Image Intensity of a Gaussian Rough-Surface in Atmospheric Turbulence

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Abstract—Based on Huygens-Fresnel principle, a general expression for the average image intensity of a heterodyne and a direct-detection imaging system in turbulent media is derived under the assumption of a Gaussian rough-surface model. From the formulation, we find that the object size, turbulence strength, wavelength, and object roughness affect image intensity dramatically in the image plane.

1. INTRODUCTION

Laser range finders and imaging radars offer new technical options for a variety of target detection and imaging scenarios. Such systems will, of necessity, be subject to the vagaries of atmospheric optical wave propagation, i.e., absorption and scattering. For imaging through turbulence, the quality of the image is degraded by turbulence with its associated random refractive-index fluctuation. Irradiance interferometry and speckle techniques have been widely used for improved imaging quality.

Most of the research analyzed the degree of image degradation focused on the modulation transfer function (MTF). The turbulence effect on the degradation of MTF has been covered in many papers [1–5], including imaging systems in both homogeneous and inhomogenous media. They found that MTF is related to second-order statistics of the field. In contrast to most of research, Fante has studied the imaging of incoherent sources through turbulence, and the results show that high spatial frequencies are not significantly affected by the turbulence; however, the lower frequencies may be affected considerably [6]. Yang et al. have given imaging expression of point object and plane mirror in turbulent medium [7]. Wang and Plonus have discussed average intensity and signal-to-noise ratio of the field reflected by a target or radiated by an object which is usually incoherent [8]. Idell and Webster have computed coherence optical imaging resolution in spatial-frequency domain [9]. Some subsequent scholars have dealt with bispectrum which contains diffraction limited information about the space of interest in turbulence [10, 11].

It is well known that the average image of an incoherently illuminated object is blurred through turbulence. This is due in part to the fact that when the object is coherently illuminated, the analysis of imaging through turbulence is more complicated than the incoherent case [12, 13]. According to image, researchers can analyze states of targets from backgrounds [14, 15]. Many researchers discuss image intensities of point targets and diffuse targets. In practice, most targets are considered rough on the scale of an optical wavelength; it is modeled as partially diffuse, somewhere between the limiting case of a smooth reflector, and a fully diffuse target. However, to the best of our knowledge, effects of turbulence on image intensity of Gaussian rough-surface illuminated by laser have not been studied.

In this paper, we put forward a model of average image intensity of Gaussian rough-surface and analyze the effect of turbulence on image formation in the spatial domain, and the analysis is extended to the more practical case of a finite-size object defined by a general reflectivity.

2. THEORY MODEL

The problem is that of forming an intensity image of a self-luminous object that emits monochromatic light which propagates through the atmosphere and is then collected by an ideal thin lens and recorded in the image plane. Fig. 1 shows the basic geometry of the problem. Without loss of generality, we will restrict our discussion to a linear system with turbulence using Taylor's frozen-in hypothesis. $\vec{r}$ is at the object plane; $\vec{v}$ is at the lens plane; a is the thin lens radius; and $\vec{p}$ is at the image plane.
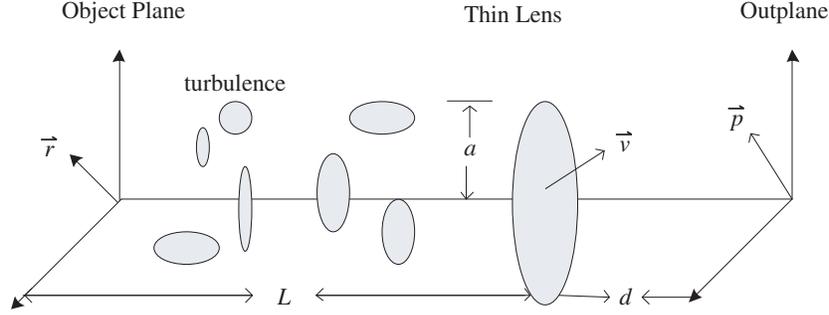

Figure 1. Geometry of the problem of imaging through turbulence.

So, we can present the field through the turbulence incident on the thin lens by the extended Huygens-Fresnel principle, that is

$$U(\vec{v}) = \frac{\exp(ikL)}{i\lambda L} \int d\vec{r}\, g(\vec{r}) \exp\left[\frac{ik}{2L}(\vec{v} - \vec{r})^2 + \psi(\vec{v}, \vec{r})\right] \tag{1}$$

where λ is the wavelength; $k = 2\pi/\lambda$ is the wavenumber; $g(\vec{r})$ is the object reflectivity function; $k = 2\pi/\lambda$ is the wavenumber; $\psi(\vec{v}, \vec{r})$ is the random field perturbation due to turbulence; and L is the distance between the object and the thin lens.

The field in the image plane is given by

$$U(\vec{p}) = \frac{\exp(ikd)}{i\lambda d} \int d\vec{v}\, U(\vec{v}) T(\vec{v}) \exp\left[\frac{ik}{2d}(\vec{p} - \vec{r})^2 - \frac{ik}{2f}\nu^2)\right] \tag{2}$$

Because lens size is smaller than the Fresnel zone $\sqrt{\lambda L}$, we can neglect the quadratic phase term of ν. d is the distance between the thin lens and image plane. The relationship between the image plane and thin lens is governed by the lens law $L^{-1} + d^{-1} = f^{-1}$. f is the focal length of the thin lens, and phase change of the thin lens is expressed by $\exp(-ik\nu^2/2f)$. The function $T(\vec{v})$ is pupil function which represents the amplitude transmission function of the aberration-free lens. For simplicity, gaussian-taper is used for the pupil function of the receiver lens, i.e.,

$$T(\vec{v}) \approx \exp\left(-\nu^2/a^2\right) \tag{3}$$

Calculations [16] show that when an undistorted plane wave is incident on the thin lens, the approximate of Eq. (3) gives an intensity distribution in the focal plane.

Substituting Eqs. (3) and (1) into Eq. (2), the intensity in the image plane can then be obtained

$$\langle I(\vec{p})\rangle = \left\langle |U(\vec{p})|^2 \right\rangle = \frac{1}{\lambda^4 L^2 d^2} \iint \iint d\vec{v}_1 d\vec{v}_2 d\vec{r}_1 d\vec{r}_2 T(\vec{v}_1) T^*(\vec{v}_2) \langle g(\vec{r}_1) g^*(\vec{r}_2)\rangle$$
$$\times \exp\left[\frac{ik}{2L}(r_1^2 - r_2^2) - \frac{ik}{L}(\vec{r}_1 \cdot \vec{v}_1 - \vec{r}_2 \cdot \vec{v}_2) - \frac{ik}{d}\vec{p} \cdot (\vec{v}_1 - \vec{v}_2)\right]$$
$$\times \left\langle \exp[\psi(\vec{v}_1, \vec{r}_1) + \psi^*(\vec{v}_2, \vec{r}_2)]\right\rangle \tag{4}$$

We assume that the statistical character of object reflectivity is independent of the fluctuations of the random field $\exp(\psi)$. For weak and moderate turbulence, the random field perturbation ψ is

assumed to be Gaussian with zero mean. The mutual coherence function of the random field can be expressed in terms of structure function D_ψ [17].

$$\langle \exp[\psi(\vec{v}_1, \vec{r}_1) + \psi^*(\vec{v}_2, \vec{r}_2)] \rangle \cong \exp\left[-\frac{1}{2}D_\psi(\vec{v}_1 - \vec{v}_2, \vec{r}_1 - \vec{r}_2)\right] \tag{5}$$

where

$$\exp\left[-\frac{1}{2}D_\psi(\vec{v}_1 - \vec{v}_2, \vec{r}_1 - \vec{r}_2)\right] \approx \exp\left(-\frac{\left|\vec{v}_1 - \vec{v}_2\right|^2 + \left|\vec{v}_1 - \vec{v}_2\right| \cdot \left|\vec{r}_1 - \vec{r}_2\right| + \left|\vec{r}_1 - \vec{r}_2\right|^2}{\rho_0^2}\right) \tag{6}$$

where $\rho_0 = (0.54C_n^2 k^2 L)^{-3/5}$ is the coherence length of a spherical wave propagating in the turbulent medium, and C_n^2 is the refractive-index structure constant. In Eq. (6) the quadratic approximation is made for the wave structure function.

Assume that a finite-size object with arbitrary degree of coherence defined by its complex reflectivity function $g(\vec{r})$ and its second-order moment is expressed by [18]

$$g(\vec{r}) = \frac{2\beta\sqrt{\pi}}{k}\exp[i\varphi(\vec{r})]\exp(-r^2/a_g^2) \tag{7}$$

where $\vec{r}$ is a transverse vector in the target plane, a_g the effective radius of the target, β a constant, and φ a random phase induced by the target.

$$\langle g(\vec{r}) \rangle = 0 \tag{8}$$

and

$$\langle g(\vec{r}_1)g^*(\vec{r}_2) \rangle = \frac{4\pi\beta^2}{k^2}\exp\left(-\frac{\left|\vec{r}_1 - \vec{r}_2\right|^2}{l_c^2}\right)\exp\left(-\frac{r_1^2 + r_2^2}{a_g^2}\right) \tag{9}$$

where $*$ denotes complex conjugate, and l_c is the correlation radius imposed on the reflected wave. We define $\beta^2 = k^2/4\pi$ for smooth targets and $\beta^2 = T_0^2/l_c^2\pi$ for fully diffuse or Lambertian target $(l_c \to 0)$, where T_0 is the reflection of the target. The first exponential function carries the object information, and the second exponential function represents the coherence function of object.

Substituting Eqs. (3), (6), and (9) into Eq. (4), the formula is given by

$$\langle I(\vec{p}) \rangle = \left\langle |U(\vec{p})|^2 \right\rangle = \frac{1}{\lambda^4 L^2 d^2}\frac{4\pi\beta^2}{k^2}\iint\iint d\vec{v}_1 d\vec{v}_2 d\vec{r}_1 d\vec{r}_2 \exp\left(-\frac{v_1^2 + v_2^2}{a^2}\right)$$

$$\times \exp\left(-\frac{\left|\vec{r}_1 - \vec{r}_2\right|^2}{l_c^2}\right)\exp\left(-\frac{r_1^2 + r_2^2}{a_g^2}\right)$$

$$\times \exp\left[\frac{ik}{2L}(r_1^2 - r_2^2) - \frac{ik}{L}(\vec{r}_1 \cdot \vec{v}_1 - \vec{r}_2 \cdot \vec{v}_2) - \frac{ik}{d}\vec{p} \cdot (\vec{v}_1 - \vec{v}_2)\right]$$

$$\times \exp\left(-\frac{\left|\vec{v}_1 - \vec{v}_2\right|^2 + \left|\vec{v}_1 - \vec{v}_2\right| \cdot \left|\vec{r}_1 - \vec{r}_2\right| + \left|\vec{r}_1 - \vec{r}_2\right|^2}{\rho_0^2}\right) \tag{10}$$

Making the change of variables

$$\vec{v}_1 - \vec{v}_2 = \vec{v}, \quad \vec{r}_1 - \vec{r}_2 = \vec{r}$$
$$\vec{v}_1 + \vec{v}_2 = \vec{V}, \quad \vec{r}_1 + \vec{r}_2 = \vec{R} \tag{11}$$

We can then obtain the average intensity received at the position p in the image plane, i.e.,

$$\langle I(\vec{p}) \rangle = \left\langle |U(\vec{p})|^2 \right\rangle = \frac{1}{\lambda^4 L^2 d^2}\frac{4\pi\beta^2}{k^2}\iint\iint d\vec{v}d\vec{V}d\vec{r}d\vec{R}\exp\left(-\frac{v^2 + V^2}{2a^2}\right)$$

$$\times \exp\left(-\frac{r^2}{l_c^2}\right)\exp\left(-\frac{r^2 + R^2}{2a_g^2}\right)$$

$$\times \exp\left[\frac{ik}{2L}(\vec{r}\cdot\vec{R}) - \frac{ik}{L}(\frac{\vec{R}+\vec{r}}{2}\cdot\frac{\vec{V}+\vec{v}}{2} - \frac{\vec{R}-\vec{r}}{2}\cdot\frac{\vec{V}-\vec{v}}{2}) - \frac{ik}{d}\vec{p}\cdot\vec{v}\right]$$

$$\times \exp\left(-\frac{v^2 + \vec{v}\cdot\vec{r} + r^2}{\rho_0^2}\right) \tag{12}$$

Carrying out the integration, Eq. (12) is simplified into to following expression

$$\langle I(\vec{p})\rangle = \frac{4\pi^2 a}{\lambda^4 L^2 d^2 \sqrt{32fmc + \frac{2(k+2ck\rho_0^2)^2}{cf\rho_0^4 L^2}}} \times \exp\left[-\frac{4c^2 fk^2\rho_0^4 L^2 p^2}{d^2(k+2ck\rho_0^2)^2 + 16c^2 mf\rho_0^4 L^2 d^2}\right] \tag{13}$$

$$c = \frac{1}{2a_g^2} + \frac{1}{l_c^2} + \frac{1}{\rho_0^2} + \frac{ak^2}{8L^2} \tag{14}$$

$$m = \frac{1}{2a^2} + \frac{1}{\rho_0^2} - \frac{1}{4c\rho_0^4} \tag{15}$$

$$f = \frac{k^2}{4cL^2} + \frac{2}{a_g^2} \tag{16}$$

Analysis of Eq. (8) shows that $\langle I(\vec{p})\rangle$ vanishes when $\rho_0 \to 0$, i.e., when turbulence becomes very strong. Therefore, the object information is totally lost in strong turbulence. This result exactly corresponds to the result given in [19] for a self-luminous object that emits incoherent monochromatic light through the turbulence.

According to above formula

$$\frac{\langle I(\vec{p})\rangle}{\langle I(0)\rangle} = \exp\left(-\frac{4c^2 fk^2\rho_0^4 L^2 p^2}{d^2(k+2ck\rho_0^2)^2 + 16c^2 mf\rho_0^4 L^2 d^2}\right) \tag{17}$$

3. NUMERICAL ANALYSIS

The averaged intensity distribution $\langle I(\vec{p})\rangle / \langle I(0)\rangle$ is computed from Eq. (17). It shows that a Gaussian profile for the average intensity is retained in the image plane after imaging through turbulence. Eq. (17)
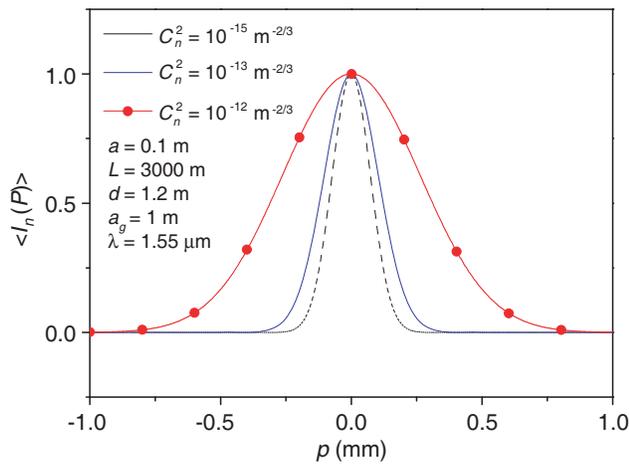

Figure 2. Average image intensity versus the lateral distance from the axis with different turbulence.
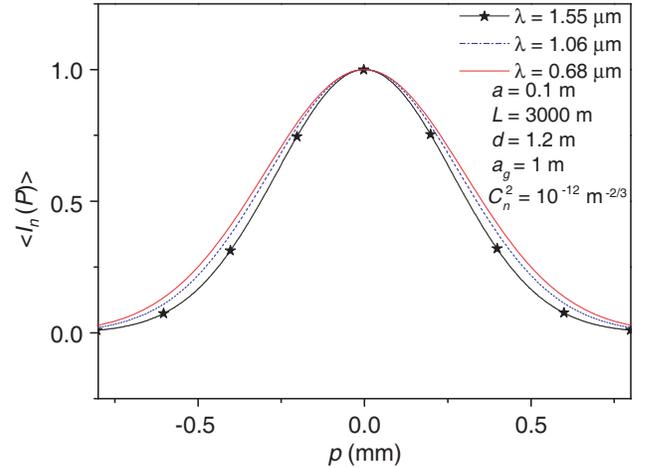

Figure 3. Average image intensity versus the lateral distance from the axis with different wave length.

shows that the average intensity is connected with object size, wavelength, target roughness, and the strength of turbulence. In Figs. 2–4, we plot average intensity as a function of off-axis distance. With the increase of off-axis distance, average image intensity will decrease gradually. The dependence of the average intensity distribution on the turbulence strength is shown in Fig. 2. The results show that with the increase of the strength of turbulence, the spot size increases, indicating the broadening effect of turbulence on the average image spot size.

Figure 3 compares average intensity of image plane with different wavelengths. From Fig. 3 we can see that with increasing wavelength, average intensity decreases gradually. It shows that a larger wavelength is less subject to turbulence than a smaller wavelength. Fig. 4 gives average image intensity $\langle I(\vec{p})\rangle / \langle I(0)\rangle$ of image plane as a function of off-axis distance with different target sizes. As the target size parameter increases, spot size becomes large, i.e., the average intensity of image plane decreases.
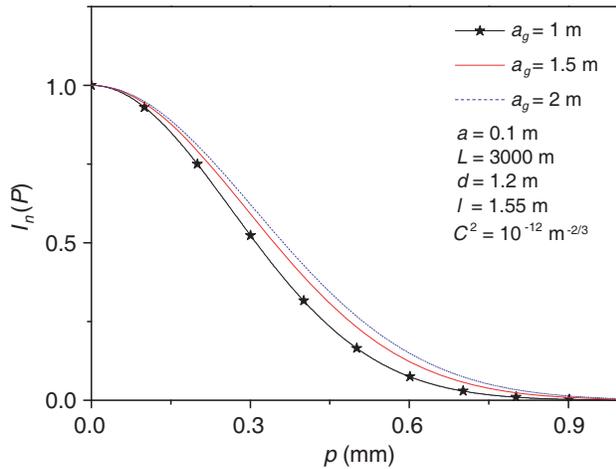

Figure 4. Average image intensity versus the lateral distance from the axis with different target size.
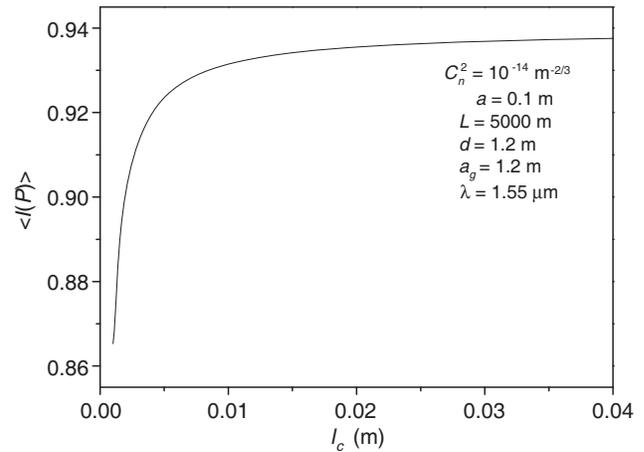
Figure 5. Average image intensity as a function of the roughness surface.

Figure 5 gives the image intensity of the ratio of turbulence to free space which is illustrated target roughness parameter. Here l_c represents the lateral correlation length of the surface, and when l_c is larger, target becomes smoother. As the target surface becomes less rough, average intensity increases as expected. When l_c increases to a point, the target becomes smooth, and average intensity maintains an almost constant value without variation. It shows that smooth target leads to a much larger image intensity.

4. CONCLUSIONS

In summary, we have derived analytical formulas for the average image intensity of the incident field reflected off arbitrary roughness target propagating through turbulent atmosphere with the help of extended Huygens-Fresnel integration and quadratic structure function. Based on the derived formulas, we present a numerical study of evolution properties of reflected beam in atmospheric turbulence on propagation collected by an ideal thin lens and recorded in the image plane. It is found that when $\rho_0 \to 0$, the image information is completely lost in turbulence. When the turbulence increases, average image will decrease. It has been found that image intensity is affected not only by wavelength but also by target size. In general, with smaller a_g and larger λ, a larger average intensity in image plane is affected by the turbulence. Additionally, target roughness affects image intensity. As the target roughness parameter l_c becomes large, image intensity increases gradually and tends to be constant.

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