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High-Order FDTD with Exponential Time Differencing Algorithm for Modeling Wave Propagation in Debye Dispersive Materials

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Abstract—A high-order (HO) finite-difference time-domain (FDTD) method with exponential time differencing (ETD) algorithm is proposed to model electromagnetic wave propagation in Debye dispersive material in this paper. The proposed method introduces an auxiliary difference equation (ADE) technique which establishes the relationship between the electric displacement vector and electric field intensity with a differential equation in Debye dispersive media. The ETD algorithm is applied to the displacement vector and auxiliary difference variable in time domain, and the fourth-order central-difference discretization is used in space domain. One example with plane wave propagation in a Debye dispersive media is calculated. Compared with the conventional ETD-FDTD method, the results from our proposed method show its accuracy and efficiency for Debye dispersive media simulation.

1. INTRODUCTION

Over the past decade, many FDTD algorithms for dispersive media have been proposed, such as the recursive convolution method [1], Z transform method [2], and auxiliary differential equation method [3]. However, when modeling high loss and/or large dielectric constants these methods become unstable. To improve the stability and calculation accuracy for high dielectric dispersive media, the high-order finite difference scheme in space domain and exponential time differencing algorithm in time domain have been used to model electromagnetic wave propagation [4–11]. In [4], a fourth-order accurate in space and second-order accurate in time FDTD scheme are presented to modeling wave propagation in lossy dispersive media. In [5], the stability property and numerical dispersion relation for high-order FDTD scheme with a Debye or Lorentz model are analyzed. In [4] and [5], the high-order scheme is used in space, and numerical dissipation is strongly dependent on the temporal resolution. In [6], the ETD scheme for FDTD is proposed to model electromagnetic wave propagation in an isotropic homogeneous lossy dielectric with electric and magnetic conductivities σ and $\sigma*$, respectively. In [7], the electromagnetic propagation in dispersive magnetoplasma medium is modeled using the FDTD method based on the ETD. In [9] and [10], an efficient ETD algorithm for general dispersive media in FDTD is introduced. However, these methods use ETD scheme only to electric field intensity or auxiliary variable. To improve the stability and computational accuracy, both the displacement vector and auxiliary difference variable apply ETD algorithm to model Debye and conductive dispersion in high dielectric material in [11].

To develop ETD methods further, in this paper a more accurate ETD-FDTD method with fourth-order central difference in space domain is proposed to model wave propagation in Debye dispersive media. The proposed method introduces an auxiliary difference equation (ADE) technique which establishes the relationship between the electric displacement vector and electric field intensity with a differential equation in high dielectric dispersive media. The ETD algorithm is applied to the

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displacement vector and auxiliary difference variable in time domain, and the fourth-order central-difference discretization is used in space domain. One example with plane wave propagation in Debye dispersive media is calculated. Compared with the conventional ETD-FDTD method, the results from our proposed method show its accuracy and efficiency for Debye dispersive media simulation.

2. MATHEMATICAL FORMULATION

With lossy and dispersive media, the Maxwell's equations can be written as

$$\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} = \nabla \times \mathbf{H}(\mathbf{r},t) - \sigma \mathbf{E}(\mathbf{r},t) - \mathbf{J}(\mathbf{r},t) \tag{1}$$

$$\frac{\partial \mathbf{H}(\mathbf{r},t)}{\partial t} = -\frac{1}{\mu_0}\nabla \times \mathbf{E}(\mathbf{r},t) \tag{2}$$

where σ is the electrical conductivity of material, and $\mathbf{J}$ is the electric current density. The electric displacement vector $\mathbf{D}$ is related to the electric field intensity $\mathbf{E}$ though the relative dielectric constant of the local tissue by

$$\mathbf{D}(\mathbf{r},\omega) = \varepsilon_0 \varepsilon_r(\mathbf{r},\omega)\mathbf{E}(\mathbf{r},\omega) \tag{3}$$

where ε_0 is the electric permittivity of free space. The relative dielectric constant ε_r of the Debye material in the frequency domain can be written as

$$\varepsilon_r(\omega) = \varepsilon_\infty + \frac{\varepsilon_s - \varepsilon_\infty}{1 + j\omega\tau} \tag{4}$$

where ε_s and ε_∞ are the static permittivity and infinite dielectric constant; ω represents the angular frequency; τ is the relaxation time. Substituting Eq. (4) into Eq. (3) and introducing an auxiliary difference variable $\mathbf{S}$, we get

$$\mathbf{E}(\mathbf{r},\omega) = \frac{\mathbf{D}(\mathbf{r},\omega) - \mathbf{S}(\mathbf{r},\omega)}{\varepsilon_0\varepsilon_\infty} \tag{5}$$

$$\mathbf{S}(\mathbf{r},\omega) = \varepsilon_0 \frac{\varepsilon_s - \varepsilon_\infty}{1 + j\omega\tau}\mathbf{E}(\mathbf{r},\omega) \tag{6}$$

With the transition relationship from frequency domain to time domain ($j\omega \to \partial/\partial t$), Eqs. (5) and (6) can be written as

$$\mathbf{E}(\mathbf{r},t) = \frac{\mathbf{D}(\mathbf{r},t) - \mathbf{S}(\mathbf{r},t)}{\varepsilon_0\varepsilon_\infty} \tag{7}$$

$$\mathbf{S}(\mathbf{r},t) + \tau\frac{\partial \mathbf{S}(\mathbf{r},t)}{\partial t} = \varepsilon_0(\varepsilon_s - \varepsilon_\infty)\mathbf{E}(\mathbf{r},t) \tag{8}$$

Multiplying both sides of Eq. (7) by σ and substituting it into Eq. (1), we can get

$$\frac{\partial \mathbf{D}(\mathbf{r},t)}{\partial t} + \frac{\sigma}{\varepsilon_0\varepsilon_\infty}\mathbf{D}(\mathbf{r},t) = \nabla \times \mathbf{H}(\mathbf{r},t) + \frac{\sigma}{\varepsilon_0\varepsilon_\infty}\mathbf{S}(\mathbf{r},t) - \mathbf{J}(\mathbf{r},t) \tag{9}$$

To derive the ETD scheme, multiplying Eq. (9) by $e^{\sigma t/\varepsilon_0\varepsilon_\infty}$ and integrating the equation over a single step from $t = n\Delta t$ to $t = (n+1)\Delta t$, we get

$$\mathbf{D}\big|_{\mathbf{r}}^{n+1} = \mathbf{D}\big|_{\mathbf{r}}^{n} e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}} + \frac{\varepsilon_0\varepsilon_\infty}{\sigma}\left(1 - e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}}\right)\left[\nabla \times \mathbf{H}\Big|_{\mathbf{r}}^{n+1/2} + \frac{\sigma}{\varepsilon_0\varepsilon_\infty}\mathbf{S}\Big|_{\mathbf{r}}^{n+1/2} - \mathbf{J}\Big|_{\mathbf{r}}^{n+1/2}\right] \tag{10}$$

By the same procedure, multiplying Eq. (8) by $e^{t/\tau}$ and integrating the equation over a single step from $t = n\Delta t$ to $t = (n+1)\Delta t$, we get

$$\mathbf{S}\big|_{\mathbf{r}}^{n+1} = \mathbf{S}\big|_{\mathbf{r}}^{n} e^{-\frac{\Delta t}{\tau}} + \varepsilon_0(\varepsilon_s - \varepsilon_\infty)\mathbf{E}\big|_{\mathbf{r}}^{n}\left(1 - e^{-\frac{\Delta t}{\tau}}\right) \tag{11}$$

According to Yee grid, the discretize formulations for High-Order ETD-ADE-FDTD can be given by introducing the fourth-order central-difference scheme. For the sake of simplicity, in the following sections we will employ a 2-D TE_z case, and the formulation is given by

$$\begin{aligned} D_x\Big|_{i,j}^{n+1} &= D_x\Big|_{i,j}^{n} e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}} + \frac{\varepsilon_0\varepsilon_\infty}{\sigma}\left(1-e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}}\right)\left[\frac{9}{8}\frac{H_z^{n+0.5}|_{i,j} - H_z^{n+0.5}|_{i,j-1}}{\Delta y}\right. \\ &\left. -\frac{1}{24}\frac{H_z^{n+0.5}|_{i,j+1} - H_z^{n+0.5}|_{i,j-2}}{\Delta y} + \frac{\sigma}{\varepsilon_0\varepsilon_\infty}\frac{S_x\Big|_{i,j}^{n+1} + S_x\Big|_{i,j}^{n}}{2} - J_x\Big|_{i,j}^{n+0.5}\right] \end{aligned} \tag{12}$$

$$\begin{aligned} D_y\Big|_{i,j}^{n+1} &= D_y\Big|_{i,j}^{n} e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}} + \frac{\varepsilon_0\varepsilon_\infty}{\sigma}\left(1-e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}}\right)\left[-\frac{9}{8}\frac{H_z^{n+0.5}|_{i,j} - H_z^{n+0.5}|_{i-1,j}}{\Delta x}\right. \\ &\left. +\frac{1}{24}\frac{H_z^{n+0.5}|_{i+1,j} - H_z^{n+0.5}|_{i-2,j}}{\Delta x} + \frac{\sigma}{\varepsilon_0\varepsilon_\infty}\frac{S_y\Big|_{i,j}^{n+1} + S_y\Big|_{i,j}^{n}}{2} - J_y\Big|_{i,j}^{n+0.5}\right] \end{aligned} \tag{13}$$

$$\begin{aligned} H_z^{n+0.5}|_{i,j} &= H_z^{n-0.5}|_{i,j} - \frac{\Delta t}{\mu_0}\left[\left(\frac{9}{8}\frac{E_y^n|_{i+1,j} - E_y^n|_{i,j}}{\Delta x} - \frac{1}{24}\frac{E_y^n|_{i+2,j} - E_y^n|_{i-1,j}}{\Delta x}\right)\right. \\ &\left. -\left(\frac{9}{8}\frac{E_x^n|_{i,j+1} - E_x^n|_{i,j}}{\Delta y} - \frac{1}{24}\frac{E_x^n|_{i,j+2} - E_x^n|_{i,j-1}}{\Delta y}\right)\right] \end{aligned} \tag{14}$$

$$E_\xi\Big|_{i,j}^{n+1} = \frac{D_\xi\Big|_{i,j}^{n+1} - S_\xi\Big|_{i,j}^{n+1}}{\varepsilon_0\varepsilon_\infty} \tag{15}$$

$$S_\xi\Big|_{i,j}^{n+1} = S_\xi\Big|_{i,j}^{n} e^{-\frac{\Delta t}{\tau}} + \varepsilon_0(\varepsilon_s - \varepsilon_\infty)E_\xi\Big|_{i,j}^{n}\left(1-e^{-\frac{\Delta t}{\tau}}\right) \tag{16}$$

In order to eliminate $\mathbf{S}^{n+1}$ in Eqs. (12) and (13), substituting Eq. (16) into Eqs. (12) and (13), we have

$$\begin{aligned} D_x\Big|_{i,j}^{n+1} &= D_x\Big|_{i,j}^{n} e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}} + \frac{\varepsilon_0\varepsilon_\infty}{\sigma}\left(1-e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}}\right)\left[\frac{9}{8}\frac{H_z^{n+0.5}|_{i,j} - H_z^{n+0.5}|_{i,j-1}}{\Delta y}\right. \\ & -\frac{1}{24}\frac{H_z^{n+0.5}|_{i,j+1} - H_z^{n+0.5}|_{i,j-2}}{\Delta y} + \frac{\sigma}{2\varepsilon_0\varepsilon_\infty}S_x\Big|_{i,j}^{n}\left(1+e^{-\frac{\Delta t}{\tau}}\right) \\ &\left. +\frac{\sigma(\varepsilon_s-\varepsilon_\infty)}{2\varepsilon_\infty}E_x\Big|_{i,j}^{n}\left(1-e^{-\frac{\Delta t}{\tau}}\right) - J_x\Big|_{i,j}^{n+0.5}\right] \end{aligned} \tag{17}$$

$$\begin{aligned} D_y\Big|_{i,j}^{n+1} &= D_y\Big|_{i,j}^{n} e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}} + \frac{\varepsilon_0\varepsilon_\infty}{\sigma}\left(1-e^{-\frac{\sigma\Delta t}{\varepsilon_0\varepsilon_\infty}}\right)\left[-\frac{9}{8}\frac{H_z^{n+0.5}|_{i,j} - H_z^{n+0.5}|_{i-1,j}}{\Delta x}\right. \\ & +\frac{1}{24}\frac{H_z^{n+0.5}|_{i+1,j} - H_z^{n+0.5}|_{i-2,j}}{\Delta x} + \frac{\sigma}{2\varepsilon_0\varepsilon_\infty}S_y\Big|_{i,j}^{n}\left(1+e^{-\frac{\Delta t}{\tau}}\right) \\ &\left. +\frac{\sigma(\varepsilon_s-\varepsilon_\infty)}{2\varepsilon_\infty}E_y\Big|_{i,j}^{n}\left(1-e^{-\frac{\Delta t}{\tau}}\right) - J_y\Big|_{i,j}^{n+0.5}\right] \end{aligned} \tag{18}$$

From Eqs. (14)–(18), we know that the proposed method can be summarized in four steps. First, the auxiliary variables $\mathbf{S}$ are obtained from Eq. (16). Second, explicitly update the x and y components of $\mathbf{E}$ using Eq. (15). Third, explicitly update H_z by Eq. (14). Fourth, explicitly update the x and y components of $\mathbf{D}$ using Eqs. (17) and (18).

3. NUMERICAL RESULTS

In order to validate the effectiveness of the proposed method, we consider transient fields in 2D cavity with Debye dispersive media, shown in Fig. 1. A sinusoidally modulated Gaussian pulse is used as an incident electric current profile

$$J_y(t) = \exp\left[-\left(\frac{t-T_c}{T_d}\right)^2\right]\sin 2\pi f_c(t-T_c) \tag{19}$$

where $T_d = 1/(2f_c)$, $T_c = 3T_d$ and $f_c = 15\,\mathrm{GHz}$. The computational domain consists of 90×90 cells with uniform cell size of 0.2 mm (50 cells per λ, where λ is the wavelength corresponding to f_c) and is truncated by the PEC boundary in both x and y directions. The problem was solved by both the low- and high-order ETD-FDTD methods. The time step Δt is 2×10^{-13} s, and the simulations were run for $4800\Delta t$. The dispersive medium square column with $3\,\mathrm{mm} \times 3\,\mathrm{mm}$ in Fig. 1 is Debye model, where $\varepsilon_s = 35.5$, $\varepsilon_\infty = 2.05$ and $\tau = 48.3 \times 10^{-12}$. The electrical conductivity σ of the dispersive material is 0.01 s/m.

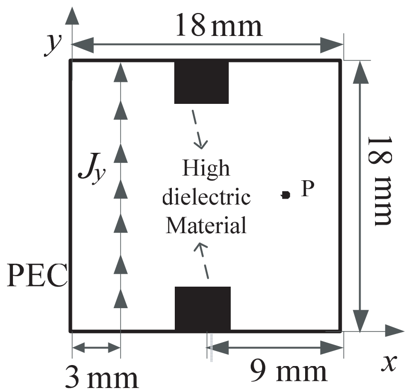

Figure 1. Diagram of computational domain for ETD-HO-FDTD.

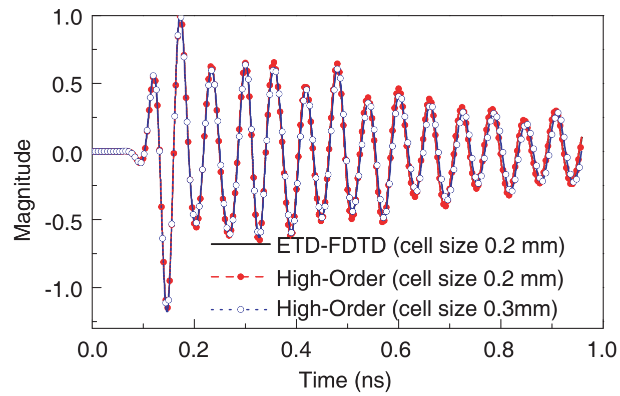

Figure 2. The transient fields calculated with ETD-FDTD and the proposed method.

Figure 2 shows the calculated results of the time domain waveform which is located at P (15 mm, 9 mm) given by the low-order ETD-FDTD and the proposed method. From their profiles, one can find that the accuracy of the proposed method is verified. For comparison, the computational domain consists of 60×60 cells with uniform cell size of 0.3 mm used. The time step Δt is 3×10^{-13} s, and the simulations were run for $3200\Delta t$. The calculated results agree with that of low-order ETD-FDTD method. From the calculated temporal electric fields, the resonant frequencies are obtained through discrete Fourier transform (DFT).

Table 1 represents the required computational resource and computing time for the numerical simulations. Compared with the low-order method, the high-order method with cell size of 0.3 mm shows the reductions of 69% and 41% on computing time and memory usage respectively while the relative error is 0.6%. The relative error of the resonant frequency is defined as: $|f_{\text{low-order-FDTD}} - f_{\text{high-order-FDTD}}|/f_{\text{low-order-FDTD}} \times 100\%$, where $f_{\text{low-order-FDTD}}$ is the reference solution from low-order-FDTD, and $f_{\text{high-order-FDTD}}$ is the resonant frequency calculated from the proposed method. All calculations have been performed on Intel (R) Core (TM) i5-4210 CPU with 8 GB RAM.

Table 1. Comparison of the computational efforts for the 2-D cavity.

Method	Δt (ps)	Cell size (mm)	Marching on steps	Memory (MB)	Resonant frequency	Relative error	CPU time (s)
Low-order	0.2	0.3	4800	0.97	16.4 GHz	- -	176
High-order	0.2	0.3	4800	0.97	16.4 GHz	0%	183
	0.3	0.2	3200	0.57	16.3 GHz	0.6%	54

4. CONCLUSION

An ETD-ADE-FDTD method based on a fourth-order central-difference discretization scheme for Debye dispersive material is presented in this paper. Compared with the low-order ETD-FDTD, the proposed method can reduce the calculation burden. One numerical example verifies the accuracy and efficiency of the proposed method.

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