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Parameter Estimation of an Inhomogeneous Medium by Scattered Electromagnetic Fields Using Nonlinear Optics and Wavelets

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Abstract—The aim of this work is to study the parameter estimation of a nonlinear medium in terms of scattered electromagnetic fields. The surface parameters are defined in terms of linear and nonlinear components of susceptibility and permeability. A set of Maxwell's equations are derived for an inhomogeneous medium using Green's function and the scattered Electromagnetic fields solving integrodifferential equations. Mathematical formulas are simplified using wavelet based method. Susceptibility and permeability is assumed as a function of wavelet basis. For parameter estimation, least square method and inner product methods are used with wavelets as a basis function, which gives solutions for nonlinear integrodifferential equation. Both time and spatial domain analysis is done using wavelets, and parameter coefficients are obtained. It is found that in both the parameter estimation methods, least square estimation gives better results. At the end of the paper statistical analysis of the scattered signals is included by calculating the mean and covariance of the signals.

1. INTRODUCTION

Different nonlinear inverse scattering theorems have been suggested for multiple scattering effects. The algorithms are solved for inhomogeneous medium by forward scattering methods, and their optimization is done using Maxwell's equations by measurement of scattered fields at discrete points [1].

Here a medium is illuminated by some incident wave, which is a monochromatic wave. Because of the incident wave, the medium-particles are energized, and perturbation in the movement of an electron causes nonlinear polarization, i.e., field dependent polarization. The linear and nonlinear polarizations play an important role in the scattering of electromagnetic waves.

Although computational complexity due to Integrodifferential equations arises in nonlinear inverse scattering algorithms, nonlinear methods more accurately define the physical properties of complex medium [1].

Integrodifferential equation derived contains the derivatives of unknown functions [2]. Mathematical modelling can be done by functional equations, PDE, Integro differential equations (IDE), and stochastic equations [2]. These equations are used to solve problems of fluid dynamics, biological models, etc. Wavelet method is one of the methods [3] to find approximate numerical solution to linear and nonlinear differential equations [2]. In this paper, algorithms to find parameters of the medium are obtained using least mean square estimation and inner product methods. These techniques are applied after getting solutions from Integrodifferential equations obtained using Maxwell's equations and Green's function.

Three-dimensional wavelet functions are used to present the basis functions. These functions estimate the solutions of integral equations. Daubchies 6 wavelet is an orthogonal wavelet with compact support and is used in different numerical approximation problems.

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The nonlinearity and susceptibility of a medium play an important role in the generation of second harmonic and third harmonic waves, which is used in imaging [4–6]. Terahertz technology is also applied to image processing using a minimum entropy criterion for estimating and compensating linear phase error [2]. By the nonlinear interaction of light and matter, THz waves are produced. These THz waves can be used in nondestructive detection, medical imaging and standoff personnel screening [2]. A two-dimensional imaging of CW THz radiation using electro-optical detection was done by Nahata et al. [4]. A 3D imaging system was worked out by Chattopadhyay et al. [6] involving THz sources and heterodyne detection techniques in submillimeter frequency modulated carrier wave.

In this work, we consider a slab of inhomogeneous medium. An algorithm is given to find the medium parameters susceptibility and permeability in terms of scattered electromagnetic fields. Using wavelet basis functions in least square estimation and inner product method using Method of Moment, the inverse solutions and parameters are obtained in terms of basis functions.

Paper organization is as follows. In Section 2, computational algorithms is given to estimate the parameters of a medium in terms of time domain algorithms using first order susceptibility. Section 3 defines the algorithm using second order susceptibility, and Section 4 gives the wavelet solutions for integral equations. In Section 5 the parameter estimation method Least square estimation and Inner product methods are defined. Stochastic method is also derived for random parameter estimation. Section 6 shows result and simulation where the results of least square estimation and inner product are shown. Section 7 concludes this work.

2. COMPUTATIONAL ALGORITHMS TO ESTIMATE THE PARAMETERS OF A MEDIUM

2.1. Time Domain Algorithm for First Order Susceptibility

In majority of inverse scattering algorithms, the illumination of an object or medium is done by an incident wave which may be generated by an array of antennas or an ultra-wideband pulse [8]. For analyzing we need a setup to use time domain solver [1]. Using the computer model forward scattering data are generated.

Forward Solver:

An electromagnetic wave is incident on the medium. Maxwell's equations are written for the medium, which gives nonlinear Helmholtz equation [9] in terms of electric E and magnetic field H. Here susceptibility kernel of the medium is assumed on some prior approximate data for the forward solver in computer model. Generalized Helmholtz equation is obtained for electric as well as magnetic fields of a nonlinear random medium. In the forward solver of computer program we can approximate the scattered waves E^1, H^1, E^2, H^2 in terms of the incident electric or magnetic fields. For this, we need some basic knowledge of the scatterers. For this calculation a set of Maxwell's equations are derived changing the permeability and permittivity of the medium to inhomogeneous parameters of the medium.

This method includes two parts:

1. Using integral equation formulation of linear PDE or solving PDE.
2. Equations are resolved in terms of Green's function and method of moment.

From Maxwell's equations —

$$\nabla(\epsilon E) = 0 \tag{1}$$

Here we assume that permittivity $\chi_e(x, y)$ and permeability $\chi_{mn}(x, y)$ both are functions of x, y and are perturbed by δ then

$$\mu(x, y) = \mu_0(1 + \delta\chi_{mn}(x, y)) \tag{2}$$

$$\epsilon(x, y) = \epsilon_0(1 + \delta\chi_e(x, y)) \tag{3}$$

δ is a small amount of perturbation due to applied electric field.

$$E = \left(E^0 + \delta E^1 + \delta^2 E^2 + \ldots\right) \ [1] \tag{4a}$$

$$H = \left(H^0 + \delta H^1 + \delta^2 H^2 + \ldots\right) \tag{4b}$$

In Equation (1), we put Equations (3) and (4) —

$$\operatorname{div}(\epsilon_0(1+\delta\chi_e))\left(E^0+\delta E^1+\delta^2 E^2+\ldots\right)=0 \tag{5}$$

From Equation (5), we get

$$\begin{aligned}
\epsilon_0 \operatorname{div} E^0 &= 0 && (6)\\
\left(\operatorname{div} E^1\right)+\left(\nabla\left(\chi_e, E^0\right)\right) &= 0 && (7)\\
\chi_e &= \chi_e(x,y) &&\\
\epsilon_0\left(\operatorname{div} E^2\right)+\nabla\left(\chi_e, E^1\right)+\chi_e \operatorname{div} E^1 &= 0 && (8)
\end{aligned}$$

Opening Maxwell's equation for magnetic field —

$$\begin{aligned}
\nabla\cdot(B) &= 0 && (9)\\
\nabla\cdot(\mu H) &= 0 && (10)\\
\mu &= \mu_0(1+\delta\chi_m(x,y)) && (11)\\
\nabla\cdot(\mu_0(1+\delta\chi_m(x,y))H) &= 0 && (12)
\end{aligned}$$

If the field H is perturbed by δ then again use Taylor series expansion in H.

We put $H=(H^0+\delta H^1+\delta^2 H^2+\ldots))$ in Equation (12)

$$\begin{aligned}
\nabla\cdot(\mu_0 H)=\nabla\left(\mu_0\left(1+\delta\chi_{mn}(x,y)\left(H^0+\delta H^1+\delta^2 H^2+\ldots\right)\right)\right) &= 0 && (13)\\
\delta\nabla\cdot\left(\mu_0 H^0\right) &= 0 && (14)\\
\delta^1\mu_0\left(\left(\operatorname{Div}\cdot H^1\right)+\left(\nabla\chi_{mn}(x,y)\,H^0\right)\right) &= 0 && (15)\\
\nabla\cdot\mu_0 H^2+\nabla\left(\chi_{mn}\cdot H^1\right) &= 0 && (16)\\
\operatorname{Div}\cdot H^2+\nabla\left(\chi_{mn} H^1\right)+\chi_{mn}\nabla H^1 &= 0 && (17)
\end{aligned}$$

Using curl equations —

$$\nabla\times(\nabla\times E)=\nabla\times(-j\omega\mu H)=-J \tag{18}$$

$$\nabla(\operatorname{div} E)-\nabla^2 E=-j\omega((\nabla\mu)\times H+\mu\nabla\times H) \tag{19}$$

$$\begin{aligned}
&\nabla^2 E-\nabla(\operatorname{div} E)+j\omega(\nabla\mu_0(1+\delta\chi_m)\\
&\times\left(H^0+\delta H^1+\delta^2 H^2\right)+\mu_0(1+\delta\chi_m)\nabla\times(H^0+\delta H^1+\delta^2 H^2+\ldots))=0
\end{aligned} \tag{20}$$

The following equation is written in the form of permittivity and permeability functions, which will vary as a function of x, y. We should have prior knowledge of the scatterers so as to generate computer based data or forward solution.

Substituting

$$\nabla\times H=j\omega\epsilon E \tag{21}$$

we get —

$$\begin{aligned}
&\nabla^2 E-\nabla(\operatorname{div} E)+j\omega\left\{\nabla\mu_0(1+\delta\nabla\chi_m)\times(H^0+\delta H^1+\delta^2 H^2+\ldots)\right\}\\
&-j\omega\mu(j\omega\epsilon_0(1+\delta\chi_e(x,y))\,(E^0+\delta E^1+\delta^2 E^2+\ldots)=0
\end{aligned} \tag{22}$$

Solving Equation (19) we get E^1, E^2, H^1, H^2 in terms of E^0 and H^0

$$\begin{aligned}
\nabla^2 E^0+K^2 E^0 &= 0 && (23)\\
\nabla^2 E^1+\nabla\left(\nabla\chi_e, E^0\right)-j\omega\mu_0\nabla\chi_m\times H^0+K^2\left(\chi_m E^0+\chi_e E^0\right)+K^2 E^1 &= 0 && (24)
\end{aligned}$$

In medium, we get perturbation in electric field in terms of the fundamental field E_0 [10].

If initial electric field is taken as radiation from a small dipole, Green's function and current density are assumed equal to 1 [10]. Then

$$E^0(t,r)=\int f\left(\hat{n},t-\frac{\hat{n},r}{c}\right)d\,\Omega(\hat{n}) \quad \text{which is equal to} \quad \int f(\hat{n},\omega)e^{-i\omega\frac{\hat{n}\cdot r}{c}}d\,\Omega(\hat{n}) \tag{25}$$

The perturbed electromagnetic field is given by Taylor series expansion —
(Using duality $E \rightarrow H$, $H \rightarrow -E$, $\epsilon \rightarrow \mu$)
For δ^0

$$\nabla^2 H^0 + K^2 H^0 = 0 \tag{26}$$

For δ^1

$$\nabla^2 H^1 + \nabla\left(\nabla\chi_m, H^0\right) - j\omega\mu_0\nabla\chi_e \times E^0 + K^2 H^1 + K^2(\chi_m + \chi_e)E^0 = 0 \tag{27}$$

$$\nabla\times(\nabla\times H) = \nabla\times(-j\omega\epsilon E) = -j\omega(\nabla\times\epsilon E + \epsilon\nabla\times E) = -j\omega(\nabla\times(1+\delta\chi_e)\times E) + (1+\delta\chi_e)\,(\nabla\times E) \tag{28}$$

$$\begin{aligned}\nabla^2 H - \nabla\,(\nabla\cdot H) &= -j\left(\nabla\times\epsilon_0\,(1+\delta\chi_e)\left(E^0 + \delta E^1 + \delta^2 E^2 + \ldots\right)\right) + \epsilon_0(1+\delta\chi_e)(-j\omega\mu H)\}\\ &= j\omega\left\{\nabla\times E^0 + \delta\nabla\times\left(\chi_e\left(E^0 + \delta E^1 + \delta^2 E^2 + \ldots\right)\right) + j^2\omega^2\mu_0\epsilon_0(1+\delta\chi_e)(1+\delta\chi_m)\left(H^0 + \delta H^1 + \delta^2 H^2 + \ldots\right)\right\}\end{aligned} \tag{29}$$

$$\nabla^2(H^0 + \delta H^1 + \delta^2 H^2 + \ldots) - \nabla\left(\nabla\cdot\left(H^0 + \delta H^1 + \delta^2 H^2 + \ldots\right)\right) = RHS \tag{30}$$

$$\nabla^2 H^0 + K^2 H^0 = 0$$

$$\nabla^2 H^1 + K^2 H^1 = \nabla\left(\nabla\cdot H^1\right) - j\omega\epsilon_0\left(\nabla\times\chi_e E^0\right) - K^2\left(\chi_m H^0 + \chi_e H^0\right) \tag{31}$$

$\nabla H^1 = -\nabla\cdot(\chi_m H^0)$ put in Equation (31)

$$\nabla^2 H^1 + K^2 H^1 = -\left\{\nabla\left(\nabla\chi_m, H^0\right) + K^2(\chi_m + \chi_e)\right\} + j\omega\mu_0\left(\nabla\times\chi_e E^0\right)\} \tag{32}$$

Same equation exists for H^1 —
Hence using

$$E^0(r) = \int E^0(k)\exp(-jkr)d^3r \tag{33}$$

which is perpendicular to the direction of propagation hence using $(k, E^0(r)) = 0$

$$\nabla\times E^0\,(r) = -j\omega\mu H^0 \tag{34}$$

From above equation incident magnetic field is H^0 which is computed —

$$H^0 = \sum_1^P\left(\frac{K_\alpha\times E_0(K_\alpha)}{\omega\mu_0}\right) \tag{35}$$

E^1 using Green's function will be evaluated in terms of E^0, similarly, E^2 in terms of E^0 and iterative solution in terms of lower order fields.

$$E^1 = -\int G_k\,(r-r')\left[\nabla\left(\nabla\chi_e, E^0\right)(r') + j\omega\mu_0\left(\nabla\chi_m\times H^0\right)(r') + K^2\left(\chi_e(r') + \chi_m(r')\right)E^0\,(r')\right]d^3r' \tag{36}$$

This E^1 is in terms of E^0 and H^0.
Similarly, for E^2

$$\begin{aligned}&\nabla^2 E^2 - \nabla\mathrm{div}E^2 + j\omega\left(\nabla\times\chi_m H^1\right) - j^2\omega^2\mu_0\epsilon_0\left(\chi_e\,(r')\,\chi_m\,(r')\,E^0\,(r')\right)\\ &+ K^2 E^2 + K^2(\chi_e\,(r') + \chi_m\,(r')\,E^1) = 0\end{aligned} \tag{37}$$

From Equation (7) —

$$\epsilon_0\mathrm{div}E^2 = -\epsilon_0\nabla(\chi_e E^1) - \epsilon_0\chi_e\mathrm{div}E^1 \tag{38}$$

Put in Equation (38)

$$\begin{aligned}&\nabla^2 E^2 + \nabla\left\{\left(\nabla\chi_e\,(r'), E^1\,(r')\right) + \chi_e\,(r')\,E^1\,(r')\right\} + K^2 E^2\\ &= -j\omega\left(\nabla\times\left(\chi_m, H^1\right) - K^2\left(\chi_m + \chi_e\right)E^1 - K^2\chi_e\chi_m E^0\right.\end{aligned} \tag{39}$$

So

$$\left(\nabla^2 + K^2\right)E^2 = -\nabla\left\{\nabla\left(\chi_e, E^1\right) + \chi_e\left(\nabla\cdot E^1\right)\right\} - j\omega\left(\nabla\times\chi_m, H^1\right) - K^2\left(\chi_m + \chi_e\right)E^1 - K^2\chi_e\chi_m E^0 \tag{40}$$

(The above equation holds true for permittivity χ_e and permeability χ_m).

If $\chi_m = 0$ for nonmagnetic material, then equation is reduced to

$$\left(\nabla^2+K^2\right) E^2 = -\nabla \left\{\nabla \left(\chi_e, E^1\right) + \chi_e \left(\nabla \cdot E^1\right)\right\} - K^2 \left(\chi_m + \chi_e\right) E^1 \tag{41}$$

$$E^2 = -\int G_k\left(r-r'\right)\left[\nabla\left(\nabla\chi_e, E^1\right)\left(r'\right) - j\omega\mu_0\left(\nabla\chi_m \times H^1\right)\left(r'\right) + K^2\left(\chi_e\left(r'\right)E^1\left(r'\right)\right)\right] d^3r' \tag{42}$$

E^2 is in terms of E^1 and H^1.

In Equation (24), also if $\chi_m = 0$, the material is assumed as nonmagnetic, and the equation is reduced to

$$\left(\nabla^2 E^1 + K^2 E^1\right) = -\left[\nabla\left(\nabla\chi_e, E^0\right)\left(r'\right) + K^2\left(\chi_e E^0\right)\right] \tag{43}$$

First order-perturbed field in terms of the Green's function is written as —

$$E^1 = -\int G_k\left(r-r'\right)\left[\nabla\left(\nabla\chi_e, E^0\right)\left(\omega, r'\right) + K^2\left(\chi_e E^0\right)\left(\omega, r'\right)\right] d^3r' \tag{44}$$

$$E^1\left(\omega, r\right) = -\frac{1}{4\pi}\int \frac{e^{-jk(r-r')}}{\left|r-r'\right|}\left[\nabla\left(\nabla\chi_e, E^0\right)\left(\left(\omega, r'\right)\right) + K^2\left(\chi_e E^0\right)\left(\omega, r'\right)\right] d^3r' \tag{45}$$

Hence E^1 and E^2 are iterative solutions where E^1 depends on E^0, and E^2 depends on E^1. So if E^1 is calculated, we can put the values in Equation (43) to calculate E^2. However for a magnetic material, the computations are a bit lengthy.

Similarly, H^2 can be calculated from Equations (29), (30) and (31). In terms of χ_m, χ_e

$$\nabla^2 H^2 + K^2 H^2 = \nabla\left(\nabla \cdot H^2\right) - j\omega\epsilon_0\left(\nabla \times \left(\chi_e E^1\right) + j^2\omega^2\mu_0\epsilon_0(\chi_m + \chi_e)H^1 + j^2\omega^2\mu_0\epsilon_0(\chi_m\chi_e)H^0\right. \tag{46}$$

$$\nabla^2 H^2 + K^2 H^2 = -\mu_0\left(\nabla\left(\chi_m H^1\right) + \chi_m\nabla\cdot H^1\right) - j\omega\epsilon_0\left(\nabla\times\left(\chi_e, E^1\right) - K^2\chi_e\chi_m H^0 - K^2(\chi_e+\chi_m)H^1\right. \tag{47}$$

$$= \left\{\mu_0\nabla\left(\left(\nabla\cdot\chi_m, H^1\right) + \chi_m\nabla\cdot H^1\right) + K^2(\chi_e+\chi_m)H^1 + j\omega\epsilon_0\left(\nabla\times\left(\chi_e, E^1\right) + K^2\chi_e\chi_m H^0\right)\right. \tag{48}$$

Equations (46) and (33) tell us how to compute the first order scattered fields E^1, H^1 from the incident fields E^0, H^0, and Equations (43) and (47) tell us how to compute the second order scattered fields E^2, H^2 that is the next higher order corrections to the scattered fields in terms of E^0, H^0 and E^1, H^1. We have been using second order perturbation theory.

3. FORWARD SOLVERS IF THERE IS A SECOND ORDER SUSCEPTIBILITY χ^2

$$\operatorname{div}\left(E + \delta\chi^1 E^1 + \delta^2\chi^2 E^2 \pm \ldots\right) = 0 \ [11]$$

Using Einstein summation convention over space β is defined as

$$\left(\chi^2\left(E\otimes E\right)\right)\left(\omega, r\right) = \int \chi^2(\omega_1, \omega-\omega_1, r)(E(\omega_1, r)\otimes E(\omega-\omega_1, r))d\omega_1 \tag{49}$$

$$= \left(\left(\int \chi^2_{\alpha\beta\gamma}\left(\omega_1, \omega-\omega_1, r\right) E_\beta\left(\omega_1, r\right) E_\gamma\left(\omega-\omega_1, r\right) d\omega_1\right)\right)_\alpha \tag{50}$$

α, β, γ are the indices.

Again Einstein summation convention over $(\beta\gamma)$ is implied

$$E = E^0 + \delta E^1 + \delta^2 E^2 + O(\delta^3) \tag{51}$$

$$\operatorname{div} E^0 = 0,$$

$$\operatorname{div} E^1 + \operatorname{div}\left(\chi^1 \cdot E^0\right) = 0, \tag{52}$$

$$\operatorname{div} E^2 + \operatorname{div}(\chi^1 E^1) + \operatorname{div}(\chi^2 E^0 \otimes E^0) = 0 \tag{53}$$

$$\chi^1_{\alpha\beta}\left(\omega, r\right) = \sum_{m=1}^{N} \theta^1_{\alpha\beta m}\psi^1_m\left(\omega, r\right) \tag{54}$$

$$\chi^2_{\alpha\beta\gamma}\left(\omega_1, \omega_2, r\right) = \sum_{m=1}^{N} \theta^2_{\alpha\beta\gamma m}\psi^2_m\left(\omega_1, \omega_2, r\right) \tag{55}$$

Remark1: The coefficients $\theta^1_{\alpha\beta m}$ and $\theta^2_{\alpha\beta\gamma m}$ in these expressions are the parameters to be estimated. ψ^1_{km}, ψ^2_{km} known as basis functions.
If E^0 is represented by —

$$E^0(\omega, r) = \int F(\omega, \hat{n}) \exp(jk\hat{n}\cdot r)\, d\Omega(\hat{n})\, (\hat{n}, F(\omega,\hat{n})) = 0 \tag{56}$$

(Since $\text{div}E^0 = 0$)

$$\nabla \times E = -j\omega\mu H \tag{57}$$
$$\nabla \times H = j\omega\epsilon_0(E + \delta\chi^1 E + \delta^2\chi^2(E \otimes E)) \tag{58}$$
$$\nabla \cdot H = 0 \tag{59}$$

Hence

$$\nabla(\nabla\cdot E) - \nabla^2 E = k^2(E + \delta\chi^1 E + \delta^2\chi^2(E\otimes E)) \tag{60}$$
$$\text{Propagation constant } k^2 = \omega^2\epsilon_0\mu \tag{61}$$
$$\nabla^2 E^0 + K^2 E^0 = 0, \tag{62}$$
$$\nabla^2 E^1 + K^2 E^1 + k^2\chi^1 E^0 + \nabla\left(\text{div}\left(\chi^1 E^0\right)\right) = 0, \tag{63}$$
$$\nabla^2 E^2 + k^2 E^2 + k^2\chi^1 E^1 + k^2\chi^2\left(E^0\otimes E^0\right) + \nabla\left(\text{div}\left(\chi^1 E^1\right)\right) + \nabla\left(\text{div}\left(\chi^2\left(E^0\otimes E^0\right)\right)\right) = 0 \tag{64}$$

Equations are represented in the form of basis function and parameter variation defined in indices.

In Equation (37), we put susceptibility in matrix form and define it in terms of indices $\alpha\beta\gamma m$

$$E^1_\alpha(\omega, r) = -\frac{1}{4\pi}\int G_\omega(r, r')\left[\left(\left(\chi^1_{\gamma\beta}(\omega, r'), E^0_\beta(\omega, r')\right), \gamma\alpha\right) d^3r' + \left(K^2\chi^1_{\alpha\beta}(\omega, r')E^0_\beta(\omega, r')\right)\right] d^3r' \tag{65}$$

Remark2: 1 — A symbol like $F(\omega, r)_{,\eta}$ means $\frac{\partial F(\omega,r)}{\partial X^\eta}$ $\eta = 1, 2, 3$.

and a symbol like $\frac{\partial^2 F(\omega,r)}{\partial X^\eta \partial X^\alpha}$ *means* $\eta, \alpha = 1, 2, 3$.

2 — All through here calculates the indices $\alpha\beta\gamma k$ *run over 1, 2, 3 and we adopt the summation convention, i.e., if a repeated index appears then it means we are summing over that index.*

Defining a susceptibility matrix in the form of wavelet basis $\psi^1(\omega, r')$ function, we can express the scattered signal as —

$$\begin{aligned} E^1_\alpha(\omega, r) = & -\sum_{k,\gamma,\beta}\theta^1_{\gamma\beta k}\int\left[\left(\left(\psi^1_k(\omega, r'), E^0_\beta(\omega, r')\right), \gamma\alpha\right) d^3r'\right. \\ & \left. + \left(\delta_{\gamma\alpha}K^2\psi^1_k(\omega, r')E^0_\beta(\omega, r')\right)\right] G_\omega(r, r')\, d^3r' \end{aligned} \tag{66}$$

$\theta^1_{\gamma\beta k}$ are parameters in terms of wavelet coefficients.

Green function

$$G_\omega(r, r') = \frac{e^{-ik|r-r'|}}{4\pi|r - r'|} \tag{67}$$

Or if electromagnetic field in frequency domain is given by

$$\begin{aligned} E^1_\alpha(\omega, r) = & \sum_{\alpha\beta\gamma}\theta^1_{\gamma\beta k}\left[\int F_\beta(\omega\hat{n})d\Omega(\hat{n})\right. \\ & \left.\times\int\left(\delta_{\gamma\alpha}k^2\psi^1_k(\omega, r')\, e^{ik\hat{n}\cdot r'} + \left(\psi^1_k(\omega, r')\, e^{ik\hat{n}\cdot r'}\right)_{,\alpha\gamma}\right)\right] G_\omega(r, r')\, d^3r' \end{aligned} \tag{68}$$

$$\begin{aligned} E^1_\alpha(\omega, r) = & \sum_{\alpha\beta\gamma}\theta^1_{\gamma\beta k}\left[\int F_\beta(\omega\hat{n})d\Omega(\hat{n})\right. \\ & \left.\times\int\left\{\left(k^2\delta_{\gamma\alpha}\psi^1_k(\omega, r')\, e^{ik\hat{n}\cdot r'} + \psi^1_{k,\gamma}(\omega, r')\, e^{ik\hat{n}_\alpha} - \psi^1_{k,\gamma}(\omega, r')\, k^2 n_\alpha n_\gamma\right) e^{ik\hat{n}\cdot r'}\right\}\right] \\ & G_\omega(r, r')\, d^3r' + \left(\psi^1_{k,\alpha\gamma}(\omega, r')\right) + \left(\psi^1_{k,\alpha}(\omega, r')\right)e^{ik\hat{n}_\gamma} \end{aligned} \tag{69}$$

We can write this as —

$$E_{\alpha}^{1}(\omega, r) = \sum_{\alpha\beta\gamma} \theta_{\gamma\beta k}^{1} \int F_{\beta}(\omega, \hat{n}) K_{\alpha\gamma}(\omega, \hat{n_{\alpha}} r') \tag{70}$$

where

$$\begin{aligned} K_{\alpha\gamma}(\omega, \hat{n_{\alpha}}, r') \; = & \int k^{2} \psi_{k,\gamma}^{1}(\omega, r') (\delta_{\gamma\alpha} - n_{\alpha} n_{\gamma}) \exp(jk\hat{n} \cdot r') G(r, r') d^{3}r' \\ & + \int jk \left(\psi_{k,\alpha}^{1}(\omega, r') n_{y} + \psi_{k,\gamma}^{1}(\omega, r') n_{\alpha}\right) \exp(jk\hat{n} \cdot r')) G_{\omega}(r, r') d^{3}r' \\ & + \int \psi_{k,\alpha\gamma}^{1}(\omega, r') \exp(jk\hat{n} r') G_{\omega}(r, r') d^{3}r' \end{aligned} \tag{71}$$

n is a unit vector representing direction defined for a region L.

The above equations give the forward solver data and it as an iterative process. Other higher order fields can also be calculated.

4. WAVELETS FOR NUMERICAL SOLUTIONS OF INTEGRAL EQUATIONS

The integral equations provide an important tool for modeling a numerous phenomenon and processes. Many numerical methods have been developed for one-dimensional integral equation, and fewer methods are known for two- and three-dimensional integral equations.

Many different basic functions are used to estimate the solution of integral equations, such as orthogonal functions and wavelets. Daubchies wavelets are orthogonal wavelets with compact support, and they have been used in different numerical approximation methods [14].

The orthogonal basis $\psi_n(t)$ of one-dimensional Daubchies wavelet for the compact support space $L^2[0, 10]$ consists of

$$\psi_{k,n}(t) = |a|^{-\frac{1}{2}} \psi\left(a_0^k t - n b_0\right)\Big|_{[0,1]} \tag{72}$$

where $n = 1, 2 \ldots, 0 \leq k \leq 2^n - 1$.

It forms a basis for $L^2(R) \cdot a_0 = 2$ and $b = 1$.

In this work we apply three-dimensional Haar wavelet construction on $[0, 10] \times [0, 10] \times [0, 10]$ to solve the least square estimation method and inner product method.

The integer 2^k indicates the level of the wavelet, and nk_0 is the translation parameter. By simple calculations

$$\int_0^1 \psi_m(r) \psi_n(r) = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \tag{73}$$

Any function $\psi_n(x) \in C[0, 1]$ can be expressed [12] as $\sum_n \langle f, \ \psi_n \rangle \psi_n$ where

$$\langle f, \psi_n \rangle = \int_0^1 f(r) \psi_n(r) dr \tag{74}$$

If $\psi_k(r')$ is a basis function which is one-dimensional wavelet on $[0, 1]$, then for three-dimensional analysis we have taken the same wavelet in three dimensions x, y, z.

The expansion of $f(x, y, z)$ is defined over $[0, 1] \times [0, 1] \times [0, 1]$ expanded by the three dimensional Daubchies wavelet $Db6$.

For simplification we assume that if wavelet basis is as $\psi_k(r')$, then define susceptibility and permeability in one dimension as —

$$\chi_{\mathrm{e}}(r') = \sum_{1}^{d} a_k \psi_k(r') \quad \text{And} \quad \chi_{\mathrm{m}}(r') = \sum_{1}^{d} b_k \psi_k(r') \tag{75}$$

a_k, b_ks are constants. For three dimensions we take summation in the given region space integrating over the region L. Here we have taken L from 0 to 1 for simulation.

Substituting $\chi_e(r')$ by above values in Equation (37), the fields are solved in terms of E_0 and H_0.

For simplification let a_k and b_k be assumed parameters of the medium (as measured values will be depending on it). Both $E^1(r)$ and $H^1(r)$ are written in the form of E^0 and H^0 from Equations (33) and (46).

$$E^1(r) = \sum_{k=1}^{d} a_k \lambda_k^1(r) + \sum_{k=1}^{d} b_k \lambda_k^2(r) \tag{76}$$

$E^1(r)$ is the scattered field, and $\lambda_k^1(r)$, $\lambda_k^2(r)$ are the integral equations in terms of E^0 and H^0, respectively. Similarly, for H field

$$H^1(r) = \sum_{k=1}^{d} a_k \lambda_k^3(r) + \sum_{k=1}^{d} b_k \lambda_k^4(r) \tag{77}$$

$H^1(r)$ is the scattered field, and $\lambda_k^3(r)$, $\lambda_k^4(r)$ are the integral equations in terms of E^0 and H^0, respectively.

We have E^0 and H^0 expressed in terms of wavelets.

$$H^0 = \sum_{1}^{p} \frac{\alpha_k E_0(k_\alpha)}{\omega \mu_0} \tag{78}$$

$$\lambda_k^1(r) = -\int G_k(r-r')\left[\nabla\left(\nabla\chi_e, E^0\right)(r') + K^2\left(\chi_e + \chi_m(r')\right)E^0(r)\right]d^3r' \tag{79}$$

$$\lambda_k^2(r) = \int G_k(r-r')\left\{j\omega\mu_0\left(\nabla\chi_m \times H^0\right)(r')\right\}d^3r' \tag{80}$$

$$\lambda_k^3(r) = \int G_k(r-r')\left\{j\omega\mu_0\left(\nabla\times\chi_e, E^0\right)\right\}d^3r' \tag{81}$$

$$\lambda_k^4(r) = \int \left\{\nabla\left(\nabla\chi_m, H^0\right) + K^2(\chi_m)H^0\right\}d^3r' \tag{82}$$

In the first experiment, we assume χ_e & χ_m a matrix and calculate the scattered outputs. In our earlier work, this has been taken as centrosymmetric and noncentrosymmeric matrices, and the scattered outputs were calculated [13].

First order scattered fields E^1 and H^1 and second order scattered fields E^2 and H^2 are calculated (Figure 1 [13]) using Maple.

5. PARAMETER ESTIMATION METHODS

5.1. Least Square Estimation

The scattered or measured data are as follows — which depend upon two parameters — θ_1, θ_2 of the medium [14]

$$E^0(t, r') = \theta_1 X_1(r') + \theta_2 X_2(r'), \tag{83}$$

$$E^1(t, r) = \theta_1 F_1(\hat{r}) + \theta_2 F_2(\hat{r}) \tag{84}$$

In order to find the error between the measured data and computer generated data, by applying least mean square error [14]

$$\sum\nolimits_k \left\|E^1(\omega, \hat{r_k}) - \alpha_1 F_1(\hat{r}_k) - \alpha_2 F_2(\hat{r}_k)\right\|^2 = X_n \tag{85}$$

The minimization of the equation uses Eq. (86) —

$$a_k b_k \left\{ w_{1j} \left\| E^1(r_j) - \sum_{k=1}^{d} a_k \lambda_k^1(r_j) - b_k \lambda_k^2(r_j) \right\|^2 + w_{2j} \left\| H^1(r_j) - \sum_{k=1}^{d} a_k \lambda_k^1(r_j) - b_k \lambda_k^2(r) \right\|^2 \right\} \tag{86}$$

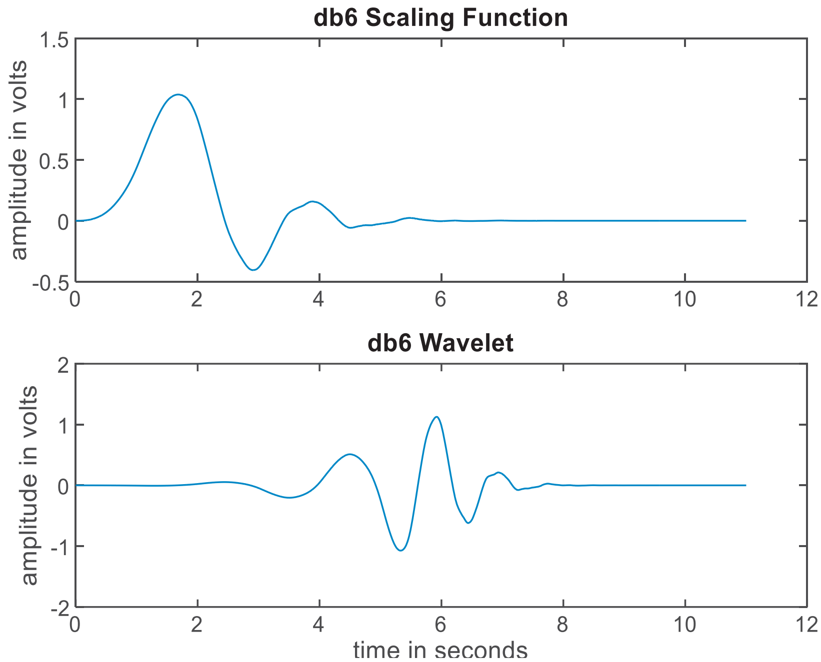


Figure 1. Db6 wavelet (using Mat lab).

Minimization and approximation are done by taking the derivative of error with respect to parameters X_1 and X_2, and we get the matrix and set it to zero.

$$\text{Then } \sum_k \left\|E^1(t,r) - \alpha_1 F_1(\hat{r}_k) - \alpha_2 F_2(\hat{r}_k)\right\|^2 = Y_n \tag{87}$$

$$\text{Finding } \frac{dY_n}{d\alpha_1} \text{ and } \frac{dY_n}{d\alpha_2} \text{ and setting it to zero} \tag{88}$$

Parameters are defined in terms of α_1 and α_2.

We get $\alpha'_1\alpha'_2$ in terms of $F_1(\hat{r})$ and $F_2(\hat{r})$ in matrix form. These are the parameters of the nonlinear material

$$\begin{aligned}\begin{bmatrix}\alpha'_1\\ \alpha'_2\end{bmatrix}^T &= \text{argmin}_{(\alpha'_1\alpha'_2)} \sum_{l\le k\le p} \left((W_\psi Y)_{n_{k,l}}) - (W_\psi Y)^T_{n_{k,l}}\right)^2\\ &= \left[\sum_{n,k} (W_\psi X)_{n_{k,l}} (W_\psi X)^T_{n_{k,l}}\right]^{-1} \left[\sum_{k,l} (W_\psi Y)_{n_{k,l}}\right]\end{aligned} \tag{89}$$

Error minimization leads to the following matrix

$$-\begin{bmatrix} 2\left\|X_1(\hat{r}_k)\right\|^2 & 2\sum_k \text{Re}\left(X_1(\hat{r}_k)\, X_2(\hat{r}_k)\right)\\ 2\sum_k Re\left(X_1(\hat{r}_k)\, X_2(\hat{r}_k)\right) & 2\left\|X_2(\hat{r}_k)\right\|^2 \end{bmatrix}^{-1} \times \begin{bmatrix} 2\sum_k \text{Re}\left(E^1(\hat{r}_k)\, X_1(\hat{r}_k)\right)\\ 2\sum_k \text{Re}\left(E^1(\hat{r}_k)\, X_2(\hat{r}_k)\right)\end{bmatrix} \tag{90}$$

The fields are in terms of magnetic component and electric field components. Error minimization is done by the following equation —

Minimizing

$$a_k\beta_k w_{1j}\left\|E^1(r_j) - \sum_{k=1}^{d} a_k\lambda_k^1(r_j) - \sum_{k=1}^{d} b_k\lambda_k^2(r_j)\right\|^2 + w_{2j}\left\|H^1(r_j) - \sum_{k=1}^{d} a_k\lambda_k^3(r_j) - \sum_{k=1}^{d} b_k\lambda_k^4(r_j)\right\|^2 \tag{91}$$

The results are discussed in Section 6.

5.2. Inner Product with Integral Equations

Another method is by taking inner product of fields generated by forward solver of E^1, H^1 to $\lambda_m^1(r_j)$ and $\lambda_m^3(r_j)$, respectively. It is also used as a basis function as scattered waves are presented in its form. Here we get two sets of equations for E field and H field.

$$\left\langle E^1(r_j), \lambda_m^1(r_j)\right\rangle = \sum_{k=1}^{d} a_k \left\langle \lambda_k^1(r_j), \lambda_m^1(r_j)\right\rangle + \sum_{k=1}^{d} b_k \left\langle \lambda_k^2(r_j), \lambda_m^1(r_j)\right\rangle \tag{92}$$

$$\left\langle H^1(r_j), \lambda_m^3(r_j)\right\rangle = \sum_{k=1}^{d} a_k \left\langle \lambda_k^2(r_j), \lambda_m^3(r_j)\right\rangle + \sum_{K=1}^{d} \lambda_k \left\langle \lambda_k^4(r_j), \lambda_m^3(r_j)\right\rangle \tag{93}$$

By adding them, we get the following equations —

RHS of the following equation generates data in a forward solver called computer-generated data where fields are represented in the form of integral equations. It is an inner product between integral equations.

From Equations (93) and (94) —

$$\begin{aligned}
&\sum_{j=1}^{K} w_{1j} \left\langle E^1(r_j), \lambda_m^1(r_j)\right\rangle + \sum_{j=1}^{K} w_{2j} \left\langle H^1(r_j), \lambda_m^3(r_j)\right\rangle \\
&= \sum_{K=1}^{d} a_k \sum_{j=1}^{K} w_{1j} \left\langle \eta_k^1(r_j), \eta_m^1(r_j)\right\rangle + \sum_{K=1}^{d} a_k \sum_{j=1}^{K} w_{2j} \left\langle \eta_k^2(r_j), \eta_m^3(r_j)\right\rangle + \sum_{K=1}^{d} b_k \sum_{j=1}^{K} w_{1j} \left\langle \eta_k^2(r_j), \eta_m^1(r_j)\right\rangle \\
&\quad + \sum_{K=1}^{d} b_k \sum_{j=1}^{K} w_{2j} \left\langle \eta_k^4(r_j), \eta_m^3(r_j)\right\rangle
\end{aligned} \tag{94}$$

This gives a scattered field matrix

$$\xi^E = A_{EE}\alpha + A_{EH}\beta \tag{95}$$

$$\xi^H = A_{HE}\alpha + A_{HH}\beta \tag{96}$$

In addition, parameters can be calculated by using inverse of A matrix with the scattered field matrix.

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} A_{EE} & A_{EH} \\ A_{HE} & A_{HH} \end{bmatrix} \begin{bmatrix} \xi^E \\ \xi^H \end{bmatrix} \tag{97}$$

where field matrix scattered is multiplied with the integral equations $\lambda_m^1(r_j)$ in time domain.

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} \begin{bmatrix} \left\langle \lambda_k^1(r_j), \lambda_m^1(r_j)\right\rangle & \left\langle \lambda_k^2(r_j), \lambda_m^1(r_j)\right\rangle \\ \left\langle \lambda_k^2(r_j), \lambda_m^3(r_j)\right\rangle & \left\langle \lambda_k^4(r_j), \lambda_m^3(r_j)\right\rangle \end{bmatrix} = \begin{bmatrix} \xi^E \\ \xi^H \end{bmatrix} \tag{98}$$

$$\begin{bmatrix} \alpha_k \\ \beta_k \end{bmatrix} = A^{-1} \begin{bmatrix} \xi^E \\ \xi^H \end{bmatrix} \tag{99}$$

$$\xi^E = \sum_{j=1}^{K} w_{1j} \langle\langle E^1(r_j), \lambda_m^1(r_j)\rangle \tag{100}$$

$$\xi^H = \sum_{j=1}^{K} w_{2j} \langle\langle H^1(r_j), \lambda_m^1(r_j)\rangle \tag{101}$$

Here again put $\chi_m = 0$ for a nonmagnetic material.

$$\left\langle E^1 \eta_k^1\right\rangle + \left\langle H^1 \eta_k^3\right\rangle = \begin{bmatrix} a_k & b_k \end{bmatrix} \begin{bmatrix} w_i \left\langle a_k \lambda_k^1(r), a_k \lambda_k^1(r)\right\rangle & \langle w_{ij} = 0\rangle \\ w_{2j} \left\langle a_k \lambda_k^1(r), a_k \lambda_k^1(r)\right\rangle & w_{2j} \left\langle a_k \lambda_k^3(r), a_k \lambda_k^4(r)\right\rangle \end{bmatrix} \tag{102}$$

We get —

$$\left|a_k\lambda_k^1(r)\right|^2 = \left|\nabla\left(\nabla\chi_e, E^0\right)(r') + K^2(\chi_e)E^0(r)\right|^2 \tag{103}$$

$$\left\langle a_k\lambda_k^1(r_j), a_k\lambda_k^2(r_j)\right\rangle = 0 \tag{104}$$

$$\left|a_k\lambda_k^3(r_j)\right|^2 = \left|j\omega\epsilon_0\nabla\times\chi_e, E\right|^2 \tag{105}$$

$$\left\langle a_k^2\lambda_k^3(r_j), \lambda_k^4(r_j)\right\rangle = (i\omega\epsilon_0\nabla\times\chi_e, E)\left(\nabla\left(\nabla\chi_m, H^0\right) + K^2(\chi_m+\chi_e)H^0\right) \text{ where } \chi_m = 0 \tag{106}$$

$\begin{bmatrix} A_{EE} & A_{EH} \\ A_{HE} & A_{HH} \end{bmatrix}$ is a computer generated forward solver. $\begin{bmatrix} \xi^E \\ \xi^H \end{bmatrix}$ is the measured scattered field inner product with the basis functions. $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ are the parameters of the medium.

Future Scope

5.3. The Statistical Parameters of the Random Medium Over Dimension L

Nonlinear medium behaves as a harmonic oscillator [5], and the scattering is random. Random variables whose matrix is estimated are calculated by estimating the mean value of $E(\omega, r)$ and correlations using local ergodicity. Ensemble averages can be replaced by local frequency and spatial averaging defined by correlations of the scattered fields. If $\theta'_{\gamma\beta m}$ are random variables whose statistics is to be estimated, then we find the expectation $E(\theta'_{\gamma\beta m})$ and $E(\theta'_{\gamma\beta m}\theta'_{\gamma'\beta' m'})$ estimating the mean value of $E'_\alpha(\omega r)$ and its correlations using local ergodicity. Here ensemble averages are replaced by local frequency and spatial averages.

$$E[E'_\alpha(\omega, r)] = \sum E\left(\theta'_{\gamma\beta m}\right)L_{\alpha\beta\gamma m}(\omega, r) \tag{107}$$

And

$$E\left[E'_\alpha(\omega, r)E'_{\alpha'}\left(\omega', r'\right)\right] = \sum_{\alpha\beta\gamma m\alpha'\beta'\gamma' m'} E\left(\theta'_{\alpha\gamma\beta m}\theta'_{\alpha'\gamma'\beta' m'}\right)L_{\alpha\gamma\beta m}(\omega, r)L_{\alpha'\gamma'\beta' m'}\left(\omega', r'\right) \tag{108}$$

where $E'_\alpha(\omega, r) = \sum\limits_{\gamma\beta m}\theta_{\gamma\beta m}L_{\alpha\beta\gamma m}(\omega, r)$.

From Equation (69)

$$\begin{aligned} L_{\alpha\beta\gamma n}(\omega, r) \;=\; & \int F_\beta(\omega, \hat{n})\sum_{\alpha\beta\gamma}\theta^1_{\gamma\beta k}\left[\int F_\beta(\omega, \hat{n})d\Omega(\hat{n})\right. \\ & \left.\times\int\left(\delta_{\gamma\alpha}k^2\psi_k^1\left(\omega, r'\right)e^{ik\hat{n}\cdot r'} + \left(\psi_k^1\left(\omega, r'\right)e^{ik\hat{n}\cdot r'}\right)_{,\alpha\gamma}\right)\right]G_\omega\left(r, r'\right)d^3r' \end{aligned} \tag{109}$$

The aim here would be to evaluate the mean and covariance of parameters $\theta^1_{\beta k\gamma}$ from the mean and covariance of the scattered electric fields. The mean and covariance of the electric field can be estimated using spatial and frequency averages assuming ergodicity.

Here we assume that the parameters are θ. For example

$$\Big| E\left[E'_\alpha(\omega, r)E'_{\alpha'}\left(\omega', r'\right)\right] \approx \frac{1}{N}\int_{\xi,\eta\in B} E'_\alpha(\omega+\xi, r+\eta)E'_{\alpha'}\left(\omega'+\xi, r'+\eta\right)d\xi d\eta\,, \tag{110}$$

where $N = \int_B d\xi d\eta$.

This is the expansion of the scattered electric field in terms of susceptibility expansion coefficients $\theta_{\gamma\beta m}$, which are assumed to be random variables for characterizing the randomness of the susceptibility fluctuations.

6. RESULTS AND SIMULATIONS

6.1. Inner Product with Integral Equations

Defined by

$$\left\langle E^1, \lambda_m^1 \right\rangle \tag{111}$$

Using wavelet basis functions —

ψ_k is a basis function.

Find $\nabla\psi_k$ first, then take inner product with E^0

$$E^0 = \sum \psi_k \tag{112}$$

If r is a vector space $r = \hat{x} + \hat{y} + \hat{z}$

$$\nabla\psi_k(r) = \frac{\partial}{\partial x}\psi_x + \frac{\partial}{\partial y}\psi_y + \frac{\partial}{\partial z}\psi_z \tag{113}$$

$$\begin{aligned}(\nabla(\nabla\psi_k(r)) =\ & \frac{\partial}{\partial x^2}^2\psi_k + \frac{\partial}{\partial x}\frac{\partial}{\partial y}\psi_k + \frac{\partial}{\partial x}\frac{\partial}{\partial z}\psi_k + \frac{\partial}{\partial y}\frac{\partial}{\partial x}\psi_k \\ & + \frac{\partial}{\partial y^2}^2\psi_k + \frac{\partial}{\partial y}\frac{\partial}{\partial z}\psi_k + \frac{\partial}{\partial x\partial z}^2\psi_k + \frac{\partial}{\partial y}\frac{\partial}{\partial z}\psi_k + \frac{\partial}{\partial z^2}^2\psi_k\end{aligned} \tag{114}$$

From Equation (100) are computer generated data from forward solver, and $[\begin{smallmatrix}\xi^E \\ \xi^H\end{smallmatrix}]$ are measured scattered field.

In step one we define susceptibility χ_e and permeability χ_m of the medium in terms of wavelet basis functions.

$$\chi_e(r) = \sum_{k=1}^{d} a_k\psi_k(r) \tag{115}$$

In direct least square-based estimation on spatial samples at points, $r_1, r_2, r_3, \ldots, r_N$ we get parameters by using the equation

$$\begin{aligned}\left(\hat{\theta}_1, \hat{\theta}_2\right)^T &= \text{argmin}_{(\theta_1,\theta_2)} \sum_{k=1}^{N} \left(Y(r_k) - \theta^T X(r_k)\right)^2 \\ &= \left(\sum_{k=1}^{N}(X(r_k)X(r_k)^T\right)^{-1} \left(\sum_{K=1}^{N} Y(r_k)X(r_k)\right)\end{aligned} \tag{116}$$

Heavy computations are required for large N. On the other hand, if we know that parameters $X_1(r), X_2(r)$ have dominant wavelet coefficients at the resolution indices $\{n_1, n_2, n_p\}$ only then, we can use the model.

Basis function Daubchies wavelet (see Figure 1) is designed in Matlab and in Maple (see Figure 2). Initially the scattered electromagnetic fields are E^1, E^2 simulated in Maple [13].

6.2. Outputs Using Least Square Estimation

Using least square estimation method, we estimated state space representation of the susceptibility and permeability [14].

The incident electric and magnetic fields were designed, and scattered electric fields are simulated using self-phase modulation, which is defined by third order nonlinearity [14].

The inner matrix is generated with the help of these scattered fields, and parameters are achieved. This matrix is in the form of sine cosine wave harmonics when Maple is used. This also proves that the medium behaves as a harmonic oscillator [4] and as a scatterer. These scattered wave equations

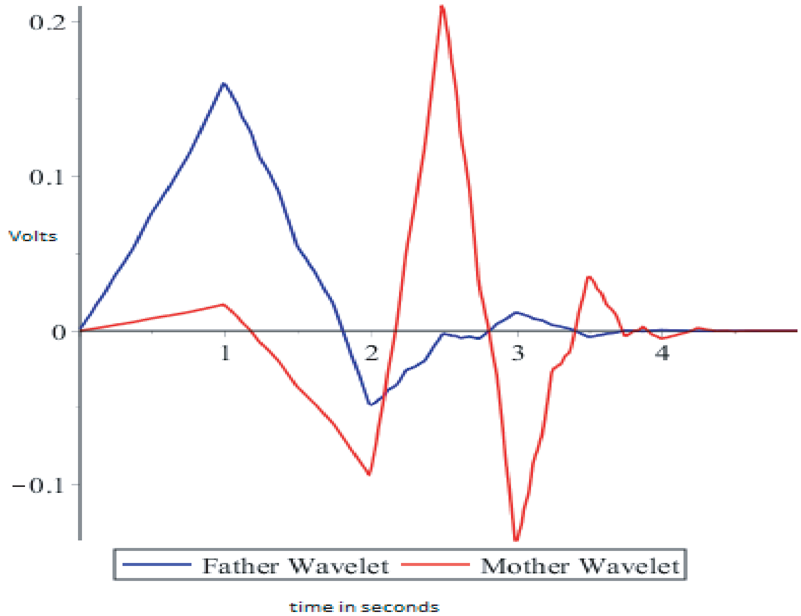


Figure 2. Wavelet basis function Db6 using Maple tool.

and above-mentioned methods can be used in imaging 1D, 2D and 3D data. Figures 1 and 2 give the Daubchies wavelet and its matrices in the workspace.

$$psi(s) = Wavelet\ Coefficients\ (\text{“}Daubechies\text{”},\ 6)$$

$$\Psi(x) = \left(\begin{bmatrix} 0.0352262918857095 \\ 0.0854412738820267 \\ -0.135011020010255 \\ -0.459877502118492 \\ 0.806891509311093 \\ -0.332670552950083 \end{bmatrix} \cdot \begin{bmatrix} 0.332670552950083 \\ 0.806891509311093 \\ 0.459877502118492 \\ -0.135011020010255 \\ -0.0854412738820267 \\ 0.0352262918857095 \end{bmatrix} \right)$$

The scattered fields are calculated at 4.7 GHz [14] frequency. Figure 3 gives the scattered field amplitude and phase variation, which is fed to Equation (90).

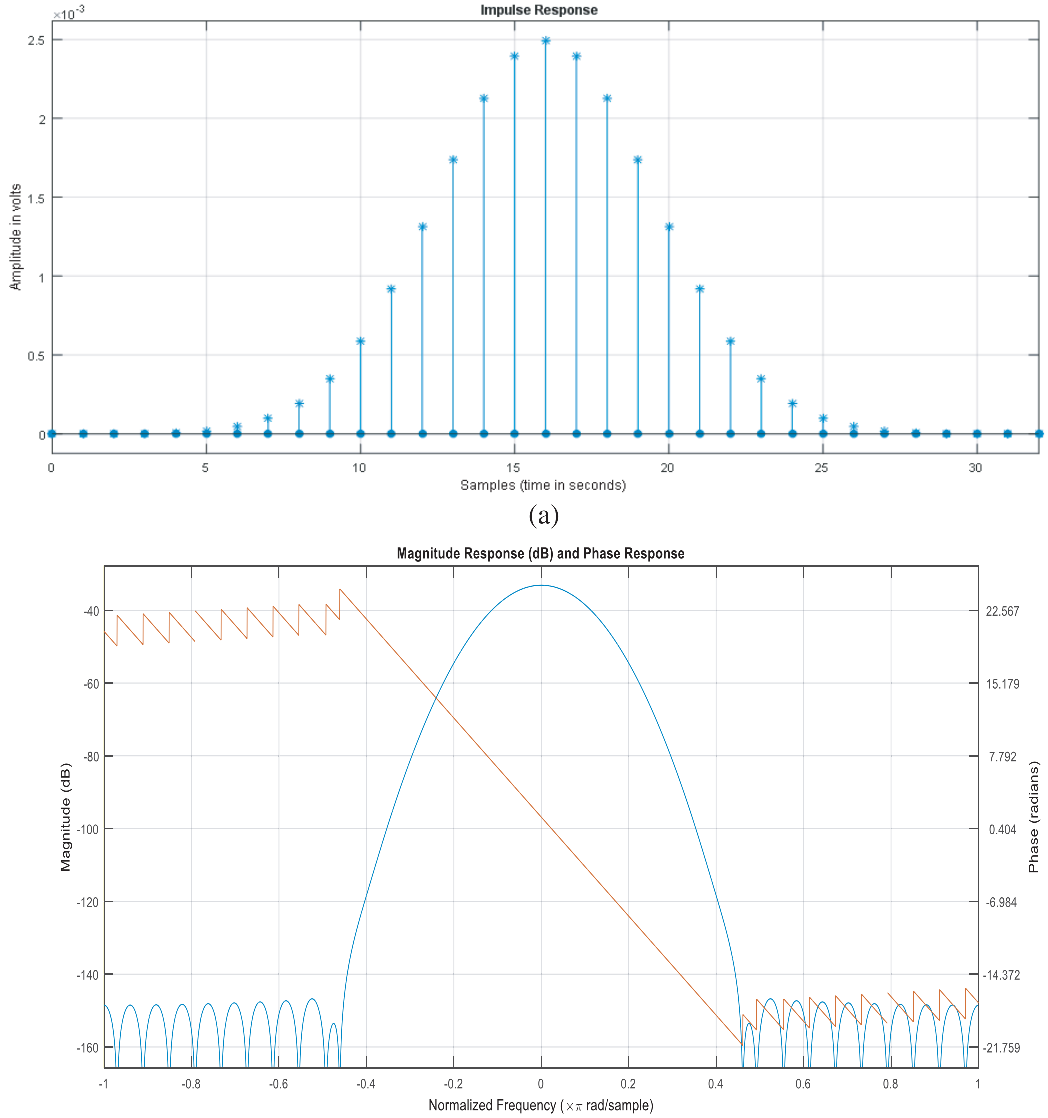
The least square estimation gives permeability and permittivity variations shown in Figures 4(a) and 4(b).

6.3. Outputs Using Inner Product Methods

A set of equations from (96) to (101) are used to find the inner product solutions. Scattered electromagnetic wave from self-phase modulated data [13] is used which is a nonlinear scattered electromagnetic wave. Using E and H fields from these second order nonlinear waves and defining three-dimensional wavelet basis functions for susceptibility, we get the set of equations. Equations from (103) to (107) are used for inverse solutions. Assuming negligible magnetic permeability, susceptibility variations are calculated. Figure 5 shows the relative susceptibility variation in the medium.

6.4. Discussions

In nonlinear inverse scattering, the sensors as antennas are required for the measurements [8]. We also need a large dataset and dielectric properties of the scattering medium. The solutions are based upon the Maxwell's equations. The scattered fields are continuous function of incident field and dielectric properties of the background. Therefore, we get a single solution in terms of the scattering fields at all positions. With the help of a sensor practical measurement of the field at a finite number of locations



(a)

(b)

Figure 3. Scattered waves E^1 generated by self-phase modulation for testing the inverse technique (Figure 3 [14]). (a) Impulse response. (b) Magnitude and phase response.

as well as limited number of frequencies is done. It has been studied that the solution is not unique for practical problems, and uniqueness of the integrating field is overcome by measuring large number of samples of the scattered field data [7]. We estimated susceptibility by the least square estimation method. The error between the measured values and computer-generated values is practically used to optimize parameters of the medium. This RF imaging technique is helpful in identifying the hidden objects and underground explosives, which can be used for security purposes.

Assuming the susceptibility matrix of the medium to be frequency and space dependent and also anisotropic, we formulate, using Maxwell's theory in such media, the basic generalized Helmholtz equation for the electric field. We then expand the inhomogeneous (i.e., the space dependent) susceptibility tensor as a linear combination of basis function with coefficients of this expansion as being unknown parameters to be estimated from the measurements of the electric field at different spatial points. Only one frequency 4.7 GHz is involved since we are using a field independent susceptibility, and therefore our partial differential equations are linear. Using the first order perturbation theory

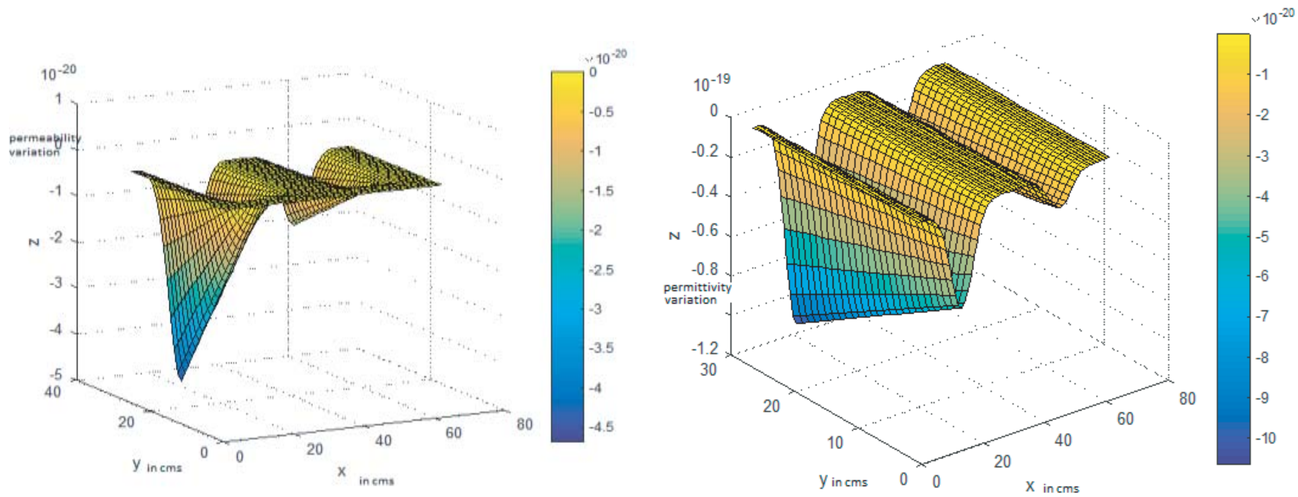


Figure 4. Spatial resolution of medium parameters at 4.7 GHz (Figure 4 [14]). (a) Relative susceptibility variation. (b) Relative permeability variation.

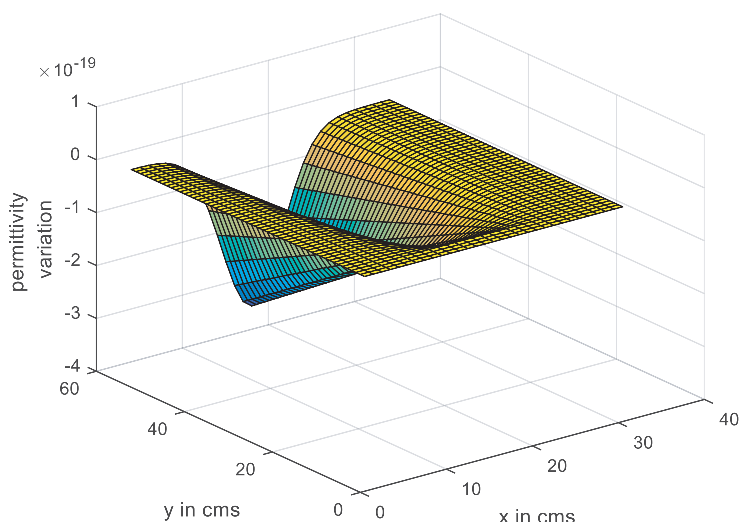


Figure 5. Relative susceptibility variation assuming medium is nonmagnetic using inner product method.

by treating the susceptibility tensor as being the first order of smallness, we express the first order perturbation to the electric field (i.e., the scattered field) as a linear combination of these expansion parameters. To do so, Green's function for the Helmholtz operator is used.

Once we have obtained such a solution for the perturbed electric field in terms of the susceptibility expansion parameters, we match this scattered field expansion to the actual measurements of the scattered electric field at different spatial points using the least squares method. The number of measurements must be far more than the number of parameters. In this way, we obtain accurate parameters estimation. Further, we also discuss the case of field dependent susceptibility. In this case, frequency mixing taking place, and our test function must depend on two frequency variables and also the spatial variables in contrast to the previous case where it depends only on one frequency and spatial variables. For the case, we have not carried out any simulation for that is the subject of another paper. Finally, we have also proposed an algorithm for estimating the parameters in the test function expansion of the susceptibility but rather the parameter statistics, i.e., mean and correlation. This is important in cases when the susceptibility undergoes rapid fluctuations.

7. CONCLUSION

Using Maxwell's equations, we have optimized the parameters of an inhomogeneous medium by scattered electromagnetic fields with a time domain algorithm. The inverse solutions using Least Square Estimation and Inner Product Methods are solved by method of moments. Parameters are obtained in terms of the basis functions. Forward solver technique is used with the first order nonlinearity and Kerr nonlinearity. Method of moment is used with wavelet bases in one-dimensional, two-dimensional and three-dimensional cases. For this, we use two-dimensional and three-dimensional wavelet functions. Computational complexity increases as the number of wavelets is increased. We can limit this problem by taking a limited number of wavelets for a set of scattered electromagnetic waves, which will reduce the computational cost. Wavelet technique requires less memory space. Wavelet basis also gives better solutions in terms of fast computations. To conclude, we have developed a computationally cheaper algorithm for estimating linear and nonlinear components of the parameters that govern the susceptibility field from measurements of the scattered Electromagnetic fields at different frequencies and spatial locations.

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