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An Investigation of the Generalised Range-Based Detector in Pareto Distributed Clutter

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Abstract—The purpose of this paper is to examine whether a generalised range-based sliding window detector provides any improved detection performance relative to a single order statistic based counterpart. This is for non-coherent target detection in an X-band maritime surveillance radar environment, and as such the intensity clutter is modelled by a Pareto distribution. It will be demonstrated mathematically that a single order statistic detector is in fact sufficient. Some numerical examples are also provided to clarify the theoretical results.

1. INTRODUCTION

The detection processes under consideration are those first examined in [1–5], which were concerned with achieving the constant false alarm rate (CFAR) property in exponentially distributed clutter; a monograph on this subject matter is [6]. These non-coherent detection processes, referred to as sliding window detectors, assume that a cell under test (CUT) is to be examined for the presence of a target. This is done by comparing it with a normalised measurement of the clutter level. This normalisation is performed in such a way that, in homogeneous clutter situations, the probability of false alarm (Pfa) does not vary with the clutter power. Achieving the CFAR property is a critical feature of a radar detection scheme. Continued interest in such detection processes, in modern operational environments, is apparent from [7–19].

The context under consideration is target detection in an X-band maritime surveillance radar environment. As such, the clutter environment of interest is assumed to follow a Pareto Type II distributional form, validated in several independent studies of such clutter [20–22]. In situations where the fitted Pareto scale parameter is smaller than unity, which has been observed to be the case in [20, 22], the Pareto Type II model can be approximated by a Pareto Type I, facilitating the design of detectors [23, 24]. Since there is continued interest in detector development in a Pareto clutter environment, this letter examines a generalised range based detector, which has not been examined previously in the context of interest. Such a detector is based upon the difference of two decreasing order statistics. Given the fact that a sum of order statistics, as the measurement of the clutter level, has been shown to produce very good results [25], it is thus of interest to see whether a difference is a suitable clutter measure. In order to answer this question, it will be shown firstly how the corresponding detector's threshold can be set, so as to achieve a desired Pfa. Then the decision rule will be examined relative to a single order statistic (OS) detector, and it will be demonstrated that it cannot provide performance gains.

2. SLIDING WINDOW DETECTORS

Sliding window detection processes assume that one has a series of amplitude or intensity clutter measurements, denoted by statistics $Z_1, Z_2, \ldots, Z_N$, which are non-negative, independent and identically

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distributed. These statistics are referred to as the clutter range profile (CRP) and a function g is applied to these to produce a single measurement of the clutter level. A CUT is then considered, which is also assumed to be independent of the CRP statistics, and is tested for the presence of a target embedded within the clutter. If the statistic of the CUT is denoted Z_0 then the test compares Z_0 with a normalised version of g, such that the Pfa remains constant in homogeneous clutter returns [6]. To formulate this statistically, if one sets H_0 to be the hypothesis that the CUT does not contain a target, and H_1 the alternative that the CUT does contain a target, then the test can be written

$$Z_0 \underset{H_0}{\overset{H_1}{\gtrless}} \tau g(Z_1, Z_2, \ldots, Z_N), \tag{1}$$

where $\tau > 0$ is the normalisation constant, referred to as the threshold multiplier, and the notation in the above means that H_0 is rejected if $Z_0 > \tau g(Z_1, Z_2, \ldots, Z_N)$. The Pfa of this test is given by

$$\mathrm{P_{FA}} = \mathrm{IP}(Z_0 > \tau g(Z_1, Z_2, \ldots, Z_N)|H_0). \tag{2}$$

If Eq. (2) does not depend on a clutter model parameter, then the test achieves the CFAR property with respect to this parameter [5]. When the clutter is modelled by exponentially distributed random variables, the test of Eq. (1) is CFAR provided that the function g is scale invariant.

The loss of the CFAR property occurs when Eq. (1) is applied in other clutter environments; a good example is the Weibull clutter scenario [26], as well as the Pareto Type I case [27]. In the Pareto Type I clutter case the CRP is modelled by statistics with distribution function

$$F_{Z_j}(t) = \mathrm{IP}(Z_j \leq t) = 1 - \left(\frac{\beta}{t}\right)^{\alpha}, \tag{3}$$

for all $1 \leq j \leq N$ and $t \geq \beta$, where $\alpha > 2$ is the Pareto shape and $\beta > 0$ the Pareto scale parameter. The lower bound on α is imposed so that the first two moments exist; based upon the fits to real data reported in [22] it was found that α tends to always exceed 4.

A useful characteristic of a random variable, distributed according to Eq. (3), is that if X_j has an exponential distribution with unity mean then one can write

$$Z_j = \beta e^{\alpha^{-1} X_j}, \tag{4}$$

which is a result used in [23], and will be useful in the analysis to follow.

In the next section the detector of interest is introduced and analysed.

3. RANGE BASED DETECTORS

A range-based sliding window detector arises from the selection of $g(Z_1, Z_2, \ldots, Z_N) = Z_{(j)} - Z_{(i)}$, where it is assumed that $j > i$ so that g is positive and well-defined. With such a choice the basic decision rule is

$$Z_0 \underset{H_0}{\overset{H_1}{\gtrless}} \tau \left(Z_{(j)} - Z_{(i)}\right). \tag{5}$$

Observe that the difference of two order statistics will be a scale invariant function, and so it is expected that Eq. (5) will be CFAR with respect to the Pareto scale parameter. This is confirmed in the following result:

Lemma 3.1 *The Pfa of Eq. (5), operating in independent Pareto Type I clutter with shape and scale parameters α and β respectively, is given by*

$$\begin{aligned} \mathrm{P_{FA}} &= \tau^{-\alpha} \left(\frac{N-i+1}{N+1}\right)(j-i)\binom{N-i}{N-j} \\ &\times \int_0^1 \psi^{N-j+1}\left[1 - \psi^{\alpha^{-1}}\right]^{-\alpha}[1-\psi]^{j-i-1}\, d\psi. \end{aligned} \tag{6}$$

Hence, it is clear that the detector in Eq. (5) is only CFAR with respect to the Pareto scale parameter, as expected. It is relatively simple to solve for τ, for a given Pfa, in Eq. (6).

The proof of Lemma 3.1 is now presented. Under H_0 the CUT statistic is also Pareto distributed, but independent of the CRP. In view of Eq. (4), one can write $Z_0 = \beta e^{\alpha^{-1} X_0}$ under H_0 and

$Z_{(k)} = \beta e^{\alpha^{-1} X_{(k)}}$ where X_0 has an exponential distribution with parameter unity and $X_{(k)}$ is the kth OS of a series of such random variables. Now the jointly distributed pairs of order statistics $X_{(i)}$ and $X_{(j)}$ has density function

$$f_{X_{(i)},X_{(j)}}(x,y) = \frac{N!}{(i-1)!(j-i-1)!(N-j)!} \times e^{-x-y}(1-e^{-x})^{i-1} \times (e^{-x}-e^{-y})^{j-i-1} e^{-y(N-j)}, \tag{7}$$

for $y \geq x$, which can be found in [28]. Thus the desired Pfa is

$$\begin{aligned}
\mathrm{P_{FA}} &= \mathrm{IP}\left(e^{\alpha^{-1}X_0} > \tau\left[e^{\alpha^{-1}X_{(j)}} - e^{\alpha^{-1}X_{(i)}}\right]\right) \\
&= \mathrm{IP}\left(X_0 > \alpha \log \tau\left[e^{\alpha^{-1}X_{(j)}} - e^{\alpha^{-1}X_{(i)}}\right]\right) \\
&= \int_{x=0}^{\infty}\int_{y=x}^{\infty} f_{(X_{(i)},X_{(j)})}(x,y) \times \mathrm{IP}\left(X_0 > \alpha \log \tau\left[e^{\alpha^{-1}y} - e^{\alpha^{-1}x}\right]\right) dydx \\
&= \tau^{-\alpha}\int_{x=0}^{\infty}\int_{y=x}^{\infty} f_{(X_{(i)},X_{(j)})}(x,y) \times \left[e^{\alpha^{-1}y} - e^{\alpha^{-1}x}\right]^{-\alpha} dydx,
\end{aligned}$$

where the fact that the CUT is independent of the CRP has been used, together with the exponential distribution function. For convenience let

$$\kappa = \frac{N!}{(i-1)!(j-i-1)!(N-j)!}. \tag{8}$$

Thus, by applying the density in Eqs. (7) to (8), the Pfa is

$$\begin{aligned}
\mathrm{P_{FA}} &= \kappa\tau^{-\alpha}\int_{x=0}^{\infty}\int_{y=x}^{\infty}\left[e^{\alpha^{-1}y} - e^{\alpha^{-1}x}\right]^{-\alpha} \times e^{-x-y}(1-e^{-x})^{i-1}(e^{-x}-e^{-y})^{j-i-1}e^{-y(N-j)}dydx \\
&\equiv \kappa\tau^{-\alpha}\int_{\phi=0}^{1}\int_{\theta=0}^{\phi}\left[\theta^{-\alpha^{-1}} - \phi^{-\alpha^{-1}}\right]^{-\alpha}(1-\phi)^{i-1} \times (\phi-\theta)^{j-i-1}\theta^{N-j}d\theta d\phi,
\end{aligned} \tag{9}$$

where a change of variables $\phi = e^{-x}$, followed by $\theta = e^{-y}$, has been applied. The final result follows by changing variables in the second integral with $\psi = \frac{\theta}{\phi}$, with ϕ constant, and applying the definition of the beta function. Thus the result of Lemma 3.1 can be applied to Eq. (5), provided α is known *a priori*.

The pertinent question to now address is whether a range-based detector of the form in Eq. (5) provides any improvements on a detector based upon a single OS. Hence it is relevant to consider the decision rule

$$Z_0 \underset{H_0}{\overset{H_1}{\gtrless}} \nu Z_{(j)}, \tag{10}$$

for some $1 \leq j \leq N$, operating in Pareto Type I clutter. Now it is demonstrated in [27] that the threshold multiplier in Eq. (10) is

$$\nu = \left[\frac{N-j+1}{N+1}\mathrm{P_{FA}}^{-1}\right]^{\frac{1}{\alpha}}, \tag{11}$$

when this detector is operating in the clutter environment of interest. In order to compare the performance of Eqs. (10) and (5), one can appeal to the detector comparison lemma introduced in [29]. To express this in the current context, write the detector of Eq. (10) in the form

$$T_1(Z_0, Z_1, \ldots, Z_N) = \frac{Z_0}{Z_{(j)}} \tag{12}$$

with $T_1(Z_0, Z_1, \ldots, Z_N) \underset{H_0}{\overset{H_1}{\gtrless}} \nu$ the equivalent form of (10). Similarly, express Eq. (5) as

$$T_2(Z_0, Z_1, \ldots, Z_N) = \frac{Z_0}{Z_{(j)} - Z_{(i)}} \tag{13}$$

so that the test is $T_2(Z_0, Z_1, \ldots, Z_N) \underset{H_0}{\overset{H_1}{\gtrless}} \tau$. Then the main result in [29] states that, for the same Pfa, the statistic

$$\eta(Z_0, Z_1, \ldots, Z_N) = \frac{\nu}{\tau} \frac{T_2(Z_0, Z_1, \ldots, Z_N)}{T_1(Z_0, Z_1, \ldots, Z_N)} \tag{14}$$

exceeding unity is equivalent to T_2 always having a larger probability of detection (Pd) than T_1.

Now with reference to expression (6), observe that in the integral, since $\psi \in [0, 1]$ and $\alpha > 1$ it follows that

$$0 < 1 - \psi^{\alpha^{-1}} < 1 - \psi < 1. \tag{15}$$

Hence, based upon Eq. (15) it follows that

$$(1 - \psi^{\alpha^{-1}})^{-\alpha} > (1 - \psi)^{-1} > 1. \tag{16}$$

Therefore an application of Eqs. (16) to (6) yields

$$\begin{aligned} \tau^\alpha &> {\mathrm{P_{FA}}}^{-1} \left(\frac{N - i + 1}{N + 1} \right) (j - i) \binom{N - i}{N - j} \times \int_0^1 \psi^{N-j+1} (1 - \psi)^{j-i-2} d\psi \\ &= {\mathrm{P_{FA}}}^{-1} \left(\frac{N - j + 1}{N + 1} \right) [(N - i + 1)(j - i - 2)], \end{aligned} \tag{17}$$

where an application of the definition of the beta function has been applied, together with simplification of the final result. Thus, based upon Eq. (17), one can conclude that

$$\frac{\tau}{\nu} > \max \left(1, [(N - i + 1)(j - i - 2)]\right)^{\frac{1}{\alpha}}, \tag{18}$$

where the fact that the ratio of threshold multipliers exceeds unity follows by bounding $(1 - \psi^{\alpha^{-1}})^{-\alpha}$ from below by unity in Eq. (6) and proceeding with a similar derivation. In most cases of interest it is found that Eq. (18) tends to exceed 2. As an example, if one selects $N = 32, i = 1$ and $j = 4$ (so that $j - i - 2$ is nonzero) then this term is exactly 2.3784 with the selection of $\alpha = 4$.

Observe that

$$\begin{aligned} \eta(Z_0, Z_1, \ldots, Z_N) &= \frac{\nu}{\tau} \frac{Z_{(j)}}{Z_{(j)} - Z_{(i)}} \\ &= \frac{\nu}{\tau} \left[1 + \frac{Z_{(i)}}{Z_{(j)} - Z_{(i)}} \right] \\ &> \frac{\nu}{\tau} \left[1 + \frac{Z_{(i)}}{Z_{(j)}} \right], \end{aligned} \tag{19}$$

and Eq. (19) exceeds unity when the ratio of $Z_{(i)}$ and $Z_{(j)}$ exceeds $\frac{\tau}{\nu} - 1$. By an application of Eq. (4) it follows that this will occur when

$$X_{(j)} - X_{(i)} < -\alpha \log \left(\frac{\tau}{\nu} - 1 \right). \tag{20}$$

Note that $\log(\frac{\tau}{\nu} - 1) > 0$ when $\tau > 2\nu$. As remarked previously, this tends to be the case, and so the inequality of Eq. (20) will never hold, since $j > i$. Therefore, the detector in Eq. (5) will always be inferior to the single OS detector in Eq. (10).

To illustrate these results, consider the case where $N = 32$ and the $\mathrm{P_{FA}} = 10^{-4}$. Assume that the Pareto shape parameter is $\alpha = 4.7241$, while the scale parameter is $\beta = 0.0446$. These values have been selected on the basis of real fits to data [22]. Figure 1 plots the ratio of threshold τ, obtained via Lemma 3.1, and threshold ν from Eq. (11). This figure plots the intensity, in a logarithmic scale, of the ratio of the thresholds as functions of the indices i and j, such that $j > i$. The plot shows that the ratio of thresholds always exceeds 2.345, which is the minimum value in the plot.

To provide a different perspective, Figure 2 provides a snapshot for the case where $i = 1$, with j ranging from 2 to 32.

Two examples of detector performance are now provided. In this situation, Monte Carlo sampling with 10^6 runs is used to estimate the Pd for a series of signal to clutter ratios (SCRs). The target model

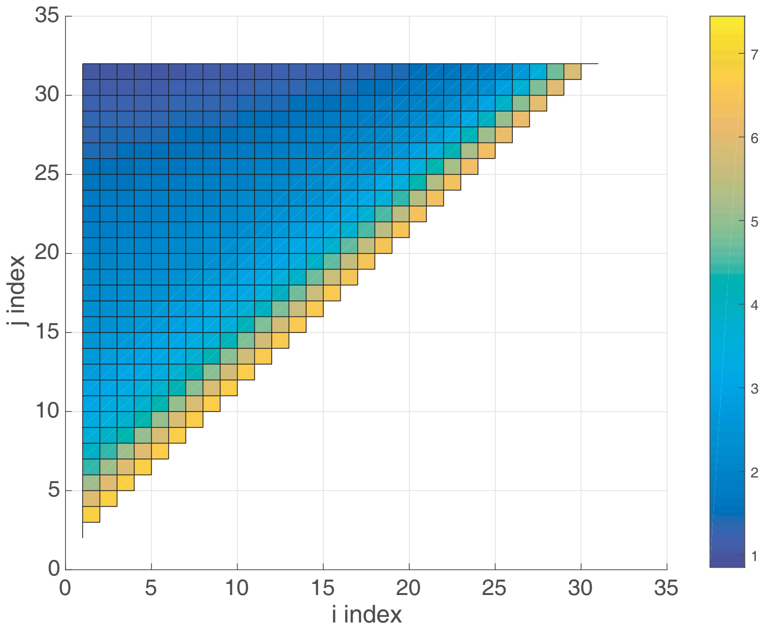

Figure 1. Ratio of thresholds plot, in a logarithms. The least value, in standard units, occurs when $j = 32$ and $i = 1$ and is 2.345.

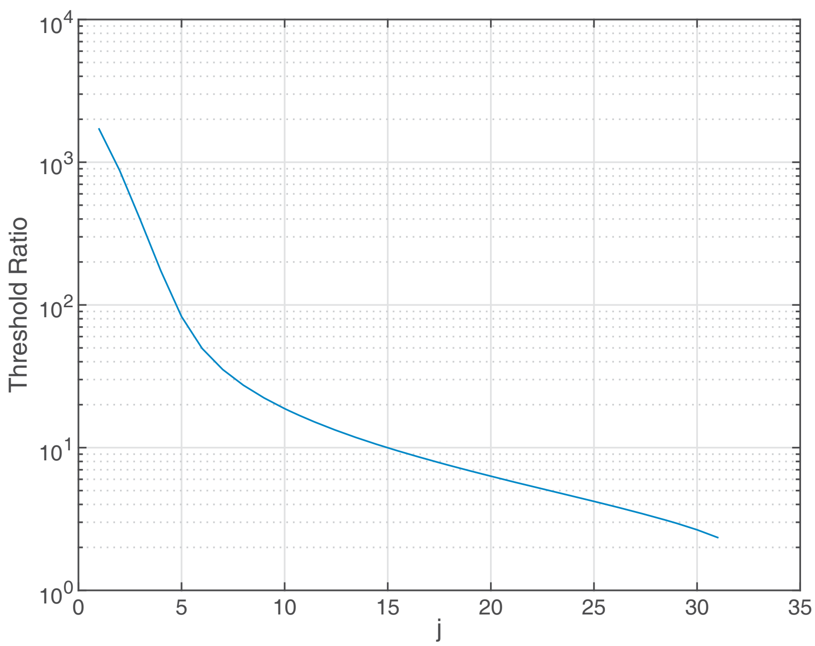

Figure 2. Ratio of thresholds plot. The least value, in standard units, occurs when $j = 32$ and $i = 1$ and is 2.345.

used is Swerling I, which has been added to the clutter in the complex domain. As before the Pfa is set to 10^{-4}, with $N = 32$ and with the same Pareto shape and scale parameters. Figure 3 shows the case where $i = 1$ and $j = 15$, while Figure 4 is for the case where $i = 10$ and $j = 15$. It is clear from these results that the OS-based detector has better performance than a range-based decision rule.

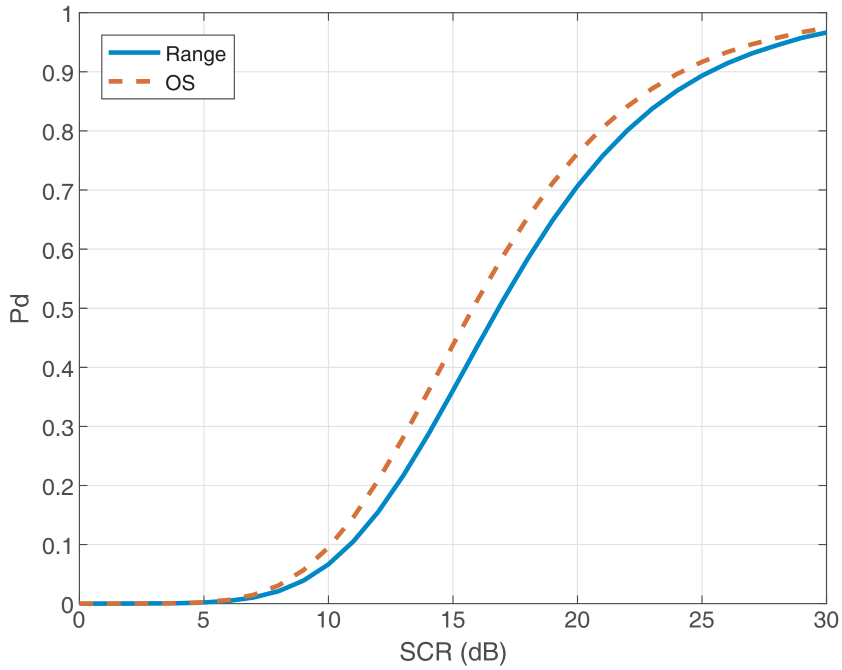

Figure 3. Detector performance when $i = 1$ and $j = 15$.

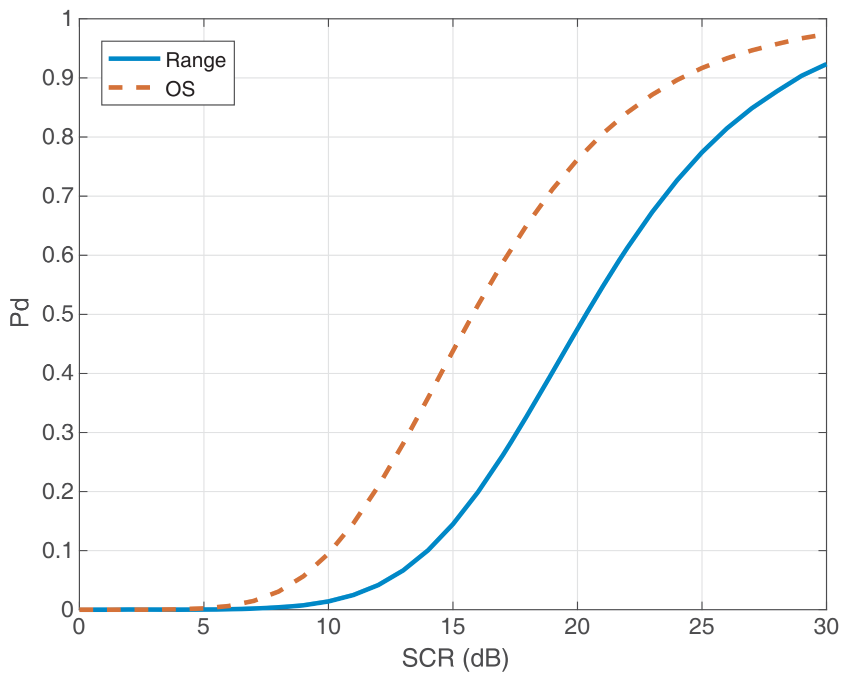

Figure 4. A second example of detector performance, in the situation where $i = 10$ and $j = 15$.

4. CONCLUSIONS

This paper investigated whether a range-based detector, operating in Pareto distributed clutter, provided any performance gains on a single-order statistic decision rule counterpart. It was shown mathematically that it did not, and some examples were used to clarify this.

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