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The Use of the Fractional Derivatives Approach to Solve the Problem of Diffraction of a Cylindrical Wave on an Impedance Strip

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Abstract—Earlier we considered the use of the apparatus of fractional derivatives to solve the two-dimensional problem of diffraction of a plane wave on an impedance strip. We introduced the concept of a "fractional strip". A "fractional strip" is understood as a strip on the surface, which is subject to fractional boundary conditions (FBC). The problem under consideration on the basis of various methods has been studied quite well. As a rule, this problem is studied on the basis of numerical methods. The proposed approach, as will be shown below, makes it possible to obtain an analytical solution of the problem for values of fractional order $\nu = 0.5$ and for fractional values of the interval $\nu \in [0, 1]$, the general solution will be investigated numerically.

1. FORMULATION OF THE PROBLEM

We arrange a two-dimensional strip of width $2a$ on the plane $y = 0$. The strip along the z-axis is infinite. The source of the cylindrical wave $\vec{J}_e = \vec{z} J_e \delta(x - x_o)\delta(y - y_o)$ is located at the point (x_o, y_o) (see Fig. 1).

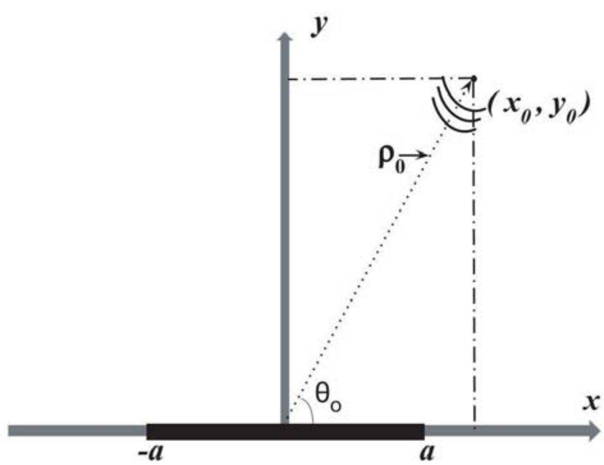

Figure 1. Geometry of the problem.

Let us consider the case of an E-polarized wave, i.e., $\vec{E}_z^i(0, 0, E_z), \vec{H}_z^i(H_x, H_y, 0)$. In this case, the source field has the form

$$\vec{E}_z^i(x, y) = -\vec{J}_e \frac{\eta_0 k}{4} H_0^{(1)}\left(k\sqrt{(x - x_o)^2 + (y - y_o)^2}\right) \tag{1}$$

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Here $H_0^{(1)}(kx)$ is the Hankel function of the first kind and zero order, η_0 the impedance of free space, and $k = \frac{2\pi}{\lambda}$ the wave number. We set the time dependency as $e^{-i\omega t}$ and then omit it. The complete field can be represented as a superposition of the fields below

$$\vec{E}_z = \vec{E}_z^i + \vec{E}_z^s \tag{2}$$

where, $\vec{E}_z^i$ is the source field, and $\vec{E}_z^s$ describes the scattered field. To find the scattered field $\vec{E}_z^s$, it is necessary to subject the total field, as noted above, to a new boundary condition [1, 2], which we call the fractional boundary condition (FBC).

$$D_{ky}^{\nu} E_z\left(x, y\right)\Big|_{y=\pm 0} = 0 \tag{3}$$

where $x, -a < x < a$ and ν is a fractional order (FO). Further, the fractional derivative D_{ky}^{ν} will be determined from the Riemann-Liouville equation [3] which has the form

$$D_y^{\nu} f\left(y\right) = {}_{-\infty} D_y^{\nu} f\left(y\right) = \frac{1}{\Gamma\left(1-\nu\right)} \frac{d}{dy} \int\limits_{-\infty}^{y} \frac{f\left(t\right)}{\left(y-t\right)^{\nu}} dt \tag{4}$$

The fractional order ν varies from 0 to 1, and $\Gamma\ (\nu)$ is the Gamma function. For the value $\nu = 0$, the strip with FBC in Eq. (3) corresponds to a perfect electrical conducting (PEC) strip, and for $\nu = 1$ a strip with perfect magnetic conductivity (PMC) is obtained [3]. For intermediate values $0 < \nu < 1$, FBC describes a fractional boundary with specific properties, which is investigated in this article. FBC leads to the use of the fractional Green's function (FGF) $G^{\nu}(x)$ [5, 6] and the fractional Green theorem [1, 5, 6]. In this case, the scattered field can be represented as [1]

$$E_z^s\left(x, y\right) = \int\limits_{-\infty}^{\infty} f^{1-\nu}\left(x'\right) G^{\nu}\left(x - x', y\right) dx' \tag{5}$$

Here, $f^{1-\nu}(x')$ is an unknown function, which we will call the fractional density of the potential, and the fractional Green's function $G^{\nu}(x)$ has the form [3]

$$\begin{aligned} G^{\nu}\left(x - x', y\right) &= -\frac{i}{4} D_{ky}^{\nu} H_0^{(1)}\left(k\sqrt{\left(x-x'\right)^2 + y^2}\right) \\ &= -i\frac{e^{\pm i\frac{\pi}{2}\nu}}{4\pi} \int\limits_{-\infty}^{\infty} e^{ik\left[\alpha(x-x') \pm y\sqrt{1-\alpha^2}\right]} \left(1-\alpha^2\right)^{\frac{\nu-1}{2}} d\alpha, \quad y \gtrless 0 \end{aligned} \tag{6}$$

Representing Eq. (5) for the scattered field by taking Fourier transform, we obtain

$$E_z^s\left(x, y\right) = -\frac{e^{\pm i\frac{\pi}{2}\nu}}{4\pi} \int\limits_{-\infty}^{\infty} F^{1-\nu}(\alpha) e^{ik\left[\alpha x \pm y\sqrt{1-\alpha^2}\right]} \left(1-\alpha^2\right)^{\frac{\nu-1}{2}} d\alpha \tag{7}$$

where,

$$\begin{aligned} &F^{1-\nu}\left(\alpha\right) = \int\limits_{-1}^{1} \tilde{f}^{1-\nu}(\xi) e^{-i\varepsilon\alpha\xi} d\xi, \quad \tilde{f}^{1-\nu}\left(\xi\right) = a f^{1-\nu}\left(\xi\right) \\ &\varepsilon = ka, \quad \xi = \frac{x}{a}, \quad \tilde{f}^{1-\nu}\left(\xi\right) = \frac{\varepsilon}{2\pi} \int\limits_{-\infty}^{\infty} F^{1-\nu}(\alpha) e^{i\varepsilon\alpha\xi} d\alpha \end{aligned} \tag{8}$$

2. SOLUTION OF THE PROBLEMS

Now, subjecting the total field $\vec{E}_z$ to the FBC in Eq. (3) and taking into account Eqs. (7) and (8) to determine the fractional Fourier transform $F^{1-\nu}(\alpha)$, we obtain the integral equation (IE) of the following form

$$\int_{-\infty}^{\infty} F^{1-\nu}(\alpha) \frac{\sin\varepsilon(\alpha-\beta)}{\alpha-\beta}\left(1-\alpha^2\right)^{\nu-\frac{1}{2}} d\alpha$$

$$= -4B\pi e^{-i\frac{\pi}{2}\nu} \int_{-\infty}^{\infty} e^{i\left[-kx_0\alpha+ky_0\sqrt{1-\alpha^2}\right]} \frac{\sin\varepsilon(\alpha-\beta)}{\alpha-\beta}(1-\alpha^2)^{\frac{\nu-1}{2}} d\alpha \tag{9}$$

where,

$$B = -J_e \frac{\eta_0 k}{4\pi}$$

As noted above, for values of fractional order $\nu = 0.5$, IE in Eq. (9) has an analytic solution that has the form

$$F^{0.5}(\alpha) = -4Be^{-i\frac{\pi}{4}} \int_{-\infty}^{\infty} \frac{\sin\varepsilon(\beta-\alpha)}{(\beta-\alpha)} e^{i\left[(-kx_0\beta)+ky_0\sqrt{1-\beta^2}\right]} \left(1-\beta^2\right)^{-\frac{1}{4}} d\beta \tag{10}$$

Accordingly, the density of the fractional potential has the form

$$\tilde{f}^{0.5}(\xi) = -2\varepsilon Be^{-i\frac{\pi}{4}} \int_{-\infty}^{\infty} e^{i\left[(\varepsilon\alpha\xi-kx_0\alpha)+ky_0\sqrt{1-\alpha^2}\right]} \left(1-\alpha^2\right)^{-\frac{1}{4}} d\alpha \tag{11}$$

Analytic solutions in Eqs. (10) and (11) will be analysed below. Now we construct the solution of the IE in Eq. (9) for fractional values of $0 < \nu < 1$. Fractional Fourier Transform of density function [1, 2] can be written as

$$F^{1-\nu}(\alpha) = \frac{2\pi}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} (-i)^n f_n^\nu \beta_n^\nu \frac{J_{n+\nu}(\varepsilon\alpha)}{(2\varepsilon\alpha)^\nu} \tag{12}$$

Here $\beta_n^\nu = \Gamma(n+2\nu)/\Gamma(n+1)$, f_n^ν are the unknown coefficients, which are subject to the definition, and $J_{n+\nu}(\varepsilon\alpha)$ are the Bessel functions. The representation in Eq. (12) is a consequence of the fact that the fractional density of the potential $\tilde{f}^{1-\nu}(\xi)$ in order to satisfy the condition on the edge [1, 2] is represented as a uniformly convergent series in the orthogonal Gegenbauer polynomials $C_n^\nu(\xi)$

$$\tilde{f}^{1-\nu}(\xi) = (1-\xi^2)^{\nu-\frac{1}{2}} \sum_{n=0}^{\infty} f_n^\nu \frac{C_n^\nu(\xi)}{\nu} \tag{13}$$

Substituting now the representation in Eq. (12) for Fourier Transform in IE in Eq. (9), we obtain a system of linear algebraic equations (SLAE) for determining the unknown coefficients f_n^ν of the form

$$\sum_{n=0}^{\infty} (-i)^n f_n^\nu \beta_n^\nu C_{mn}^\nu = \gamma_m^\nu \tag{14}$$

where,

$$C_{mn}^\nu = \int_{-\infty}^{\infty} J_{n+\nu}(\varepsilon\alpha) J_{m+\nu}(\varepsilon\alpha)(1-\alpha^2)^{\nu-\frac{1}{2}} \frac{d\alpha}{\alpha^{2\nu}}$$

$$\gamma_m^\nu = -\frac{2}{\pi}\Gamma(\nu+\frac{1}{2})e^{-i\frac{\pi}{2}\nu} \int_{-\infty}^{\infty} e^{i\left[-kx_0\alpha+ky_0\sqrt{1-\alpha^2}\right]}(1-\alpha^2)^{\frac{\nu-1}{2}} d\alpha$$

As noted in [1, 2], SLAE allows one to determine the unknown coefficients f_n^ν with any given accuracy.

3. PHYSICAL CHARACTERISTICS OF THE SCATTERED FIELD

In this section we present the expressions for the radiation pattern, monostatic and bi-static radar cross sections (RCS). These expressions will be used to analyse the electromagnetic characteristics of the scattered field.

Let's derive the expression for the field $\vec{E}_z^s$ in the far-zone $kr \to \infty$. First in the cylindrical coordinate system (r, φ) can be found by using these relations $x = r\cos\varphi$, $y = r\sin\varphi$. Then, the scattered field in Eq. (7) is

$$E_z^s(r,\varphi) = \frac{i}{4\pi}(\pm i^\nu)\int_{-\infty}^{+\infty} F^{1-\nu}(\cos\beta)\, e^{ikr\cos(\varphi\pm\beta)}\sin^\nu\beta d\beta \tag{15}$$

where, the upper sign is chosen for the values $\varphi \in [0,\pi]$, and the lower sign for $\varphi \in [\pi,\pi]$. If $kr \to \infty$ we can use the method of stationary phase to derive the expression for $E_z^s(r,\varphi)$ as follows

$$E_z^s(r,\varphi) = A(kr)\Phi^\nu(\varphi) \quad \text{while} \quad kr \to \infty \tag{16}$$

where,

$$A(kr) = \sqrt{\frac{2}{\pi kr}} e^{ikr - i\pi/4}$$
$$\Phi^\nu(\varphi) = -\frac{i}{4}(\pm i)^\nu F^{1-\nu}(\cos\varphi)\sin^\nu\varphi$$

The function $\Phi^\nu(\varphi)$ denotes the radiation pattern (RP) of the scattered field that can be expressed via the coefficients f_n^ν

$$\Phi^\nu(\varphi) = \frac{i\pi(\pm i)^\nu}{2\Gamma(\nu+1)}\tan^\nu\varphi \sum_{n=0}^{\infty}(-i)^\nu f_n^\nu \beta_n^\nu \frac{J_{n+\nu}(\epsilon\cos\varphi)}{(2\epsilon)^\nu} \tag{17}$$

The formula for the bi-static RCS $\frac{\sigma_{2\mathrm{d}}}{\lambda}$ and monostatic RCS$\sigma_{2\mathrm{d}}$ is derived from the expression for RP $\Phi^\nu(\varphi)$ as [4]

$$\frac{\sigma_{2\mathrm{d}}}{\lambda}(\varphi) = \frac{2}{\pi}|\Phi(\varphi)|^2; \quad \sigma_{2\mathrm{d}}(monostatic) = \frac{\sigma_{2\mathrm{d}}}{\lambda}(\theta_o). \tag{18}$$

As was shown in [2], the fractional order is related to the impedance

$$\nu = \frac{1}{i\pi}\ln\frac{1-\eta}{1+\eta}, \quad \eta = -i\tan\left(\frac{\pi\nu}{2}\right) \tag{19}$$

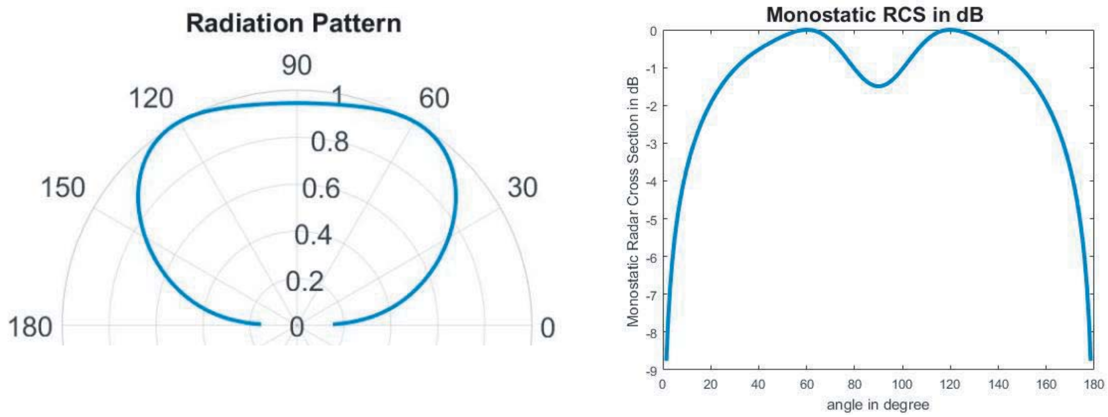
The value $\nu = 0$ corresponds to the impedance $\eta = 0$ (PEC) and $\nu = 1$ corresponds to $\eta = -i\infty$ (PMC). For the intermediate values $0 < \nu < 1$ the impedance has pure imaginary values between 0 and $-i\infty$. For a special case, when $\nu = 0.5$, $\eta = \sqrt{\frac{\mu}{\epsilon}} = -i$ by using Eq. (19).

4. NUMERICAL RESULT

In this section, we numerically analyze diffraction of a cylindrical wave on an impedance strip. We have focused on Radiation Pattern, Monostatic Radar Cross section and Total Electric Field near the strip. For $\nu = 0.5$, we have found the expression analytically as introduced in previous section. Figs. 2–5 show the RP, Monostatic RCS and field distributions in the vicinity of the impedance strip for various values of the frequency parameter ε and the distribution of the source. The parameters can be summarized by following equations which are expressing the location of source and radial distance from the source, wave number and width of strip. Here, the results are given for $\nu = 0.5$ in Figs. 2–5.

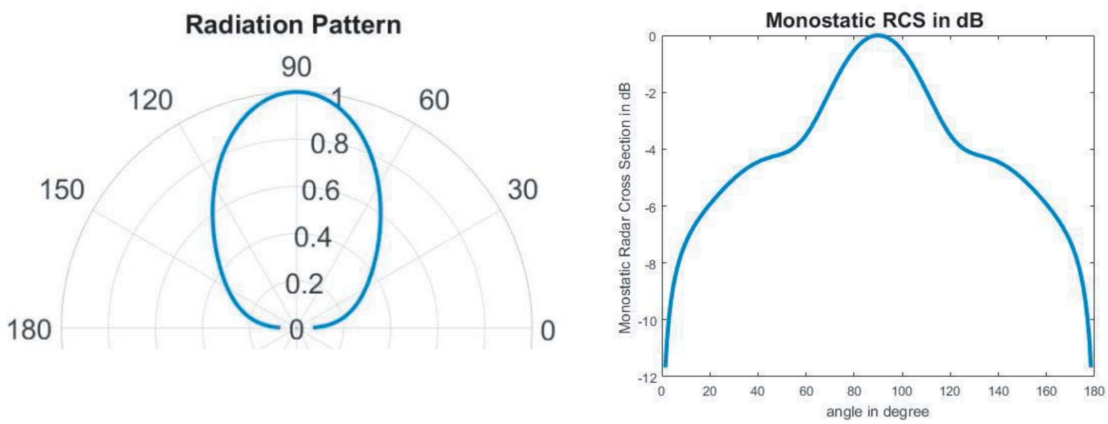
$$\begin{gathered} \varepsilon = ka \\ \kappa_1 = kx_0 = \frac{\varepsilon x_0}{a}, \quad \kappa_2 = ky_0 = \frac{\varepsilon y_0}{a} \quad \kappa_3 = k\rho_0 = k\sqrt{x_0^2 + y_0^2} = \sqrt{\kappa_1^2 + \kappa_2^2} \end{gathered} \tag{20}$$

Figure 2 shows Normalized RP and Monostatic RCS for the 'fractional strip' with the frequency parameter $\varepsilon = 2\pi$, it yields $\kappa_1 = 0$, $\kappa_2 = 0.1\varepsilon$, and $\kappa_3 = 0.1\varepsilon$ by using (20) when $\nu = 0.5$.



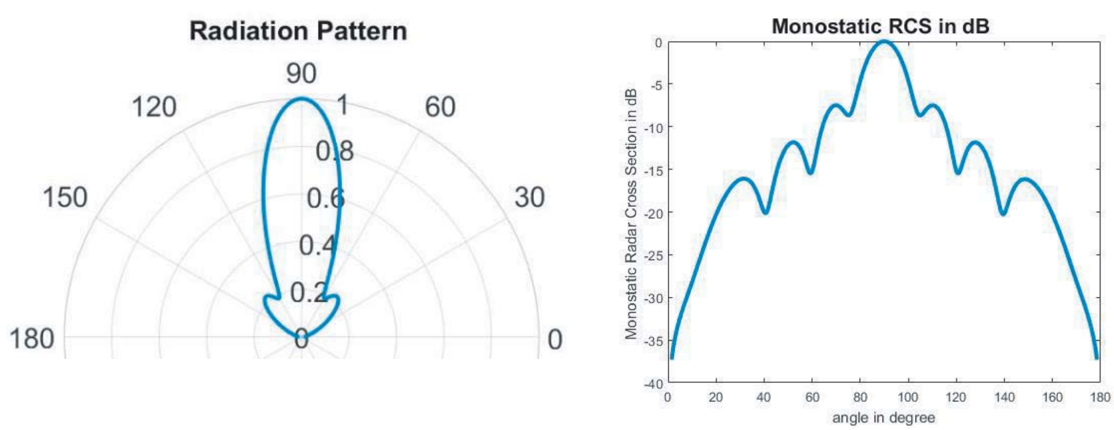
(a) (b)

Figure 2. (a) Normalized radiation pattern for $\varepsilon = 2\pi$, $\kappa_1 = 0$ and $\kappa_2 = 0.1\varepsilon$, (b) monostatic radar cross section for $\varepsilon = 2\pi$ and $\kappa_3 = 0.1\varepsilon$.



(a) (b)

Figure 3. (a) Normalized radiation pattern for $\varepsilon = \pi$, $\kappa_1 =$ and $\kappa_2 = 0.6\varepsilon$, (b) monostatic radar cross section for $\varepsilon = \pi$ and $\kappa_3 = 0.6\varepsilon$.



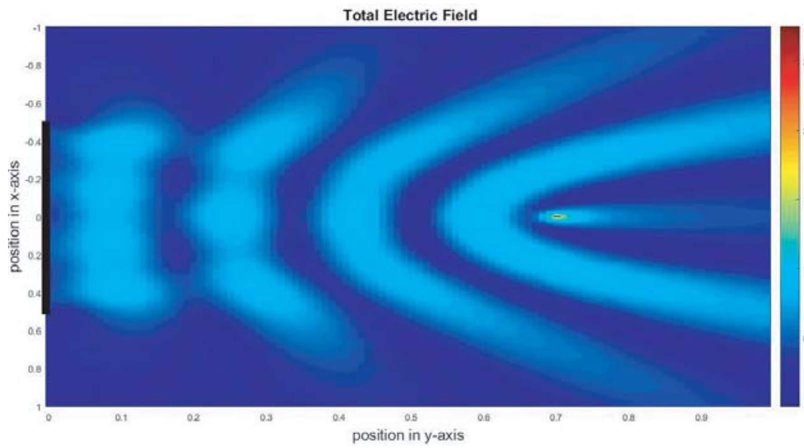
(a) (b)

Figure 4. (a) Normalized radiation pattern for $\varepsilon = \pi$, $\kappa_1 = 0$ and $\kappa_2 = 1.4\varepsilon$, (b) monostatic radar cross section for $\varepsilon = \pi$ and $\kappa_3 = 2\pi\varepsilon$.

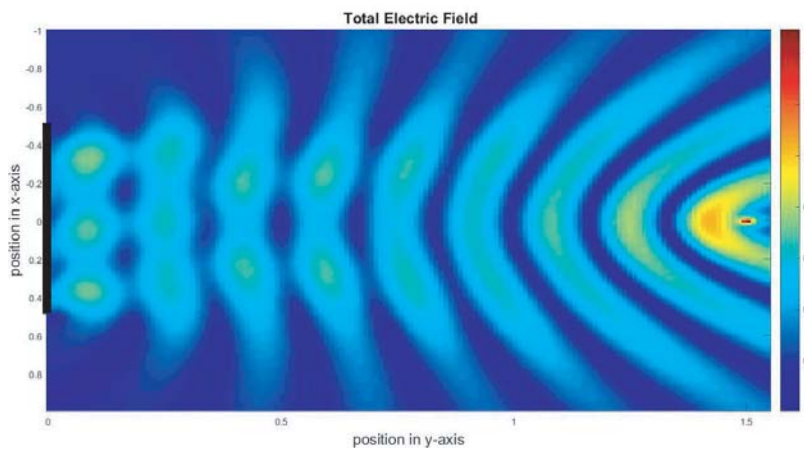
Figure 3 shows Normalized RP and Monostatic RCS for the ‘fractional strip’ with the frequency parameter $\varepsilon = \pi$, it yields $\kappa_1 = 0$, $\kappa_2 = 0.6\varepsilon$, and $\kappa_3 = 0.6\varepsilon$ by using (20) when $\nu = 0.5$.

Figure 4 shows Normalized RP and Monostatic RCS for the ‘fractional strip’ with the frequency parameter $\varepsilon = 2\pi$, it yields $\kappa_1 = 0$, $\kappa_2 = 2\pi\varepsilon$, and $\kappa_3 = 2\pi\varepsilon$ by using (20) when $\nu = 0.5$.

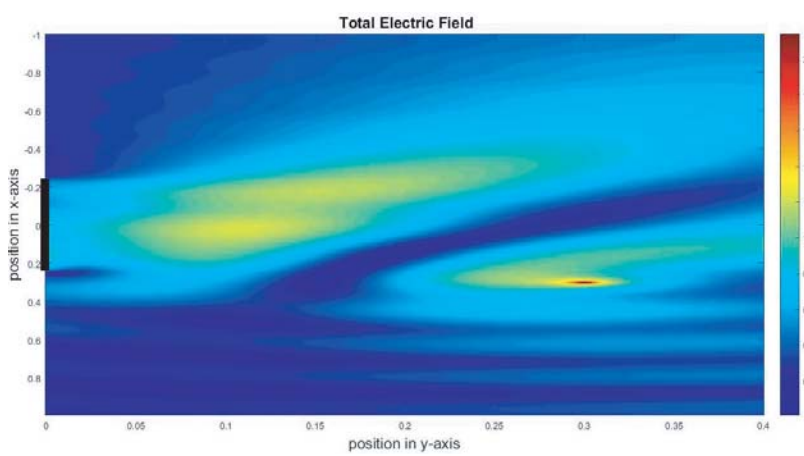
Figure 5 shows magnitude of Electric Field for the ‘fractional strip’ with the different frequency parameter ε and source location when $\nu = 0.5$.



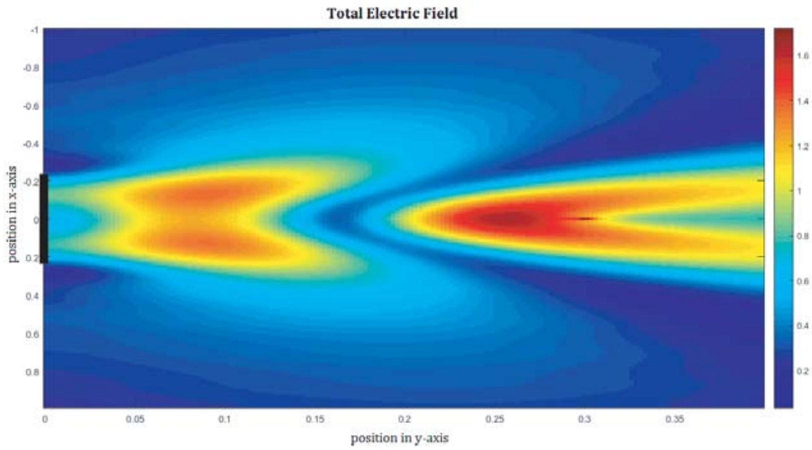
(a)



(b)



(c)



(d)

Figure 5. Magnitude of total electric field for (a) $\varepsilon = 3\pi$, $\kappa_1 = 0$ and $\kappa_2 = 1.4\varepsilon$, (b) $= 3\pi$, $\kappa_1 = 0$ and $\kappa_2 = 3\varepsilon$, (c) $\varepsilon = 1.5\pi$, $\kappa_1 = 1.2\varepsilon$ and $\kappa_2 = 1.2\varepsilon$, (d) $\varepsilon = 1.5\pi$, $\kappa_1 = 0$ and $\kappa_2 = 1.2\varepsilon$.

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