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New Formulas for Calculating Torque between Filamentary Circular Coil and Thin Wall Solenoid with Inclined Axis Whose Axes are at the Same Plane

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Abstract—In this paper we present a novel approach for calculating the torque between two filamentary circular coils with inclined axes whose centers are at the same plane. In this approach we use Grover's formula for the mutual inductance between two filamentary circular coils with inclined axes whose centers are at the same plane. The filament method is applied to the combination comprising a filamentary circular coil and a thin wall solenoid. As the comparative method we give the new formula for this coil's combination which is derived from Chester Snow's formula for two solenoids with inclined axes.

1. INTRODUCTION

The calculation of the torque between two coils, carrying current, is a subject closely related to the calculation of their mutual inductance. Evidently the torque can be calculated by simple differentiation in any case where a general formula for the mutual inductance is available [1–19]. In this approach we use Grover's formula for the mutual inductance between two filamentary circular coils with inclined axes whose centers are at the same plane to calculate the torque between them. The obtained formula is obtained by the simple integration whose kernel function is expressed over the elliptic integrals of the first and second kinds [20, 21]. The Gaussian numerical integration is used. We used this formula for calculating the torque between the filamentary circular coil and the thin wall solenoid with inclined axes whose centers are at the same plane. The filament method is applied [11–13]. The thin wall solenoid is replaced by the set of thin filamentary coils. We also give a new formula for calculating the torque between this coil's combinations which is derived from Chester Snow's torque formula between two thin inclined solenoids whose centers are at the same plane [2]. This new formula is obtained over Legendre polynomials [20, 21]. Using two new formulas we obtain the results which are in an excellent agreement. These approaches could be an excellent challenge as a benchmark problem for other methods in the treatment of the magnetic torque between no coaxial and inclined circular coils whose applications are in modern medical devices, sensor, actuators, and biotelemetry (for example, an inductive powering system capable of remotely powering implantable monitoring and stimulating devices).

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2. BASIC EXPRESSIONS

2.1. Torque Formula Derived from Grover's Formula for the Mutual Inductance between Two Inclined Coaxial Coils Whose Centers Are at the Same Plane

The mutual inductance between two filamentary circular coils with inclined axes (See Fig. 1), one with radii R_P and the other with radii R_S, with distance between coils' centers c, can be calculated as [1],

$$M = \frac{\mu_0}{\pi} \cos\theta \int_0^{\pi} \frac{\sqrt{R_P R_S}\Psi(k)}{\sqrt{V^3}} d\phi \tag{1}$$

where

$$\begin{aligned}
\alpha &= \frac{R_S}{R_P} \quad \beta = \frac{c}{R_P} \\
k^2 &= \frac{4\alpha V}{1+\alpha^2+\beta^2+2\alpha\beta\cos\phi\sin\theta+2\alpha V}, \quad V = \sqrt{1-\cos^2\phi\sin^2\theta} \\
\Psi(k) &= \left[\left(\frac{2}{k}-k\right)K(k) - \frac{2}{k}E(k)\right]
\end{aligned}$$

The centers of these coils are at the same plane.

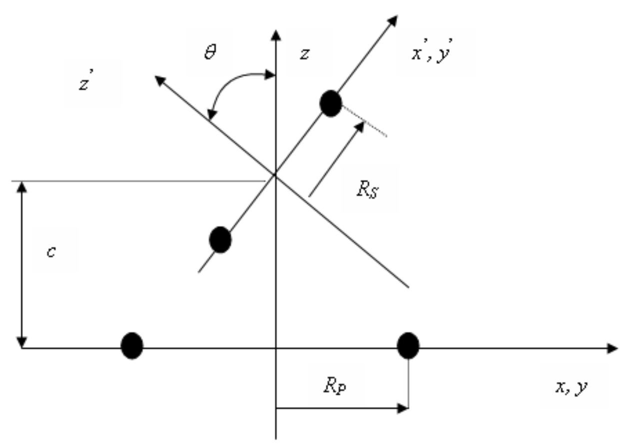

Figure 1. Filamentary circular coils with angular misalignment (centers are at the same plane).

If the coils carry currents of I_1 and I_2 the torque on either coil tending to decrease θ [2] is

$$\tau = -I_1 I_2 \frac{\partial M}{\partial \theta} \tag{2}$$

Applying Eq. (1) in Eq. (2) we obtained the torque between a filamentary circular coils whit inclined axes whose centers are at the same plane,

$$\tau = \frac{\mu_0}{2\pi} I_1 I_2 \sqrt{R_P R_S} \int_0^{\pi} \frac{F_1(\phi,\theta)\Psi(k) - F_2(\phi,\theta)\Phi(k)}{V^{7/2}} d\phi \tag{3}$$

where

$$\begin{aligned}
F_1(\phi,\theta) &= \sin\theta\,(2\sin^2\phi - \cos^2\phi\cos^2\theta) \\
F_2(\phi,\theta) &= \frac{k\cos\phi\cos^2\theta}{4\alpha V}\left[(1+\alpha^2+\beta^2)\sin\theta\cos\phi + 2\alpha\beta\right] \\
\Phi(k) &= 2K(k) - \frac{2-k^2}{1-k^2}E(k)
\end{aligned}$$

R_P — Radii of the primary coil;
R_S — Radii of the secondary coil;
c — Distance between coils' centers;
θ — Angle between coil planes;
I_1 — Current in the primary coil;
I_2 — Current in the secondary coil;
$E(k)$ — Complete elliptic integral of the first kind [20, 21];
$K(k)$ — Complete elliptic integral of the second kind [20, 21];
$\mu_0 = 4\pi \times 10^{-7}$ H/m — The magnetic permeability of vacuum.

2.2. Torque Formula Derived from Snow's Formula for the Torque between Two Inclined Concentric Solenoids

The torque calculation between two inclined concentric solenoids, whose centers are at the same plane (See Fig. 2), is given by the next formula [2],

$$\tau = \tau_0 \left\{ 1 - 2\frac{1-\mu_1^2}{\mu_1\mu_2} \sum_{s=1}^{\infty} \left(\frac{r_2}{r_1}\right)^{2s} \frac{P_{2s+1}(\mu)P'_{2s}(\mu_1)P'_{2s+2}(\mu_2)}{2s(2s+1)(2s+2)(2s+3)} \right\} \tag{4}$$

where

$$\tau_0 = \frac{\mu_0}{200b_1}\pi N_1 N_2 I_1 I_2 \mu_1 R_S^2 \sin(\theta), \quad r_1 = \sqrt{R_P^2 + b_1^2}, \quad r_2 = \sqrt{R_S^2 + b_2^2}$$
$$\mu_1 = \frac{b_1}{r_1}, \quad \mu_2 = \frac{b_2}{r_2}, \quad \mu = \cos(\theta)$$

R_P — Radii of the bigger solenoid;
R_S — Radii of the smaller solenoid;
$2b_1$ — Length of the bigger solenoid;
$2b_2$ — Length of the smaller solenoid;
θ — Angle between solenoids' axes;
I_1 — Current in the bigger solenoid;
I_2 — Current in the smaller solenoid;
N_1 — Number of turns in the bigger solenoid;
N_2 — Number of turns in the smaller solenoid;
$P_s(\mu)$ — Legendre polynomials [17, 18];
$P'_s(\mu)$ — Differential of Legendre polynomials;
($P'_s(\mu)= dP_s(\mu)/\mathrm{d}\ \mu$) [20, 21].

From formula (4) it is possible to find the formula for the torque calculation between the filamentary circular coil and the solenoids with the inclined axes whose centers are at the same plane.

We find the limit in Eq. (4) putting $b_2 \to 0$ ($\mu_2 \to 0$). After some complicated transformations we obtained a new formula for the calculation of the torque between concentric previously mentioned coils (See Fig. 3),

$$\tau = \tau_0 \left\{ 1 - \frac{1}{\mu_1} \sum_{s=1}^{\infty} S_s \right\} \tag{5}$$

where

$$\tau_0 = \frac{\mu_0}{200b_1}\pi N_1 I_1 I_2 \mu_1 R_S^2 \sin(\theta)$$
$$S_s = (-1)^s \left(\frac{l^2}{2}\right)^s \frac{(2s+1)}{(s+1)!} \frac{[\mu P_{2s+1}(\mu) - P_{2s}(\mu)][\mu_1 P_{2s}(\mu_1) - P_{2s-1}(\mu_1)]}{(1-\mu^2)}$$
$$r_1 = \sqrt{R_P^2 + b_1^2}, \quad l = \frac{R_S}{r_1}, \quad \mu_1 = \frac{b_1}{r_1}, \quad \mu = \cos(\theta)$$

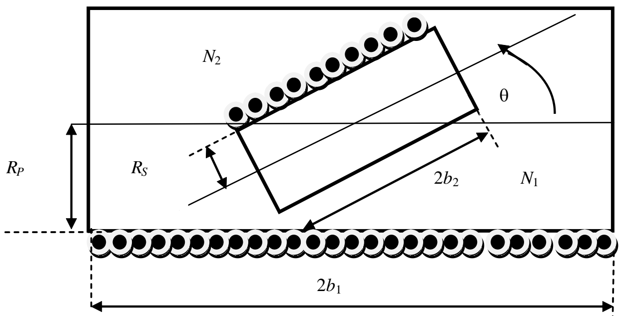

Figure 2. Configuration of mesh matrix: Two concentric thin wall solenoids with inclined exes.

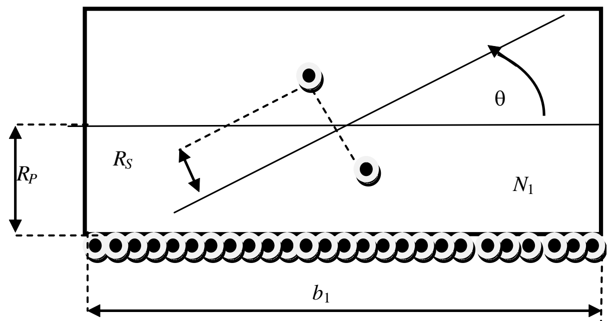

Figure 3. Configuration of mesh matrix: Concentric the filamentary coil and the solenoid with inclined axes.

R_P — Radii of the solenoid;
R_S — Radii of the filamentary circular coil;
b_1 — Length of the solenoid;
θ — Angle between inclined axes of the filamentary circular coil and the solenoid;
I_1 — Current in the solenoid;
I_2 — Current in the filamentary circular coil;
N_1 — Number of turns in the solenoid.

From the new formula (5) the torque for the filamentary circular coil and the thin wall solenoids with inclined axes may be obtained when the center of the filamentary coil lies on any points of the axis of the solenoid. The four cases can be investigated:

a) Center of the filamentary circular coil in the middle plan of the solenoid,
b) Center of the filamentary circular coil in the end plan of the solenoid,
c) Center of the filamentary circular coil on the axis of the solenoid, inside the end plane,
d) Center of the filamentary circular coil on the axis of the solenoid, outside the end plane.

We will explain how to apply each case.

Case a) For this case it is necessary to put in Eq. (5) the number of turns $N_1 = n \cdot b_1$ where n is the winding density per cm, and instead b_1 to put $b_1^{(middle)} = b_1/2$, because the filamentary circular coil is in the middle of the solenoid. Other dimensions are taken as given.

Case b) For this case it is only necessary to put in Eq. (5) the number of turns $N_1 = n \cdot (b_1)$ where n is the winding density per cm. Other dimensions are taken as given.

Case c) We calculate two torques for the values $b_1^{(1)} = b_1 - x_1 +$ and $b_1^{(2)} = x_1$, where x_1 is the distance inside the end plane. The corresponding number of turns is $N_1^{(1)} = n \cdot b_1^{(1)}$ and $N_1^{(2)} = n \cdot b_1^{(2)}$, where n is the winding density per cm. Other dimensions are taken as given. The total torque is,

$$\tau = \tau_1^{(1)} + \tau_1^{(2)} \tag{6}$$

Case d) We calculate two torques for the values $b_1^{(1)} = x_1 + b_1$ and $b_1^{(2)} = x_1$, where x_1 is the distance outside the end plane. The corresponding number of turns is $N_1^{(1)} = n \cdot b_1^{(1)}$ and $N_1^{(2)} = n \cdot b_1^{(2)}$, where n is the winding density per cm. Other dimensions are taken as given. The total torque is,

$$\tau = \tau_1^{(1)} - \tau_1^{(2)} \tag{7}$$

Thus, from new formula (5) all cases can be treated.

3. FILAMENT METHOD

The calculation of the torque between a filamentary circular coil and a solenoid whose centers lies at the axe of the solenoid.

We consider the system of the thin wall solenoid and the filamentary circular coil with inclined axes. Centers of these coils are at the axe of the solenoid (See Fig. 4). The thin wall solenoid is with N_1 number of turns of the winding. The filamentary circular coil is with one turn, $N_2 = 1$. The corresponding dimensions of this coil arrangement are shown in Fig. 3.

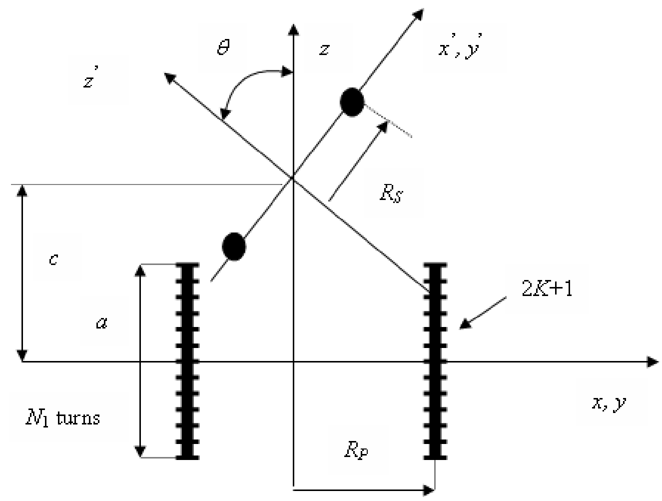

Figure 4. Configuration of mesh matrix: Thin wall solenoid — Filamentary circular coil with centers at the axe of the solenoid.

Using the same reasoning and procedures as in [4] the thin wall solenoid can be divided into $(2K + 1)$ filamentary coils. Obviously all centers of filamentary coils replacing the thin wall solenoid and the center of the thin filamentary inclined coil are lying at the axe of the solenoid. Applying the same logic for the filament method already given in [4–11] and Equation (3) the torque can be obtained in the next form,

$$\tau = \frac{N_1 I_1}{(2K+1)} \sum_{g=-K}^{g=K} \tau(g) \tag{8}$$

where

$$\begin{aligned} \tau(g) &= \frac{\mu_0}{2\pi} I_1 I_2 \sqrt{R_P R_S} \int_0^{\pi} \frac{F_1(\phi,\theta)\Psi(k) - F_2(\phi,\theta)\Phi(k)}{V^{7/2}} d\phi \\ z(g) &= c + \frac{a}{(2K+1)} g; \quad g = -K, \dots, 0, \dots, K \end{aligned}$$

R_P — Radii of the solenoid;
R_S — Radii of the filamentary circular coil;
a — Length of the solenoid;
c — Distance between coils' centers;
θ — Angle between inclined axes of the filamentary circular coil and the solenoid;
I_1 — Current in the solenoid;
I_2 — Current in the filamentary circular coil;
N_1 — Number of turns in the solenoid;
α, β, k, V, F_1, F_2, Ψ and Φ are previously defined.

4. NUMERICAL VALIDATION

Let us consider the case of a solenoid of radii 6 cm, length 12 cm, and winding density 10 turns par centimeter, so that the total number of turns is 120. A circular filament of 5 cm radii is centered at different points on the axis of the solenoid with different angles of inclination. Axes are inclined at an angle whose cosine is respectively 1, 0.9, 0.8, 0.7, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1 and 0. If the coils carry currents of strength $I_1 = I_2 = 1A$ calculate the torque between them.

Four cases will be investigated:

a1) Center of the circle in the middle plane of the solenoid,

a2) Center of the circle on the axis of the solenoid 3 cm, inside the end plane,

a3) Center of the circle in the end plane of the solenoid,

a4) Center of the circle on the axis of the solenoid 6 cm, outside the end plane.

In Tables 1, 2, 3 and 4 we give comparative results obtained by Eqs. (5) and (8) for all possible cases.

a1) Given $R_P = 6\,\mathrm{cm}$, $R_S = 5\,\mathrm{cm}$, $b_1 = 12\,\mathrm{cm}$, $c = 0$, $n = 10\,\mathrm{turns/cm}$, $I_1 = I_2 = 1A$.
We calculate the torque in the middle plane of the solenoid (**Case a**).
To apply Eq. (5) for this case: $R_P = 6\,\mathrm{cm}$, $R_S = 5\,\mathrm{cm}$, $b_1^{(middle)} = b_1/2 = 6\,\mathrm{cm}$, $N_1 = n \cdot b_1 = 120$, $I_1 = I_2 = 1A$.

Table 1. Comparison of computational efficiency.

cosθ	**τ (μNm) This Work (5)**	**τ (μNm) This Work (8)**
1.0	0.0	0.0
0.9	3.8867443	3.8867443
0.8	5.0941691	5.0941691
0.7	5.7296466	5.7296466
0.6	6.0546119	6.0546119
0.5	6.2072999	6.2072999
0.4	6.2699879	6.2699879
0.3	6.2894339	6.2894339
0.2	6.2909698	6.2909698
0.1	6.2877745	6.2877745
0.0	6.2860972	6.2860972

Table 2. Comparison of computational efficiency.

cosθ	**τ (μNm) This Work (5)**	**τ (μNm) This Work (8)**
1.0	0.0	0.0
0.9	3.8810041	3.8810041
0.8	4.7579469	4.7579469
0.7	5.1505766	5.1505770
0.6	5.3739087	5.3739105
0.5	5.5127075	5.5127073
0.4	5.6001803	5.6001771
0.3	5.6542422	5.6542393
0.2	5.6860437	5.6860435
0.1	5.7025622	5.7025648
0.0	5.7076565	5.7076602

a2) Given $R_P = 6\,\mathrm{cm}$, $R_S = 5\,\mathrm{cm}$, $a = 12\,\mathrm{cm}$, $c = 3\,\mathrm{cm}$, $n = 10\,\mathrm{turns/cm}$, $I_1 = I_2 = 1A$.
We calculate the torque inside the solenoid (**Case c**).
To apply Eq. (5) for this case: $R_P = 6\,\mathrm{cm}$, $R_S = 5\,\mathrm{cm}$, $x_1 = 3\,\mathrm{cm}$ inside the end plane. Practically we have two coils' configurations for which $b_1^{(1)} = b_1 - x_1 = 9\,\mathrm{cm}$, $N_1^{(1)} = n \cdot b_1^{(1)} = 90$ and $b_1^{(2)} = x_1 = 3\,\mathrm{cm}$, $N_1^{(2)} = n \cdot b_1^{(2)} = 30$, $I_1 = I_2 = 1A$.
The total torque is calculated by Eq. (6).

a3) Given $R_P = 6\,\mathrm{cm}$, $R_S = 5\,\mathrm{cm}$, $a = 12\,\mathrm{cm}$, $c = 6\,\mathrm{cm}$, $n = 10\,\mathrm{turns/cm}$, $I_1 = I_2 = 1A$.
We calculate the torque in the end plane of the solenoid (**Case b**).
To apply Eq. (5) for this case: $R_P = 6\,\mathrm{cm}$, $R_S = 5\,\mathrm{cm}$, $b_1 = 12\,\mathrm{cm}$. $N_1 = n \cdot (b_1) = 120$, $I_1 = I_2 = 1A$.

a4) Given $R_P = 6\,\mathrm{cm}$, $R_S = 5\,\mathrm{cm}$, $a = 12\,\mathrm{cm}$, $c = 12\,\mathrm{cm}$, $n = 10\,\mathrm{turns/cm}$, $I_1 = I_2 = 1A$.
We calculate the torque outside the solenoid (**Case d**).
To apply Eq. (5) for this case: $R_P = 6\,\mathrm{cm}$, $R_S = 5\,\mathrm{cm}$, $x_1 = 6\,\mathrm{cm}$ outside the end plane. Practically we have two coils' configurations for which $b_1^{(1)} = x_1 + b_1 = 18\,\mathrm{cm}$, $N_1^{(1)} = n \cdot b_1^{(1)} = 180$ and $b_1^{(2)} = x_1 = 6\,\mathrm{cm}$, $N_1^{(2)} = n \cdot b_1^{(2)} = 60$, $I_1 = I_2 = 1A$.

Table 3. Comparison of computational efficiency.

cosθ	**τ (μNm) This Work (5)**	**τ (μNm) This Work (8)**
1.0	0.0	0.0
0.9	2.0061582	2.0061583
0.8	2.7364249	2.7364249
0.7	3.2269131	3.2269131
0.6	3.5820105	3.5820105
0.5	3.8443818	3.8443818
0.4	4.0372549	4.0372549
0.3	4.1751271	4.1751271
0.2	4.2675447	4.2675447
0.1	4.3206730	4.3206730
0.0	4.3380045	4.3380045

Table 4. Comparison of computational efficiency.

cosθ	**τ (μNm) This Work (5)**	**τ (μNm) This Work (8)**
1.0	0.0	0.0
0.9	0.1199897	0.1199893
0.8	0.2855164	0.2855164
0.7	0.4979987	0.4979981
0.6	0.7307931	0.7307933
0.5	0.9558282	0.9558282
0.4	1.1531408	1.1531407
0.3	1.3117482	1.3117485
0.2	1.4266483	1.4266488
0.1	1.4959461	1.4959463
0.0	1.5190795	1.5190794

From Tables 1, 2, 3 and 4 we can see that all results are in an excellent agreement by using the new formulas (5) and (8). We have to mention that using formula (5) we use the same expression for treating all calculation of the torque in the inside, in the outside, in the middle plane and in the end plane of the solenoid. Using expression (8) we have to apply this formula for four cases a1), a2), a3) and a4) that was explained previously.

In Appendix A and Appendix B we give MATLAB code for formulas (5) and (8) so that the potential readers, who are not familiar with the special functions, could easily verify the new formulas for given example in the paper.

5. CONCLUSIONS

In this paper we propose new simple formulas for calculation of the torque between two filamentary circular coils with inclined axes whose centers are at the same plane. We took from the history great Grover's formula for calculating the mutual inductance between filamentary circular coils with inclined axes. Using the developed formulas and the filament method, the torque between a filamentary circular coil and a thin wall solenoid with inclined axes whose centers are at the same plane has been calculated. Formula (8) is given by a simple integral whose kernel function is expressed by the complete elliptic integrals of the first and second kinds. Also we give modified formula (5) for calculating the torque between a filamentary circular coil and a thin wall solenoid with inclined axes, which is derived from Chester Snow's formula (4) for calculating the torque between two solenoids with axes whose centers are at the same plane. This new formula is obtained by the convergent series expressed by Legendre polynomials. We find these formulas very simple and useful for calculating the torque of the inclined circular coil and short solenoid, which are currently popular in the RFID and biomedical domain, because of the accuracy and short computational time. This analytic technique can be widely applied to inductive wireless power transfer links without the limitations imposed by numerical methods.

Results obtained by these formulas are in an excellent agreement. As we know these two formulas appear in the literature for the first time. The presented methods can be used in the large scale of practical applications either for microcoils or for large coils.

APPENDIX A.

MATLAB code for the formula (5)

```
% Torque between the filamentary coil and the solenoid with
% inclined axes whose centers are
% at the same plane. This formula is derived from Chester Snow's
```

```
% formula for the torque
% calculations between two inclined solenoids whose centers are
% at the same plane
% a1 is the radii of the solenoid
% a2 is the radii of the solenoid
% b1 is the length of the solenoid
% theta is the angle between inclined axes
% N1 is the number of turns of the solenoid
% I1 is the current in the solenoid
% I2 is the current in the filamentary coil
% ------------------------------------------------------------------------------------
% Case a)
% Center of the circle in the middle plane of the solenoid
  a1 = 6;
  a2 = 5;
  b1 = 6;
  N1 = 120;
  I1 = 1;
  I2 = 1;
% ------------------------------------------------------------------------------------
% Case c)
% Calculation of the torque inside the solenoid
% a1 = 6;
% a2 = 5;
% b1 = 9;
% N1 = 120 * 9/12;
% a1 = 6;
% a2 = 5;
% b1 = 3;
% N1 = 120 * 3/12;
% I1 = 1;
% I2 = 1;
% T = T1 + T2
% ------------------------------------------------------------------------------------
% Case b)
% Calculation of the torque in the end plane of the solenoid
% a1 = 6;
% a2 = 5;
% b1 = 12;
% N1 = 120;
% I1 = 1;
% I2 = 1;
% ------------------------------------------------------------------------------------
% Case d)
% Calculation of the torque outside the solenoid
```

```
% a_1 = 6;
% a_2 = 5;
% b_1 = 18;
% N_1 = 120 * 18/12;
% a_1 = 6;
% a_2 = 5;
% b_1 = 6;
% N_1 = 120 * 6/12;
% I_1 = 1;
% I_2 = 1;
% T = T_1 - T_2
% -----------------------------------------------------------------------------
K = 70;
r_1 = sqrt(a_1.^2 +b_1.^2);
m_1 = b_1/r_1;
l = a_2/r_1;
theta = acos(0.7);
m = cos(theta);
s = 0;
for n = 1 : K
b_11 = eye(1, 2 * n + 2)*legendre((2 * n + 1), m);
b_22 = eye(1, 2 * n + 1)*legendre((2 * n), m);
s_11 = (2 * n + 1)*(m * b_11 - b_22)/(m.^2 - 1);
a_11 = eye(1, 2 * n + 1)*legendre((2 * n), m_1);
a_22 = eye(1, 2 * n)*legendre((2 * n - 1), m_1);
s_22 = 2 * n * (m_1 * a_11 - a_22)/(m_1.^2 - 1);
s_1 = (-1).^n * l.^(2 * n) * s|_11 * s_22 * factorial(2 * n);
s_2 = 2 * n * 2.^(2 * n + 1)*factorial(n)*factorial(n + 1);
V = s_1/s_2;
s = s + V;
end
T_0 = 2 * pi^2 * N_1 * a_2.^2 * m_1/b_1 * 10.^(-9);
T = T_0 * I_1 * I_2*sin(theta)*(1 - 2 * (1 - m_1^2)/m_1 * s)
```

APPENDIX B.

MATLAB code for the formula (8)

```
% Torque calculation between the filament coil and the thin wall
% solenoid with inclined axes whose centers are at the same
% plane. FILAMENT METHOD
% R_p is the radii of the solenoid
% R_s is the radii of the solenoid
% a is the length of the solenoid
% c is distance which determines the position of the filamentary % circular coil % along
% the axis of the solenoid
% theta is the angle between inclined axes
% N_1 is the number of turns of the solenoid
% I_1 is the current in the solenoid
```

```
% I_2 is the current in the filamentary coil
% -----------------------------------------------------------------------------
global l
p = 1000;
N_1 = 120;
I_1 = 1;
I_2 = 1;
mi = 4 * pi/10000000;
s = 0;
for l = -p : p
s = s+ Gauss('torque_cs_angle',0,pi,20);
end
T = mi * I_1 * I_2 * N_1/(2 * p + 1) * s
% -----------------------------------------------------------------------------
function q = torque_cs_angle(t)
global l
p = 1000;
R_p = 0.06;
R_s = 0.05;
a = 0.12;
c = 0.0;
theta = acos(0.7);
l_1 = R_s/R_p;
z = c + b_1/(2 * p + 1) * l;
d = z/R_p;
V = sqrt(1-sin(theta).^2.*cos(t).^2);
V1 = V.^3.*sqrt(V);
m = 4 * l_1. * V./(1 + l_1.^2 + d.^2 + 2 * l_1. * d.*sin(theta).*cos(t) + 2 * l_1. * V);
[k, e] = ellipke(m);
F_1 = (2./sqrt(m)-sqrt(m)).*k - 2./sqrt(m).*e;
F_2 = (2 - m)./(1 - m). * e - 2. * k;
F_11 = sin(theta).*(2*sin(t).^2-cos(theta).^2*cos(t).^2);
F_22 = sqrt(m).*cos(t).*cos(theta).^2.*((1 + l_1.^2 + d.^2).*sin(theta).*cos(t) + 2 * l_1. * d)./l_1./V./4;
q = sqrt(R_p. * R_s).*(F_1. * F_11 + F_2. * F_22)./V_1./pi./2;
% -----------------------------------------------------------------------------
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