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A T-Matrix Solver for Fast Modeling of Scattering from Multiple PEC Objects

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Abstract—T matrix characterizes the scattering property of a single PEC object and does not depend on the incidence. In this work, we propose a method to derive a reduced-order T matrix for a single 3D PEC object with arbitrary shape. The method is based on the vector addition theorem and the conventional EFIE, MFIE or CFIE methods. Given the T matrix for a PEC object, the scattered fields can be directly calculated from any incidence. For multiple objects, a matrix equation system is built based on the T-matrix and the position of each object. Finally, numerical examples show the accuracy and efficiency for solving the scattering of both spherical and non-spherical arrays. Compared to the moment methods, the computational cost of solving the final matrix equation is reduced by several orders of magnitude.

1. INTRODUCTION

Modeling of electromagnetic scattering from multiple objects has been studied over many years. The popular solutions for analysis of scattering from conducting objects are the finite difference method (FDM), finite element method (FEM) and moment method (MOM). Among the moment methods, electric field integral equation (EFIE), magnetic field integral equation (MFIE) and combined field integral equation (CFIE) with RWG basis functions are widely used. However, for a large problem that includes multiple PEC objects, there are some challenges for the conventional methods. First, it is well known that fast algorithms need to be applied to these methods in order to solve large-scale problems. Second, as the mesh is refined for large problems, the condition numbers of EFIE and CFIE formulations grow fast and can cause the ill-conditioned system matrices. MFIE has a well-conditioned formulation, while it can only be applied to closed objects and is ill-conditioned for interior resonance problems.

The T-matrix method is firstly discussed in [1] for solving electromagnetic scattering problems. Later, [2] extends the T-matrix method to an arbitrary number of scatterers. The total T-matrix is expressed in terms of the individual T-matrices by an iterative procedure. In order to reduce the computational cost of the total T-matrix, a recursive algorithm is proposed in [3–5]. Since then, the use of this idea for various structures has been demonstrated [9–11, 13]. Although a great deal of work has been performed on solving scattering problems of multiple objects using the T-matrix method [12, 14, 15], applying the method to the multiple PEC structures with arbitrary shapes, especially to non-spherical structures is sill limited. Among the literature work for solving the scattering electromagnetic fields from the 3D multiple objects, most of the previous work handles multiple spherical or cylindrical objects only.

In this paper, a method based on T-matrix is proposed to analyze the scattering from multiple PEC objects. In this method, we first discuss how to obtain the T-matrix for each PEC object and

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then convert it into a small matrix. Then based on the small T-matrix for each object, an algorithm considering the interactions of multiple scatterers is proposed. There are three main advantages of this method. First is that since the T-matrix for each 3D PEC object with arbitrary shape can be found to be very small, the dimension of the system matrix equation for multiple objects can be several orders of magnitude smaller than those from the method of moments. Hence, the computational cost for multi-scatterer problems is greatly reduced compared to the conventional methods. Secondly, the proposed method is not limited to the spherical objects and can be applied to any multiple PEC problems with arbitrary shapes. Finally, since the T-matrix for each object is independent of incidence, the recalculation for difference incidences can be avoided.

2. FACTORIZATION OF THE DYADIC GREEN'S FUNCTION BY VECTOR ADDITION THEOREM

A dyadic Green's function in EM can be written as

$$\overline{\mathbf{G}}(\mathbf{r}_j, \mathbf{r}_i) = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) g(\mathbf{r}_j, \mathbf{r}_i) \tag{1}$$

It can be expanded by vector wave functions in spherical coordinates [6, 7]

$$\begin{aligned}\overline{\mathbf{G}}(\mathbf{r}_j, \mathbf{r}_i) =& ik \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{l(l+1)} [\mathbf{M}_{lm}(k, \mathbf{r}_{js}) \Re g \hat{\mathbf{M}}_{lm}(k, \mathbf{r}_{is})) \\ &+ \mathbf{N}_{lm}(k, \mathbf{r}_{js}) \Re g \hat{\mathbf{N}}_{lm}(k, \mathbf{r}_{is}))]\end{aligned} \tag{2}$$

where $\mathbf{r}_j - \mathbf{r}_i = \mathbf{r}_{js} - \mathbf{r}_{is}$, $|\mathbf{r}_{js}| < |\mathbf{r}_{is}|$. $\mathbf{M}_{lm}$ and $\mathbf{N}_{lm}$ are vector wave spherical harmonics expressed in terms of spherical Hankel functions and spherical harmonics. $\Re g$ means taking the regular part of the function where the spherical Hankel function is replaced by a spherical Bessel function:

$$\begin{aligned}\mathbf{M}_{lm}(k, \mathbf{r}) &= \nabla \times \mathbf{r}\psi_{lm}(k, \mathbf{r}) \\ \mathbf{N}_{lm}(k, \mathbf{r}) &= \frac{1}{k} \nabla \times \nabla \times \mathbf{r}\psi_{lm}(k, \mathbf{r})\end{aligned} \tag{3}$$

Here, $\psi_{lm}(k, \mathbf{r})$ is the solution of the Helmholtz equation in free space, and

$$\psi_{lm}(k, \mathbf{r}) = h_l^{(1)}(kr) Y_{lm}(\theta, \phi) \tag{4}$$

Here, $h_l^{(1)}(kr)$ is the first-kind spherical Hankel function,

$$Y_{lm}(\theta, \phi) = \sqrt{\frac{(l-m)!(2l+1)}{(l+m)!4\pi}} P_l^m(\cos(\theta)) e^{im\phi} \tag{5}$$

where $P_l^m(x)$ is the Legendre's polynomial.

In the above, $\Re g$ means taking the regular part of the function, which means that the spherical Hankel function is replaced by the spherical Bessel function.

$$\begin{aligned}\Re g \hat{\mathbf{M}}_{lm}(k, \mathbf{r}') &= \nabla \times \mathbf{r}' \hat{\psi}_{lm}(k, \mathbf{r}') \\ \Re g \hat{\mathbf{N}}_{lm}(k, \mathbf{r}') &= \frac{1}{k} \nabla \times \nabla \times \mathbf{r}' \hat{\psi}_{lm}(k, \mathbf{r})\end{aligned} \tag{6}$$

Here,

$$\hat{\psi}_{lm}(k, \mathbf{r}') = j_l(kr') Y_{lm}^*(\theta, \phi) \tag{7}$$

where j_l is the l-order spherical Bessel function. Since $Y_{lm}(\theta, \phi)$ is orthonormal, $Y_{l,-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi)$.

Truncating the summation at $l = l_{\max}$, then the number of terms involved in Eq. (2) is $P = (l_{\max} + 1)^2 - 1$. Therefore, Eq. (2) can be rewritten in a more compact form

$$\overline{\mathbf{G}}(\mathbf{r}_j, \mathbf{r}_i) = \overline{\boldsymbol{\psi}}^t(\mathbf{r}_{js})_{3\times 2P} \cdot \Re g \hat{\overline{\boldsymbol{\psi}}}(\mathbf{r}_{is})_{2P\times 3} \tag{8}$$

where $\overline{\boldsymbol{\psi}}^t(\mathbf{r}_{js})$ and $\Re g\hat{\overline{\boldsymbol{\psi}}}(\mathbf{r}_{is})$ are matrices composed of ordered $\mathbf{M}_{lm}$ and $\mathbf{N}_{lm}$.

In Eq. (2), $\mathbf{M}_{lm}$ and $\mathbf{N}_{lm}$ can be further expanded by the vector theorem in spherical coordinates [7], that is

$$\begin{aligned}\mathbf{M}_{lm}(\mathbf{r}) &= \sum_{l',m'}[\Re g\mathbf{M}_{l'm'}(\mathbf{r}')A_{l'm',lm}(\mathbf{r}'') + \Re g\mathbf{N}_{l'm'}(\mathbf{r}')B_{l'm',lm}(\mathbf{r}'')] \\ \mathbf{N}_{lm}(\mathbf{r}) &= \sum_{l',m'}[\Re g\mathbf{N}_{l'm'}(\mathbf{r}')A_{l'm',lm}(\mathbf{r}'') + \Re g\mathbf{M}_{l'm'}(\mathbf{r}')B_{l'm',lm}(\mathbf{r}'')]\end{aligned} \tag{9}$$

where $\mathbf{r} = \mathbf{r}' + \mathbf{r}''$ and $|\mathbf{r}'| < |\mathbf{r}''|$ has been assumed.

Next, Substituting Eq. (9) into Eq. (2), we obtain the expansion form for the dyadic Green's function

$$\begin{aligned}\overline{\mathbf{G}}(\mathbf{r}_{ji}) =& ik\sum_{L'L}^{\infty}\frac{1}{l(l+1)}[\Re g\mathbf{M}_{L'}(k,\mathbf{r}_{js'})A_{L',L}(\mathbf{r}_{s's})\Re g\hat{\mathbf{M}}_L(k,\mathbf{r}_{is}) \\ &+ \Re g\mathbf{N}_{L'}(k,\mathbf{r}_{js'})A_{L',L}(\mathbf{r}_{s's})\Re g\hat{\mathbf{N}}_L(k,\mathbf{r}_{is}) \\ &+ \Re g\mathbf{M}_{L'}(k,\mathbf{r}_{js'})B_{L',L}(\mathbf{r}_{s's})\Re g\hat{\mathbf{N}}_L(k,\mathbf{r}_{is}) \\ &+ \Re g\mathbf{N}_{L'}(k,\mathbf{r}_{js'})B_{L',L}(\mathbf{r}_{s's})\Re g\hat{\mathbf{M}}_L(k,\mathbf{r}_{is})]\end{aligned} \tag{10}$$

It can be further written as

$$\overline{\mathbf{G}}(\mathbf{r}_{ji}) = ik\sum_{L'L}^{\infty}\frac{1}{l(l+1)}\begin{pmatrix}\Re g\mathbf{M}_{L'}(k,\mathbf{r}_{js'})\\ \Re g\mathbf{N}_{L'}(k,\mathbf{r}_{js'})\end{pmatrix}^T\cdot\begin{pmatrix}A_{L',L}(\mathbf{r}_{s's}) & B_{L',L}(\mathbf{r}_{s's})\\ B_{L',L}(\mathbf{r}_{s's}) & A_{L',L}(\mathbf{r}_{s's})\end{pmatrix}\cdot\begin{pmatrix}\Re g\hat{\mathbf{M}}_L(k,\mathbf{r}_{is})\\ \Re g\hat{\mathbf{N}}_L(k,\mathbf{r}_{is})\end{pmatrix} \tag{11}$$

Here, we use $\mathbf{r}_{js} = \mathbf{r}_{js'} + \mathbf{r}_{s',s}$, where $|\mathbf{r}_{js'}| < |\mathbf{r}_{s's}|$. In the above, $L = (l,m)$, $L' = (l',m')$. The expressions for $A_{l'm',lm}$ and $B_{l'm',lm}$ can be found in [7].

In the above, Equation (9) can be written in the compact form as below

$$\overline{\boldsymbol{\psi}}^t(\mathbf{r})_{3\times 2P} = \Re g\overline{\boldsymbol{\psi}}^t(\mathbf{r}')_{3\times 2P}\cdot\overline{\boldsymbol{\alpha}}(\mathbf{r}'')_{2P\times 2P} \tag{12}$$

When $|\mathbf{r}'| < |\mathbf{r}''|$ is assumed, it is written as

$$\overline{\boldsymbol{\psi}}^t(\mathbf{r})_{3\times 2P} = \overline{\boldsymbol{\psi}}^t(\mathbf{r}'')_{3\times 2P}\cdot\overline{\boldsymbol{\beta}}(\mathbf{r}')_{2P\times 2P} \tag{13}$$

where $\overline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\beta}}$ are translation operators defined in [6, 7].

The expansion of the dyadic Green's function can also be rewritten in the compact form as

$$\overline{\mathbf{G}}(\mathbf{r}_j,\mathbf{r}_i) = \Re g\overline{\boldsymbol{\psi}}^t(\mathbf{r}_{js'})_{3\times 2P}\cdot\overline{\boldsymbol{\alpha}}(\mathbf{r}_{s's})_{2P\times 2P}\cdot\Re g\hat{\overline{\boldsymbol{\psi}}}(\mathbf{r}_{is})_{2P\times 3} \tag{14}$$

Here, $\overline{\boldsymbol{\psi}}^t$ and $\Re g\hat{\overline{\boldsymbol{\psi}}}$ are matrices composed of ordered $\Re g\mathbf{M}_{lm}$ and $\Re g\mathbf{N}_{lm}$. $\overline{\boldsymbol{\alpha}}$ is the matrix stacked by $A_{L',L}$ and $B_{L',L}$. The detailed formulations can be found in Appendix D of [7]. Fig. 1 shows all the position vectors mentioned above.

3. DERIVATION OF THE T-MATRIX FOR A SINGLE PEC SCATTERER

The scattering solution of a PEC object is known as

$$\mathbf{E}^s(\mathbf{r}) = i\omega\mu\int\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')d\mathbf{r}' \tag{15}$$

To get the scattering solution, usually the PEC object is discretized into RWG basis, and the scattering field of the whole object is discretized into the scattering solution from each basis. Suppose that the PEC object is discretized into M RWG bases, then

$$\mathbf{E}^s(\mathbf{r}) = \sum_{i=1}^{M} i\omega\mu\int_{s_i}\overline{\mathbf{G}}(\mathbf{r},\mathbf{r}'_i)\cdot\mathbf{\Lambda}_i(\mathbf{r}'_i)d\mathbf{r}'_i\cdot a_i \tag{16}$$

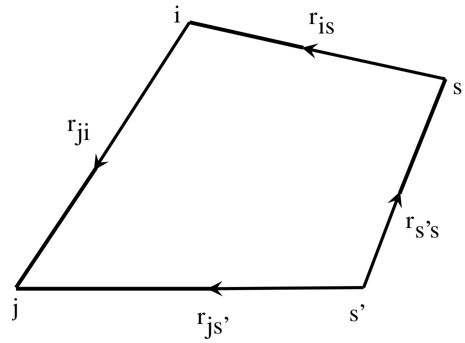


Figure 1. Configuration of position vectors.

Substituting Eq. (1) into Eq. (16) and using the property of the RWG basis, we can get

$$\mathbf{E}^s(\mathbf{r}) = \sum_{i=1}^{M} i\omega\mu \int_{s_i} g(\mathbf{r}, \mathbf{r}'_i)\mathbf{\Lambda}_i(\mathbf{r}'_i)d\mathbf{r}'_i \cdot a_i + \sum_{i=1}^{M} \frac{i}{\omega\epsilon} \int_{s_i} \nabla g(\mathbf{r}, \mathbf{r}'_i)\nabla' \cdot \mathbf{\Lambda}_i(\mathbf{r}'_i)d\mathbf{r}'_i \cdot a_i \tag{17}$$

Equation (17) is the familiar expression for EFIE equation. Rewrite $(\mathbf{r} - \mathbf{r}')$ as $(\mathbf{r} - \mathbf{r}_i) - (\mathbf{r}_i - \mathbf{r}'_i)$ in Eq. (16), and by using the expansion of dyadic Green's function in Eq. (8), Eq. (16) can be expanded into

$$\mathbf{E}^s(\mathbf{r}) = \sum_{i=1}^{M} \overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_i) \cdot \mathbf{M}_{ii} \cdot a_i \tag{18}$$

where $\mathbf{M}_{ii}$ is defined as

$$\mathbf{M}_{ii} = i\omega\mu \int_{S_i} \Re g\hat{\overline{\boldsymbol{\psi}}}(\mathbf{r}'_i - \mathbf{r}_i) \cdot \mathbf{\Lambda}_i(\mathbf{r}'_i)d\mathbf{r}'_i \tag{19}$$

In the above, $\mathbf{r}_i$ denotes the center of the i-th RWG; $\mathbf{r}'_i$ are the sampling points on the RWG; a_i is the current coefficient. In this way, each basis on the PEC surface is regarded as a subscatterer and its scattered field is expanded into outgoing wave function form. More compactly, Eq. (18) can be rewritten in a matrix form,

$$\mathbf{E}^s(\mathbf{r}) = \overline{\boldsymbol{\Psi}}^t(\mathbf{r}) \cdot \overline{\mathbf{M}}_{(1)} \cdot \mathbf{a}_{(1)} \tag{20}$$

$\overline{\boldsymbol{\Psi}}^t(\mathbf{r})$ and $\overline{\mathbf{M}}_{(1)}$ are larger matrices stacked by $\overline{\boldsymbol{\psi}}^t(\mathbf{r}-\mathbf{r}_i)$ and $\mathbf{M}_{ii}$, $i = 1, 2, \ldots, M$, respectively as below. Here M is the number of RWG bases used in the discretization of the PEC surface.

$$\overline{\boldsymbol{\Psi}}^t(\mathbf{r}) = [\overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_1), \overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_2), \ldots, \overline{\boldsymbol{\psi}}^t(\mathbf{r} - \mathbf{r}_M)], \tag{21}$$

$$\overline{\mathbf{M}}_{(1)} = [\mathbf{M}_{11}, \mathbf{M}_{22}, \ldots, \mathbf{M}_{MM}]^t. \tag{22}$$

$\mathbf{a}_{(1)}$ is the current coefficient vector. Subscript (1) indicates for one object.

Using the factorization of the Dyadic Green's function, the incident field by any source can also be expanded. Suppose that $\mathbf{J}(\mathbf{r}_s)$ are any kind of sources located at $\mathbf{r}_s$, then the incident field at $\mathbf{r}'_i$ is

$$\mathbf{E}^i(\mathbf{r}'_i) = i\omega\mu \int \overline{\mathbf{G}}(\mathbf{r}'_i - \mathbf{r}_s) \cdot \mathbf{J}(\mathbf{r}_s)d\mathbf{r}_s \tag{23}$$

Applying Eq. (14) to the dyadic Green's function above, one can get the incident field on the i-th RWG as

$$\mathbf{E}^i(\mathbf{r}'_i) = \Re g\overline{\boldsymbol{\psi}}^t(\mathbf{r}'_i - \mathbf{r}_i) \cdot \overline{\boldsymbol{\alpha}}_{is}(\mathbf{r}_i - \mathbf{r}_s) \cdot \mathbf{a}_s(\mathbf{r}'_s - \mathbf{r}_s) \tag{24}$$

where

$$\mathbf{a}_s(\mathbf{r}'_s - \mathbf{r}_s) = \int \Re g\overline{\boldsymbol{\psi}}^t(\mathbf{r}'_s - \mathbf{r}_s) \cdot \mathbf{J}(\mathbf{r}_s)d\mathbf{r}_s \tag{25}$$

Here, we consider $\mathbf{J}(\mathbf{r}_s)$ as the dipole excitation with magnitude Il. $I_x l, I_y l, I_z l$ are the excitation magnitude in $\hat{x}, \hat{y}, \hat{z}$ direction respectively. That is

$$\mathbf{J}(\mathbf{r}_s) = \mathbf{J}_x(\mathbf{r}_s) + \mathbf{J}_y(\mathbf{r}_s) + \mathbf{J}_z(\mathbf{r}_s) = (\hat{x}I_x l + \hat{y}I_y l + \hat{z}I_z l)\delta(\mathbf{r}_s) \tag{26}$$

Therefore, by the property of the δ function, we can get the x, y, z components of $\mathbf{a}_s$, respectively.

$$\begin{aligned}
[\mathbf{a}_s]_x &= i\omega\mu[\Re g\overline{\boldsymbol{\psi}}(-\mathbf{r}'_s)I_x l]_x \\
[\mathbf{a}_s]_y &= i\omega\mu[\Re g\overline{\boldsymbol{\psi}}(-\mathbf{r}'_s)I_y l]_y \\
[\mathbf{a}_s]_z &= i\omega\mu[\Re g\overline{\boldsymbol{\psi}}(-\mathbf{r}'_s)I_z l]_z
\end{aligned} \tag{27}$$

So far both the incident and scattered fields have been expanded into the wave function form. Next, a T-matrix can be defined as below to express the linear relationship between them, i.e.,

$$\mathbf{E}^s(\mathbf{r}) = \overline{\boldsymbol{\Psi}}^t(\mathbf{r}) \cdot \overline{\mathbf{T}}_{(1)} \cdot \overline{\boldsymbol{\alpha}}_s \cdot \mathbf{a}_s \tag{28}$$

where $\overline{\boldsymbol{\alpha}}_s$ is a larger matrix stacked by $\overline{\boldsymbol{\alpha}}_{is}$. We can see from Eq. (28) that the incident and scattered fields are related by T-matrix $\overline{\mathbf{T}}_{(1)}$. Given the T-matrix $\overline{\mathbf{T}}_{(1)}$, the scattered fields can be directly calculated from the incident fields.

In order to calculate the T-matrix, we first apply the boundary condition on the PEC surface. That is,

$$0 = \int_{S_i} \boldsymbol{\Lambda}_i(\mathbf{r}'_i) \cdot \{\mathbf{E}^i(\mathbf{r}'_i) + \mathbf{E}^s(\mathbf{r}'_i)\} d\mathbf{r}'_i \tag{29}$$

$i = 1, 2, \ldots, M$.

Then replacing $\mathbf{E}^s$ and $\mathbf{E}^i$ with Eqs. (16) and (24) respectively, we have

$$-\overline{\mathbf{N}}_{(1)} \cdot \overline{\boldsymbol{\alpha}}_s \cdot \mathbf{a}_s = \overline{\mathbf{S}} \cdot \mathbf{a}_{(1)} \tag{30}$$

Here, $\overline{\mathbf{S}}$ is the system matrix of the EFIE, MFIE or CFIE method. $\overline{\mathbf{N}}_{(1)}$ is a matrix with the diagonal subblocks defined as

$$\mathbf{N}_{ii} = \int_{S_i} \boldsymbol{\Lambda}_i(\mathbf{r}'_i) \cdot \Re g\overline{\boldsymbol{\psi}}^t(\mathbf{r}'_i - \mathbf{r}_i) d\mathbf{r}'_i \tag{31}$$

$i = 1, 2, \ldots, M$.

Next, substituting Eq. (20) into Eq. (28), we get

$$\overline{\mathbf{M}}_{(1)} \cdot \mathbf{a}_{(1)} = \overline{\mathbf{T}}_{(1)} \cdot \overline{\boldsymbol{\alpha}}_s \cdot \mathbf{a}_s \tag{32}$$

and from Eq. (30), we have

$$\mathbf{a}_{(1)} = -\overline{\mathbf{S}} \cdot \overline{\mathbf{N}}_{(1)} \cdot \overline{\boldsymbol{\alpha}}_s \cdot \mathbf{a}_s \tag{33}$$

Then substituting Eq. (33) into Eq. (32), we can get the formulation for the T-matrix $\overline{\mathbf{T}}_{(1)}$

$$\overline{\mathbf{T}}_{(1)_{(2PM\times 2PM)}} = -\overline{\mathbf{M}}_{(1)_{(2PM\times M)}} \cdot \overline{\mathbf{S}}^{-1}_{(M\times M)} \cdot \overline{\mathbf{N}}_{(1)_{(M\times 2PM)}} \tag{34}$$

In the above, M is the number of RWG bases on the PEC surface, and it is also the number of unknowns in the EFIE, MFIE or CFIE method. P is the number of expansion terms in the factorization of the dyadic Green's function. The dimension of the T-matrix $\overline{\mathbf{T}}_{(1)}$ is $2PM$ by $2PM$. Therefore, the T-matrix we obtain so far has larger dimensions than the final matrix of the MOM method. To reduce the dimension of the T-matrix $\overline{\mathbf{T}}_{(1)}$, we introduce it next.

The T-matrix $\overline{\mathbf{T}}_{(1)}$ derived so far is in terms of each basis on the PEC surface, so it has the dimension at least the same as that of MOM method, which is M^2. Actually, if the number of wave functions used for the field expansion of each basis is $2P$, then the dimension of $\overline{\mathbf{T}}_{(1)}$ is $(2PM)^2$, which is even larger than that of MOM method. In order to reduce the dimension of $\overline{\mathbf{T}}_{(1)}$, we apply the addition theorem in Eq. (13) to $\overline{\boldsymbol{\psi}}^t(\mathbf{r}-\mathbf{r}_i)$ in $\overline{\boldsymbol{\Psi}}^t(\mathbf{r})$ above [8]. In this way, the scattering center is pushed from the centers of all the basis to the center of the entire object. Then, the scattered field can be recalculated as

$$\mathbf{E}^s(\mathbf{r}) = \overline{\boldsymbol{\Psi}}^t(\mathbf{r}-\mathbf{r}_0) \cdot \overline{\boldsymbol{\beta}}_0 \cdot \overline{\mathbf{T}}_{(1)} \cdot \overline{\boldsymbol{\alpha}}_s \cdot \mathbf{a}_s = \overline{\boldsymbol{\Psi}}^t(\mathbf{r}-\mathbf{r}_0) \cdot \overline{\tilde{\mathbf{T}}}_{(1)} \cdot \mathbf{a}_s \tag{35}$$

Here, $\mathbf{r}_0$ is the center of the object. $\overline{\boldsymbol{\beta}}_0$ is a matrix stacked by the translation operation β. Then we define the reduced T-matrix $\tilde{\overline{\mathbf{T}}}_{(1)}$ as

$$\tilde{\overline{\mathbf{T}}}_{(1)(2P\times 2P)} = \overline{\boldsymbol{\beta}}_{0(2P\times 2PM)} \cdot \overline{\mathbf{T}}_{(1)(2PM\times 2PM)} \cdot \overline{\boldsymbol{\alpha}}_{s(2PM\times 2P)} \tag{36}$$

Since P is the number of expansions for the Dyadic Green's function, it is usually much smaller than M. Therefore, the dimension of the T-matrix is greatly reduced from $2PM$ by $2P\times M$ for $\overline{\mathbf{T}}_{(1)}$ to $2P$ by $2P$ for $\tilde{\overline{\mathbf{T}}}_{(1)}$.

From Eq. (35), we can see that the reduced-order T-matrix $\tilde{\overline{\mathbf{T}}}_{(1)}$ directly relates the incident field to the scattered field. Given the incident field vector $\mathbf{a}_s$ and the reduced-order T-matrix $\tilde{\overline{\mathbf{T}}}_{(1)}$, we can easily calculate the scattered field. Therefore, the reduced-order T-matrix $\tilde{\overline{\mathbf{T}}}_{(1)}$ characterizes the scattering properties of the scatterer and depends on the scatterer only, but not on the incident field. This makes it very useful for the calculation of the multi-scatterer problems when we know the T-matrix $\tilde{\overline{\mathbf{T}}}_{(1)}$ of each subscatterer.

4. BUILDING THE SYSTEM MATRIX FOR MULTIPLE SCATTERERS

For the multi-scatterer case, suppose that there are N PEC objects B_m, $m = 1, 2, \ldots, N$. The boundary of B_m is S_m. When a given wave is incident upon the N objects, the total field can be calculated by

$$\begin{aligned}\mathbf{E}(\mathbf{r}) =& \mathbf{E}^i(\mathbf{r}) + \mathbf{E}^s(\mathbf{r}) = \Re g\overline{\boldsymbol{\Psi}}^t(\mathbf{r}-\mathbf{r}_m)\cdot\overline{\boldsymbol{\alpha}}_{ms}\cdot\mathbf{a}_s \\ &+ \overline{\boldsymbol{\Psi}}^t(\mathbf{r}-\mathbf{r}_m)\cdot\mathbf{b}_m + \sum_{n=1,n\neq m}^{N}\Re g\overline{\boldsymbol{\Psi}}^t(\mathbf{r}-\mathbf{r}_m)\cdot\overline{\boldsymbol{\alpha}}_{mn}\cdot\mathbf{b}_n\end{aligned} \tag{37}$$

where $\mathbf{b}_m$ and $\mathbf{b}_n$ are the wave function coefficients for the m-th and n-th object respectively, and $\mathbf{r}_m$ and $\mathbf{r}_n$ are the centers of the m-th and n-th object. Here we expand the incident and scattered fields about the center of the m-th scatterer $\mathbf{r}_m$. Then by the definition of the T matrix for a single object and the boundary condition on each object, one can get the matrix equation for $\mathbf{b}_m$ as

$$\mathbf{b}_m = \tilde{\overline{\mathbf{T}}}_{m(1)}\cdot\left(\mathbf{a}_s + \sum_{n=1,n\neq m}^{N}\overline{\boldsymbol{\alpha}}_{ms}^{-1}\cdot\overline{\boldsymbol{\alpha}}_{mn}\cdot\mathbf{b}_n\right) \tag{38}$$

$m = 1, 2, \ldots, N$. N is the number of the PEC objects.

In Eq. (37), vector addition theorem below has been applied

$$\overline{\boldsymbol{\Psi}}^t(\mathbf{r}-\mathbf{r}_n) = \Re g\overline{\boldsymbol{\Psi}}^t(\mathbf{r}-\mathbf{r}_m)\cdot\overline{\boldsymbol{\alpha}}_{mn} \tag{39}$$

with condition of $|\mathbf{r}-\mathbf{r}_m| < |\mathbf{r}_m-\mathbf{r}_n|$. This enforces the method with the requirement that any two objects cannot have intersection with each other. The computational cost for the matrix Equation (38) is $(2PN)^3$, and it is much smaller than $(MN)^3$ of the MOM method since P is much smaller than M.

After solving for $\mathbf{b}_m$ in Eq. (38), the scattering wave out of region $S_1\cup S_2\cup\ldots\cup S_N$ can be calculated as

$$\mathbf{E}^s(\mathbf{r}) = \sum_{m=1}^{N}\overline{\boldsymbol{\Psi}}(\mathbf{r}-\mathbf{r}_m)\cdot\mathbf{b}_m \tag{40}$$

Once the scattered field is solved, the radar cross section can be calculated as

$$\sigma_\theta = 4\pi r^2|E_\theta^s|^2, \quad \sigma_\phi = 4\pi r^2|E_\phi^s|^2 \tag{41}$$

where,

$$E_\theta^s = \mathbf{E}^s\cdot\hat{\theta}, E_\phi^s = \mathbf{E}^s\cdot\hat{\phi} \tag{42}$$

5. NUMERICAL RESULTS

5.1. Example 1: Numerical Test of Expansion of Scattering Fields

To test the accuracy of the expansion of the scattering field, the scattering solution by one RWG basis in Eq. (18) is compared with the direct method in Eq. (17). Here we randomly picked an RWG basis from one of the 435 basis for the discretization of PEC sphere located at the origin with radius of 1 m. The current coefficient on the RWG basis a_i is set as 1.0. The frequency is 0.3 MHz. The scattering field outside the PEC surface at $x = 0.0$ m, $y \in [-1.0, 1.0]$ m, $z = 30.0$ m which is 0.01λ away from the PEC sphere is calculated. The results are shown in Fig. 2. In this example, the vector wave function truncation number P is chosen as 8 in Eq. (18). We can see a good agreement between the scattering fields by the expansion method and the direct method.

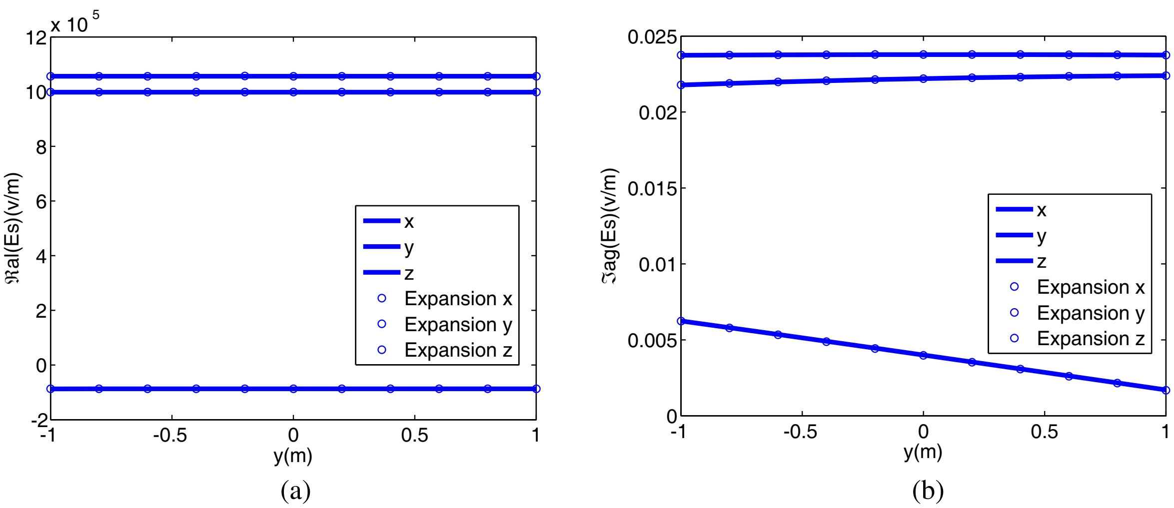


Figure 2. Scattered field from a RWG basis by the direct method and expansion method. (a) Real ($\mathbf{E}^s$). (b) Imag ($\mathbf{E}^s$).

5.2. Example 2: Scattering of Multiple PEC Spheres

This example is the scattering of a 2×2 sphere array. The centers of the four spheres are located at (0, 0, 0) m, (-3, 0, 0) m, (0, 3, 0) m and (-3, 3, 0) m respectively. The dipole incidence is at 0.03 GHz, located at (20, 1.5, 0) m and radiates in $-\hat{x}$ direction with the magnitude of 100. The radius of each sphere is 1.0 m. Each of them is discretized into 4,044 RWG basis functions. The proposed T-matrix method and CFIE with MLFMA method are used to calculate the RCS and scattered fields respectively. Fig. 3 shows the RCS results by the two methods. They have a good agreement. The near field results are shown in Fig. 4. The observation points are along a circle of 5 m around the spheres on the x-y surface. Fig. 4(a) shows the real part of the x component of the scattered field. Fig. 4(b) shows imaginary part of the x component of the scattered field. Both of them agree well with those from the CFIE with MLFMA method.

The simulation is performed on a standard PC with 8 GB memory. The total number of bases for the CFIE method is $4 \times 4,044$, which is 16,176. Using CFIE method with 3-level MLFMA, the memory usage is 6.4 GB. The matrix filling time is 12 min 38 sec and it takes 148 steps which is 1 min 28 sec to converge by GMRES method. While by the proposed T-matrix method, the memory cost for calculating the single T-matrix is 4 GB and the unknown number for the final system matrix is only 64. And it takes only 1.8 sec to solve the matrix equation. In the final matrix solving, the unknown reduction of the T-matrix method is 99.6% compared to the CFIE method.

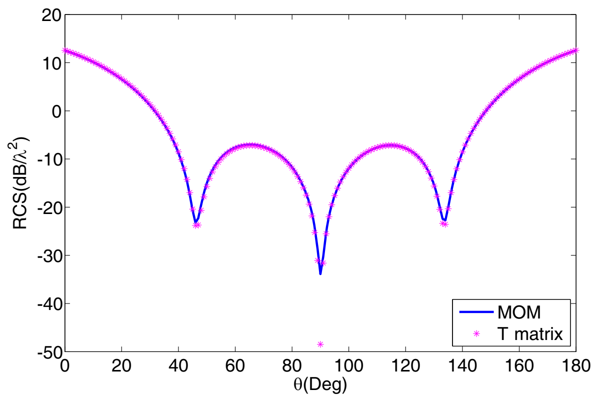
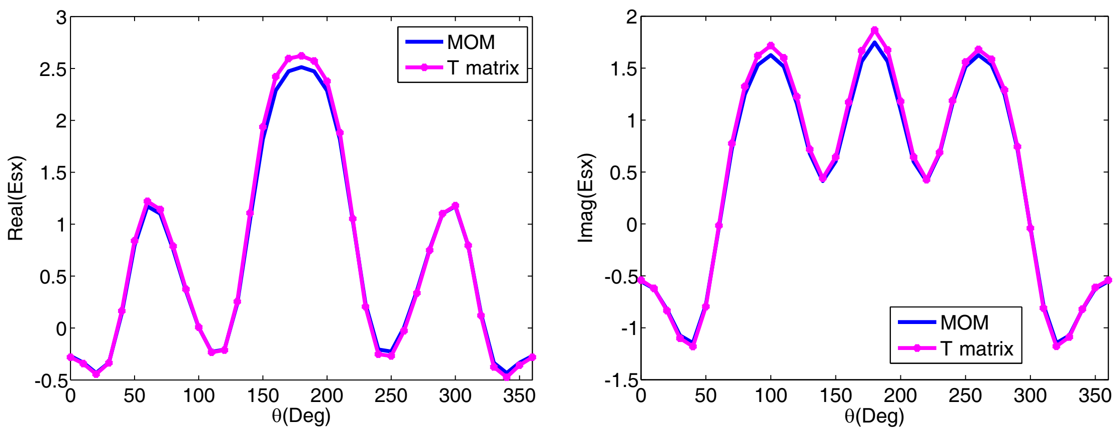


Figure 3. RCS of four PEC spheres at 0.03 GHz.



(a) (b)

Figure 4. Scattered field of four PEC spheres at 0.03 GHz. (a) Real (E_{sx}). (b) Imag (E_{sx}).

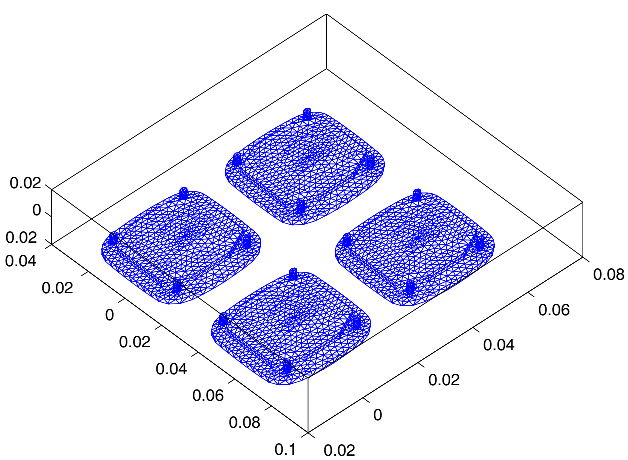


Figure 5. Mesh configuration of a 2×2 antenna array.

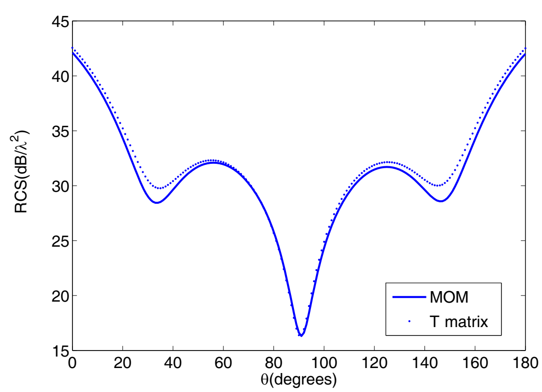


Figure 6. RCS of the 2×2 antenna array.

5.3. Example 3: Scattering of an Antenna Array

This example is a 2×2 antenna array, which is shown in Fig. 5. It is excited by an electric dipole at (4, 0.02, $-5.0e$-3) in the $-\hat{x}$ direction at the frequency of 1.5 GHz. The total number of unknowns are

2856×4. Each of them is modelled by a T matrix. P is taken as 15. The reduction in the number of unknowns is 98.95%. Fig. 6 shows that the RCS results by the proposed method and the moment method have good agreement. As for the simulation time, the total simulation time for the EFIE with 2-level MLFMA is 30 mins, and the final matrix solving time for the T-matrix method is several seconds.

6. CONCLUSION

We developed a T-matrix method for modeling the electromagnetic scattering of multiple PEC objects. The T-matrix for a single PEC object with an arbitrary shape is derived based on the vector addition theorem combined with traditional EFIE, MFIE or CFIE method. Numerical examples for both spherical and non-spherical array structures demonstrate the accuracy and efficiency of the proposed method. The main features of the proposed method can be summarized as:

- The T-matrix for a single object only depends on its size, shape and electrical properties. It does not depend on the incidence, polarization, observation point and location of the object. Hence, it is not necessary to re-derive the T-matrix for an object when the incident field or the observation point changes or the location of the object changes. For two PEC objects with the same shape and size, they have the same T-matrix.
- By combining the vector addition theorem with the traditional EFIE, MFIE or CFIE method, the proposed method is not limited to spherical objects, it can be applied to any object with an arbitrary shape.
- The size of the T-matrix is related to the electrical size of the object. As the size of the object increases, the dimension of its T-matrix increases. However, the size of the T-matrix often much smaller than that of the corresponding MOM impedance matrix, which makes it more efficient in terms of the computational cost for the final matrix solving for multiple objects.
- One major limitation for the proposed T-matrix method is that it only works for well-separated structures, it can not apply to structures with overlaps.

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