 \le i \le N-1 \\ 1 \le k \le N-1 \end{cases} \tag{18}$$

Hence, the generic element $\Phi_{i,k}$ of Eq. (17) is obtained summing the i-th element of the last column $(i, 2N)$ of Eq. (17) to the k-th element of the last row $(N, N+k)$ minus the last element of the N-th row $(N, 2N)$, and Eq. (6) can be written as

$$\varphi_{i,N+k} = \Phi_{i,k} + \psi_{i,N+k} \quad \text{with} \quad \begin{cases} 1 \le i \le N-1 \\ 1 \le k \le N-1 \end{cases} \tag{19}$$

From Eqs. (6), (16) and (17), there are $2N-1$ "free" phases of the scattering coefficients, corresponding to the elements of the last row and the last column of the matrix Φ, defined in Eq. (17). The other phases $\varphi_{i,N+k}$ in Eq. (19) are related to these values and to the values of $\psi_{i,k}$ that must be part of a polyphase system of order N as previously discussed.

The matrix Φ can be related to the presence of $2N$ transmission lines, or phase shifters, connected to the input and output ports of the coupler. In fact, if θ_p is the electrical length of a transmission line connected to port p of the $2N$-port coupler, with $p = 1, 2, \ldots, 2N$, from Eq. (17) the following relationships hold:

$$\varphi_{i,2N} = -(\theta_i + \theta_{2N}) \tag{20}$$
$$\varphi_{N,N+k} = -(\theta_N + \theta_{N+k}) \tag{21}$$
$$\varphi_{N,2N} = -(\theta_N + \theta_{2N}) \tag{22}$$
$$\Phi_{i,k} = \varphi_{N,N+k} + \varphi_{i,2N} - \varphi_{N,2N} = -(\theta_i + \theta_{N+k}) \tag{23}$$

Hence, the phase $\Phi_{i,k}$, which is a part of the overall phase of the scattering coefficient $S_{i,k}$, is equal to the phase due to two lines connected to ports i and k of lengths θ_i and θ_{N+k}.

On the contrary, the matrix Ψ is related to the particular device we are discussing, i.e., the $2N$ port hybrid directional coupler, and it represents the coupler "kernel" that imposes particular relationships to the phases of the scattering coefficients to ensure that S-matrix is unitary.

3. EXAMPLES

For a hybrid directional coupler with 5 input and 5 output ports ($N = 5$), the polyphase system of order 5 that satisfies Eqs. (9), (11), (13) and (14) and defines the matrix Ψ in Eq. (16) is constituted by the following phases: 0, $\frac{2\pi}{5}$, $\frac{4\pi}{5}$, $-\frac{4\pi}{5}$, $-\frac{2\pi}{5}$. These phases must be placed in each column and in each row of the matrix Ψ and must satisfy the conditions previously discussed, i.e., that any row, column, difference of rows and difference of columns must form a polyphase system of order 5. The 0 phases are put by definition in the last row and in the last column of the matrix Ψ, shown in Eq. (16). Hence, we can try the following starting configuration:

$$\Psi_5 = \begin{bmatrix} \frac{2\pi}{5} & \frac{4\pi}{5} & -\frac{4\pi}{5} & -\frac{2\pi}{5} & 0 \\ \frac{4\pi}{5} & \dots & \dots & \dots & 0 \\ -\frac{4\pi}{5} & \dots & \dots & \dots & 0 \\ -\frac{2\pi}{5} & \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{\text{trial}} \tag{24}$$

The second column begins with $\frac{4\pi}{5}$, and the remaining three elements must be chosen among $\frac{2\pi}{5}$, $-\frac{4\pi}{5}$, $-\frac{2\pi}{5}$ in a proper order to ensure that the difference between the second column and the first column must be a polyphase system of order 5: a possible choice is $-\frac{2\pi}{5}$, $\frac{2\pi}{5}$, $-\frac{4\pi}{5}$:

$$\Psi_5 = \begin{bmatrix} \frac{2\pi}{5} & \frac{4\pi}{5} & -\frac{4\pi}{5} & -\frac{2\pi}{5} & 0 \\ \frac{4\pi}{5} & -\frac{2\pi}{5} & \frac{2\pi}{5} & -\frac{4\pi}{5} & 0 \\ -\frac{4\pi}{5} & \frac{2\pi}{5} & -\frac{2\pi}{5} & \frac{4\pi}{5} & 0 \\ -\frac{2\pi}{5} & -\frac{4\pi}{5} & \frac{4\pi}{5} & \frac{2\pi}{5} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{\text{final}} \tag{25}$$

In fact, the difference between the first and second columns is $\frac{2\pi}{5}$, $\frac{4\pi}{5}$, $-\frac{4\pi}{5}$, $-\frac{2\pi}{5}$, 0, that satisfy the previous condition. Analogously, for the rows 2, 3, 4 of the third and fourth columns, a choice could be $\frac{2\pi}{5}$, $-\frac{2\pi}{5}$, $\frac{4\pi}{5}$ (third column) and $-\frac{4\pi}{5}$, $\frac{4\pi}{5}$, $\frac{2\pi}{5}$ (fourth column). The obtained matrix is shown in Eq. (25). It can be verified that the conditions on the rows and columns and their differences are satisfied.

The overall S^{io} matrix corresponding to matrix Ψ in Eq. (25) can be obtained from Eq. (7)

$$S^{\text{io}} = \frac{1}{\sqrt{5}} \begin{bmatrix} e^{j(\varphi_{51}+\varphi_{15}-\varphi_{55}+\frac{2\pi}{5})} & e^{j(\varphi_{52}+\varphi_{15}-\varphi_{55}+\frac{4\pi}{5})} & e^{j(\varphi_{53}+\varphi_{15}-\varphi_{55}-\frac{4\pi}{5})} & e^{j(\varphi_{54}+\varphi_{15}-\varphi_{55}-\frac{2\pi}{5})} & e^{j\varphi_{15}} \\ e^{j(\varphi_{51}+\varphi_{25}-\varphi_{55}+\frac{4\pi}{5})} & e^{j(\varphi_{52}+\varphi_{25}-\varphi_{55}-\frac{2\pi}{5})} & e^{j(\varphi_{53}+\varphi_{25}-\varphi_{55}+\frac{2\pi}{5})} & e^{j(\varphi_{54}+\varphi_{25}-\varphi_{55}-\frac{4\pi}{5})} & e^{j\varphi_{25}} \\ e^{j(\varphi_{51}+\varphi_{35}-\varphi_{55}-\frac{4\pi}{5})} & e^{j(\varphi_{52}+\varphi_{35}-\varphi_{55}+\frac{2\pi}{5})} & e^{j(\varphi_{53}+\varphi_{35}-\varphi_{55}-\frac{2\pi}{5})} & e^{j(\varphi_{54}+\varphi_{35}-\varphi_{55}+\frac{4\pi}{5})} & e^{j\varphi_{35}} \\ e^{j(\varphi_{51}+\varphi_{45}-\varphi_{55}-\frac{2\pi}{5})} & e^{j(\varphi_{52}+\varphi_{45}-\varphi_{55}-\frac{4\pi}{5})} & e^{j(\varphi_{53}+\varphi_{45}-\varphi_{55}+\frac{4\pi}{5})} & e^{j(\varphi_{54}+\varphi_{45}-\varphi_{55}+\frac{2\pi}{5})} & e^{j\varphi_{45}} \\ e^{j\varphi_{51}} & e^{j\varphi_{52}} & e^{j\varphi_{53}} & e^{j\varphi_{54}} & e^{j\varphi_{55}} \end{bmatrix} \tag{26}$$

and matrix in Eq. (26) satisfies $S \cdot S^\dagger = I$ with S as in Eq. (1). Hence, Eq. (26) is the S^{io} matrix for hybrid directional coupler with 5 input and 5 output ports.

Some remarks must be done. There are many other matrices Ψ that solve the problem. In fact, it is easy to verify that another possible S-matrix is obtained changing the sign to Ψ. Similarly, we can exchange the first 4 columns (or the first 4 rows) to obtain other Ψ matrices satisfying the problem (last column and last row must be zero). Another solution is obtained exchanging columns i and k and rows i and k that corresponds to the exchange of port i and port k. It should be noted that some combinations could be redundant.

The choice of the phases φ_{51}, φ_{52}, φ_{53}, φ_{54}, φ_{55}, φ_{15}, φ_{25}, φ_{35}, φ_{45} is free and related to the transmission lines, or phase shifters, connected to the 10 ports, as previously discussed in Eqs. (20)–(23).

From Eq. (26), it is clear that S-matrix with real scattering coefficients (or phases equal to 0 or π) cannot exist for a 10-port hybrid coupler (5 input and 5 output ports) whichever is the choice for the lines connected to the ten ports, and such consideration can be applied to any odd value of N.

Particular solutions can be found for N even. In fact, for these cases, the polyphase system of order N contains the phase π, and the solutions of Eqs. (13) and (14) could be formed with a subset of the polyphase system of order N, made with $\frac{N}{2}$ couples $(0, \pi)$. With this choice, Eqs. (13), (14) are satisfied, while Eqs. (9), (11) could be satisfied with similar couples of phases, but this occurs only when N is a power of 2.

In fact, if $N = 2$ and $N = 4$, matrix Ψ can be written as

$$\Psi_2 = \begin{bmatrix} \pi & 0 \\ 0 & 0 \end{bmatrix} \qquad \Psi_4 = \begin{bmatrix} 0 & \pi & \pi & 0 \\ \pi & 0 & \pi & 0 \\ \pi & \pi & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{27}$$

and it can be verified that Eq. (27) produces unitary S-matrices.

On the contrary, for $N = 6$ the solution based only on couples $(0, \pi)$ cannot exist. In fact, let's suppose that the first row is

$$\text{Row 1: } [\pi \;\; \pi \;\; \pi \;\; 0 \;\; 0 \;\; 0] \tag{28}$$

The second row must contain three couples $(0, \pi)$ and must satisfy Eq. (9), which implies that the difference between the second and first rows must be made by three couples of $(0, \pi)$. If we try to write the second row changing the position of only one π

$$\text{Row 2: } [\pi \;\; \pi \;\; 0 \;\; \pi \;\; 0 \;\; 0] \tag{29}$$

the difference between the second and first rows is

$$\text{Row 1} - \text{Row 2: } [0 \;\; 0 \;\; -\pi \;\; \pi \;\; 0 \;\; 0] \tag{30}$$

and we have simultaneously introduced two values equal to π at places three and four of Eq. (30). Hence, any change of 0 in π, or vice versa, in the values of the second row introduces two π values in Eq. (30). Therefore, we can never meet the requirement of the presence of exactly three π and three 0 in the difference between the first and second rows, shown in Eq. (30). Hence, a coupler with $N = 6$, or integer multiples, can never be obtained with a matrix Ψ made only by couples of $(0, \pi)$. With similar considerations, it can be shown that a matrix Ψ made by $\frac{N}{2}$ couples of $(0, \pi)$ can be obtained only if N is an integer power of 2. For these cases, the S-matrix could be real, if the matrix Φ has the last row and column made by 0 or π. The couples $(0, \pi)$ can be recognized as a polyphase system of order 2.

As a consequence of this brief discussion about the couples of solutions $(0, \pi)$ when N is even, it can be shown that the solutions of Eqs. (9), (11), (13), (14) could be obtained by m polyphase subsystems of order $\frac{N}{m}$, if N is multiple of m, with m integer. For example, for $N = 6$, we could obtain solutions based on one polyphase full system of order 6, or two polyphase subsystem of order 3 or three polyphase subsystem of order 2. In fact, one possible matrix Ψ is

$$\Psi_6 = \begin{bmatrix} 0 & \frac{2\pi}{3} & \frac{2\pi}{3} & -\frac{2\pi}{3} & -\frac{2\pi}{3} & 0 \\ 0 & -\frac{2\pi}{3} & -\frac{2\pi}{3} & \frac{2\pi}{3} & \frac{2\pi}{3} & 0 \\ \pi & 0 & \pi & 0 & \pi & 0 \\ \pi & \frac{2\pi}{3} & -\frac{\pi}{3} & -\frac{2\pi}{3} & \frac{\pi}{3} & 0 \\ \pi & -\frac{2\pi}{3} & \frac{\pi}{3} & \frac{2\pi}{3} & -\frac{\pi}{3} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{31}$$

and the corresponding S^{io} matrix is

$$S^{\text{io}} = \frac{1}{\sqrt{6}} \cdot$$
$$\begin{bmatrix}
e^{j(\varphi_{61}+\varphi_{16}-\varphi_{66})} & e^{j(\varphi_{62}+\varphi_{16}-\varphi_{66}+\frac{2\pi}{3})} & e^{j(\varphi_{63}+\varphi_{16}-\varphi_{66}+\frac{2\pi}{3})} & e^{j(\varphi_{64}+\varphi_{16}-\varphi_{66}-\frac{2\pi}{3})} & e^{j(\varphi_{65}+\varphi_{16}-\varphi_{66}-\frac{2\pi}{3})} & e^{j\varphi_{16}} \\
e^{j(\varphi_{61}+\varphi_{26}-\varphi_{66})} & e^{j(\varphi_{62}+\varphi_{26}-\varphi_{66}-\frac{2\pi}{3})} & e^{j(\varphi_{63}+\varphi_{26}-\varphi_{66}-\frac{2\pi}{3})} & e^{j(\varphi_{64}+\varphi_{26}-\varphi_{66}+\frac{2\pi}{3})} & e^{j(\varphi_{65}+\varphi_{26}-\varphi_{66}+\frac{2\pi}{3})} & e^{j\varphi_{26}} \\
-e^{j(\varphi_{61}+\varphi_{36}-\varphi_{66})} & e^{j(\varphi_{62}+\varphi_{36}-\varphi_{66})} & -e^{j(\varphi_{63}+\varphi_{36}-\varphi_{66})} & e^{j(\varphi_{64}+\varphi_{36}-\varphi_{66})} & -e^{j(\varphi_{65}+\varphi_{36}-\varphi_{66})} & e^{j\varphi_{36}} \\
-e^{j(\varphi_{61}+\varphi_{46}-\varphi_{66})} & e^{j(\varphi_{62}+\varphi_{46}-\varphi_{66}+\frac{2\pi}{3})} & e^{j(\varphi_{63}+\varphi_{46}-\varphi_{66}-\frac{\pi}{3})} & e^{j(\varphi_{64}+\varphi_{46}-\varphi_{66}-\frac{2\pi}{3})} & e^{j(\varphi_{65}+\varphi_{46}-\varphi_{66}+\frac{\pi}{3})} & e^{j\varphi_{46}} \\
-e^{j(\varphi_{61}+\varphi_{56}-\varphi_{66})} & e^{j(\varphi_{62}+\varphi_{56}-\varphi_{66}-\frac{2\pi}{3})} & e^{j(\varphi_{63}+\varphi_{56}-\varphi_{66}+\frac{\pi}{3})} & e^{j(\varphi_{64}+\varphi_{56}-\varphi_{66}+\frac{2\pi}{3})} & e^{j(\varphi_{65}+\varphi_{56}-\varphi_{66}-\frac{\pi}{3})} & e^{j\varphi_{56}} \\
e^{j\varphi_{61}} & e^{j\varphi_{62}} & e^{j\varphi_{63}} & e^{j\varphi_{64}} & e^{j\varphi_{65}} & e^{j\varphi_{66}}
\end{bmatrix} \tag{32}$$

The difference between the columns of Ψ, reported in the first row of Eq. (33), is

$$\Delta\Psi_{\text{col}} = \begin{array}{c}
\begin{array}{cccccccccc} 2-1 & 3-1 & 4-1 & 5-1 & 3-2 & 4-2 & 5-2 & 4-3 & 5-3 & 5-4 \end{array} \\
\begin{pmatrix}
\frac{2\pi}{3} & \frac{2\pi}{3} & -\frac{2\pi}{3} & -\frac{2\pi}{3} & 0 & \frac{2\pi}{3} & \frac{2\pi}{3} & \frac{2\pi}{3} & \frac{2\pi}{3} & 0 \\
-\frac{2\pi}{3} & -\frac{2\pi}{3} & \frac{2\pi}{3} & \frac{2\pi}{3} & 0 & -\frac{2\pi}{3} & -\frac{2\pi}{3} & -\frac{2\pi}{3} & -\frac{2\pi}{3} & 0 \\
-\pi & 0 & -\pi & 0 & \pi & 0 & \pi & -\pi & 0 & \pi \\
-\frac{\pi}{3} & \frac{2\pi}{3} & \frac{\pi}{3} & -\frac{2\pi}{3} & -\pi & \frac{2\pi}{3} & -\frac{\pi}{3} & -\frac{\pi}{3} & \frac{2\pi}{3} & \pi \\
\frac{\pi}{3} & -\frac{2\pi}{3} & -\frac{\pi}{3} & \frac{2\pi}{3} & \pi & -\frac{2\pi}{3} & \frac{\pi}{3} & \frac{\pi}{3} & -\frac{2\pi}{3} & -\pi \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\end{array} \tag{33}$$

while the difference between the rows, reported in the first column of Eq. (34), is

$$\Delta\Psi_{\text{rows}} = \begin{pmatrix}
\text{2-1} & 0 & \frac{2\pi}{3} & \frac{2\pi}{3} & -\frac{2\pi}{3} & -\frac{2\pi}{3} & 0 \\
\text{3-1} & \pi & -\frac{2\pi}{3} & \frac{\pi}{3} & \frac{2\pi}{3} & -\frac{\pi}{3} & 0 \\
\text{4-1} & \pi & 0 & -\pi & 0 & \pi & 0 \\
\text{5-1} & \pi & \frac{2\pi}{3} & -\frac{\pi}{3} & -\frac{2\pi}{3} & \frac{\pi}{3} & 0 \\
\text{3-2} & \pi & \frac{2\pi}{3} & -\frac{\pi}{3} & -\frac{2\pi}{3} & \frac{\pi}{3} & 0 \\
\text{4-2} & \pi & -\frac{2\pi}{3} & \frac{\pi}{3} & \frac{2\pi}{3} & -\frac{\pi}{3} & 0 \\
\text{5-2} & \pi & 0 & \pi & 0 & -\pi & 0 \\
\text{4-3} & 0 & \frac{2\pi}{3} & \frac{2\pi}{3} & -\frac{2\pi}{3} & -\frac{2\pi}{3} & 0 \\
\text{5-3} & 0 & -\frac{2\pi}{3} & -\frac{2\pi}{3} & \frac{2\pi}{3} & \frac{2\pi}{3} & 0 \\
\text{5-4} & 0 & \frac{2\pi}{3} & \frac{2\pi}{3} & -\frac{2\pi}{3} & v-\frac{2\pi}{3} & 0
\end{pmatrix} \tag{34}$$

As predicted, in Eqs. (31)–(34) we can recognize exactly:

- one polyphase full system of order 6 at rows 4, 5 and at columns 3, 5 of (31), at columns 2-1, 4-1, 5-2, 4-3 of (33) and at rows 3-1, 5-1, 3-2 and 4-2 of (34).
- two polyphase subsystems of order 3 at rows 1, 2 and at columns 2, 4 of (31), at columns 3-1, 5-1, 4-2, 5-3 of(33) and at rows 2-1, 4-3, 5-3 and 5-4 of (34).

- three polyphase subsystems of order 2 at row 3 and at column 1 of (31), at columns 3-2 and 5-4 of (33) and at rows 4-1, and 5-2 of (34).

As previously discussed, many other Ψ matrices can be found from Eq. (31), changing sign, or the position of some columns or rows or exchanging the port order.

A very simple code with a few lines can be implemented to obtain all possible Ψ matrices describing the correct relationships between the phases of the S-matrix of hybrid couplers.

4. NUMERICAL RESULTS

To verify the relationships between the elements of the matrix Ψ, representing the device "kernel" that imposes particular relationships to the phases of the scattering coefficients to ensure that S-matrix is unitary, two numerical simulations have been implemented. In fact, the 4-port and 6-port couplers shown in Figs. 1 and 3 have been designed as discussed in [9], and their S-matrices have been evaluated with CST.

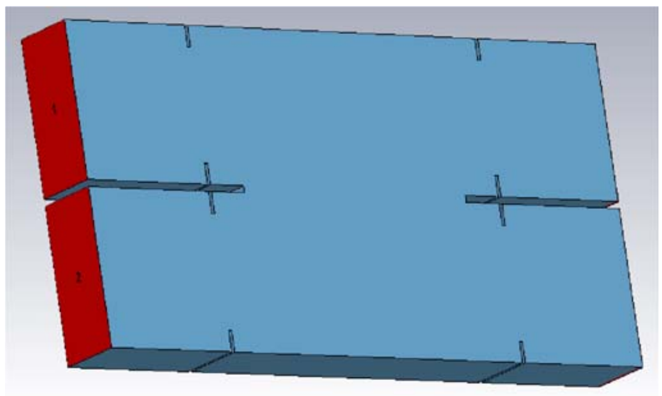


Figure 1. The 4-port coupler discussed in [9].

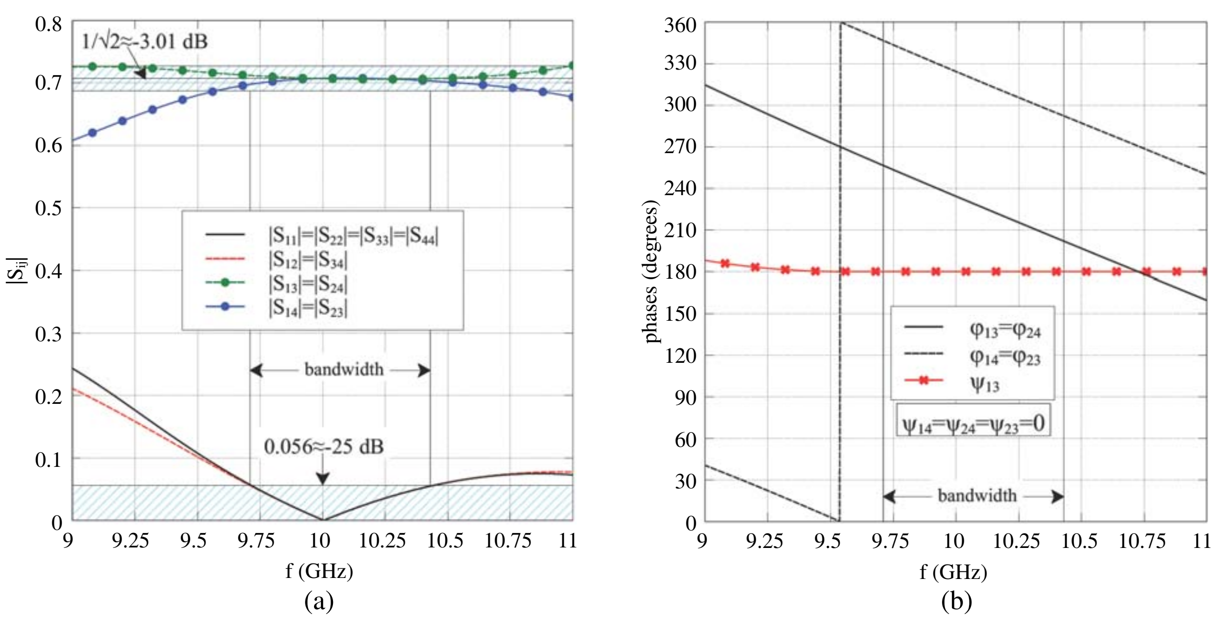


Figure 2. Amplitudes $|S_{ij}|$ (a) and phases φ_{ij} (b) of the scattering coefficients of the 4-port coupler shown in Fig. 1. Elements ψ_{ij} of the matrix Ψ are shown in (b).

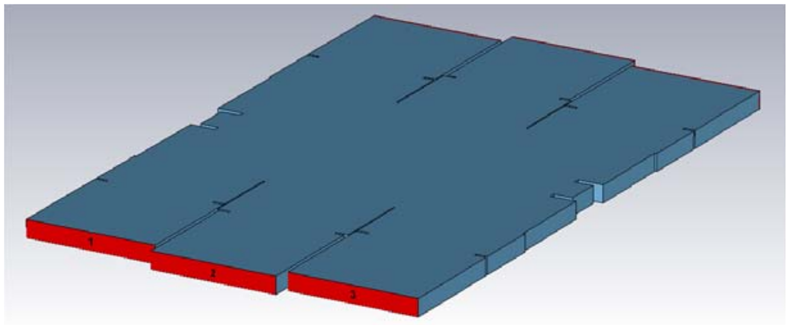


Figure 3. The 6-port coupler discussed in [9].

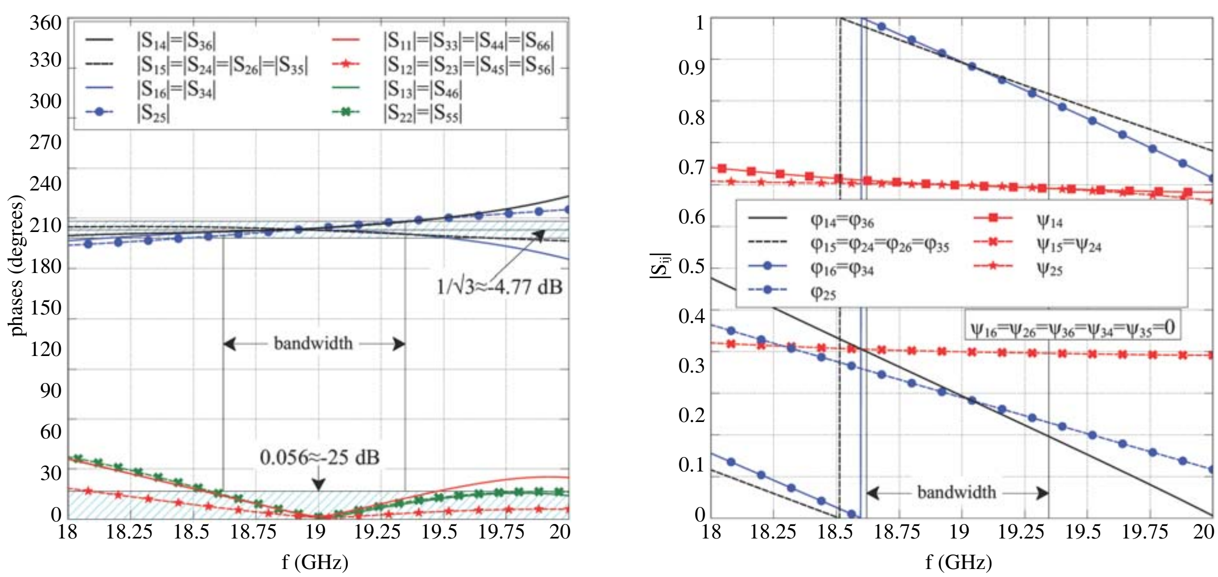
Amplitudes $|S_{ij}|$ and phases φ_{ij} of the scattering coefficients of the 4-port coupler are shown in Figs. 2(a) and 2(b). The bandwidth is obtained in the hypothesis of $|S_{11}| < -25\,\text{dB}, |S_{12}| < -25\,\text{dB}$, and $|S_{13}|, |S_{14}|$ are equal to $-3.01 \pm 0.25\,\text{dB}$, as shown in Fig. 2(a) with cyan areas.

As previously discussed, matrix Ψ can be easily evaluated from Eq. (6). Hence, from the knowledge of the overall 4-port coupler phases φ_{ij}, shown in Fig. 2(b) with black continuous and dashed lines, we can evaluate the value of ψ_{13}

$$\psi_{13} = \varphi_{13} - (\varphi_{14} + \varphi_{23} - \varphi_{24}) \tag{35}$$

that is shown in Fig. 2(b) with red line with crosses. It is evident that ψ_{13} is very close to 180° in the whole band of the coupler. The other values of Ψ are equal to zero by definition. Hence, it is verified by this simulation that matrix Ψ_2 is equal to Eq. (16) or (27).

Similar approach can be used for the 6-port coupler shown in Fig. 4. Amplitudes $|S_{ij}|$ and phases φ_{ij} of the scattering coefficients are shown in Fig. 4(a) and in Fig. 4(b), together with the band of the coupler ($|S_{11}|, |S_{12}|, |S_{13}| < -25\,\text{dB}$ and $|S_{14}|, |S_{15}|, |S_{16}| = -4.77 \pm 0.25\,\text{dB}$, highlighted with cyan areas).



(a) (b)

Figure 4. Amplitudes $|S_{ij}|$ (a) and phases φ_{ij} (b) of the scattering coefficients of the 6-port coupler shown in Fig. 4. Elements ψ_{ij} of the matrix Ψ are shown in (b).

The elements of the Ψ matrix are obtained from Eq. (6), similar to the 4-port case,

$$\psi_{14} = \varphi_{14} - (\varphi_{16} + \varphi_{34} - \varphi_{36}) \tag{36}$$
$$\psi_{15} = \varphi_{15} - (\varphi_{16} + \varphi_{35} - \varphi_{36}) \tag{37}$$
$$\psi_{24} = \varphi_{24} - (\varphi_{26} + \varphi_{34} - \varphi_{36}) \tag{38}$$
$$\psi_{25} = \varphi_{25} - (\varphi_{26} + \varphi_{35} - \varphi_{36}) \tag{39}$$
$$\tag{40}$$

that are equal to $\psi_{14} = \psi_{25} = 240^\circ = \frac{4\pi}{3}$, $\psi_{15} = \psi_{24} = 120^\circ = \frac{2\pi}{3}$ at 10 GHz, maintaining almost constant in the band of the coupler. Hence, the Ψ matrix is

$$\Psi_3 = \begin{bmatrix} \frac{4\pi}{3} & \frac{2\pi}{3} & 0 \\ \frac{2\pi}{3} & \frac{4\pi}{3} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{41}$$

and exactly satisfies the conditions previously discussed. It is also easy to verify that the differences between rows or columns of Eq. (41) give the phases of 3-order polyphase system.

5. CONCLUSIONS

A simple procedure to write the scattering matrix of hybrid coupler with N input ports and N output ports has been discussed. The phase of the scattering coefficients can be related to the presence of transmission lines, or phase shifters, connected to the ports and to the presence of polyphase systems of order N that impose the correct relationships between the phases of the scattering coefficients in order to ensure that the S-matrix is unitary. Many solutions can be found, but it is sufficient to write only one correct S-matrix to obtain all the others, changing the sign of the polyphase systems, the order of the rows/columns of Ψ or the order of the ports.

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