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Analysis of a Non-Integer Dimensional Tunnel and Perfect Electric Conductor Waveguide

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Abstract—Solutions to the Maxwell equations for a planar non-integer dimensional perfect electric conductor (NID-PEC) waveguide are obtained. The space within the guide is NID in direction normal to walls of the waveguide. Field behavior within the waveguide is noted for different values of the parameter, D, describing dimension of the NID space. For $D = 2$, classical results are recorded. The discussion is further extended by treating propagation in a tunnel within unbounded dielectric medium. The space within tunnel is also NID in direction perpendicular to walls of the tunnel. For different values of D field behaviors are also presented. It has been noted that for $D = 2$ and taking very high values of permittivity ($\epsilon \to \infty$) classical results for PEC waveguide are recorded, whereas for $\epsilon \to \infty$, field behavior within tunnel matches with NID-PEC waveguide.

1. INTRODUCTION

In scientific literature term fractal is used to describe property of auto resemblance on all scales. This term was first coined by Mandelbrot to identify an object that is broken in space or time [1]. One of the intriguing interests of studying fractal is its capacity to model objects with complex structures [2]. Fractal dimension tells how the fractal is confined in the Euclidean space or how it fills the space in which it lies. Fractional order integrals may be used to describe the properties of fractal medium, and fractal medium behaves as continuous medium in non-integer dimensional (NID) space [3]. Consequently continuum models with NID space enable us to model and explain the behavior of fractal medium.

In 1977, Stillinger introduced the concept of NID space and proposed a mathematical basis for this subject and provided expression for generalized Laplace operator [4]. Later on, Palmer and Stavrinou [5] derived equations of motion for NID space. After these valuable initial contributions, the concept of NID space gained attention of researchers and now being used in various subjects of engineering and science [6]. For example, state of atomic bound was explained by the solution of Schrodinger equation [7]. Fractal theory of heterogeneous surfaces [8], Gauss's law [9], Lagrangian formulation for field systems [10] had also been addressed in non-integer dimensional space. Solutions of Helmholtz's equation were derived in unbounded NID space for cartesian [11], cylindrical [12] and spherical coordinates [13]. Solutions of Laplaces's equation for cylindrical coordinates were derived in [14]. Quasi-static analysis of a plasmonic sphere [15], radiation from a two-dimensional buried canonical source [16], scattering from a low contrast buried cylinder [17], and buried strip [18] had also been treated when being immersed in NID space. Sadallah et al. proposed a solution of Euler-Lagrange equation in NID space [19]. Solution to the Maxwell's equations inside the NID waveguide/transmission line was studied in [20]. Tarasov discussed Gauss' law, Ampere's law, and Poisson's equation for fractal medium [21]. He proposed vector calculus for NID space, i.e., expressions for gradient, divergence, and Laplacian operator were given [22]. He also derived solutions for Poisson's equation for fractal medium, Euler-Bernoulli fractal beam, and Timoshenko beam equations for fractal material [23]. Reflection and transmission from a

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planar interface of two non-integer dimensional half spaces were investigated in [24, 27]. Study of fractal continuum electromagnetics on the basis of three-dimensional continuum fractal metric was explained in [26].

In Cartesian coordinate, Laplacian and curl operator in non-integer dimensional space are defined in Appendix A. Maxwell curl equation and Helmholtz's equation in D-dimensional space, for homogeneous and isotropic medium are given below,

$$\nabla_D \times \mathbf{E} = -j\omega\mu\mathbf{H} \tag{1a}$$

$$\nabla_D \times \mathbf{H} = j\omega\epsilon\mathbf{E} \tag{1b}$$

$$\nabla_D^2\mathbf{E} + k^2\mathbf{E} = 0 \tag{1c}$$

where $k = \omega\sqrt{\mu\epsilon}$ is the wave number. Time dependency $\exp(j\omega t)$ has been managed throughout the discussion.

In this paper, propagation of electromagnetic field in a NID-PEC waveguide and NID tunnel in dielectric medium has been treated one by one. For very high value of permittivity of the dielectric medium hosting the tunnel with $D = 2$, field behavior matches with the classical PEC waveguide result. For the sake of comparison, first, propagation of TE and TM modes in a PEC parallel plate waveguide is presented.

2. TE/TM-MODE IN PEC WAVEGUIDE

Consider a parallel plate PEC waveguide as shown in Figure 1. Two interfaces of the waveguide are located at $y = \pm b$. The medium inside the waveguide is described by the constitutive parameters ϵ and μ. Suppose that transverse electric (TE) ($E_x = 0$) mode is propagating inside the waveguide. Field inside the waveguide is written as linear combination of two uniform plane waves. Electric and magnetic fields inside the waveguide are written below

$$\begin{aligned} E_z &= A_n \exp(-j\beta x + jhy) + RA_n \exp(-j\beta x - jhy) \\ \eta H_x &= -\frac{h}{k} A_n \exp(-j\beta x)[\exp(jhy) - R\exp(-jhy)] \\ \eta H_y &= -\frac{\beta}{k} A_n \exp(-j\beta x)[\exp(jhy) + R\exp(-jhy)] \end{aligned}$$

where R is the reflection co-efficient. It may be noted that $\mathbf{k}_\pm = \beta\hat{x} \mp h\hat{y}$ are the two wave vectors.

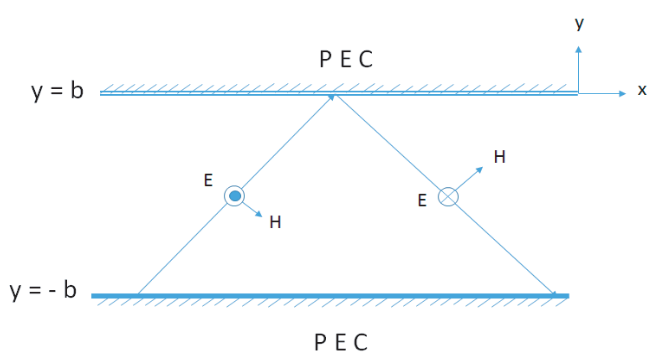

Figure 1. TE mode in PEC waveguide.

By applying boundary conditions and substituting the values of unknown, the total electric field and magnetic field inside the waveguide may be written as,

$$E_z = A_n \exp(-j\beta x)\sin\{h(b - y)\} \tag{2a}$$

$$\eta H_x = -\frac{h}{k} A_n \exp(-j\beta x)\cos\{h(b - y)\} \tag{2b}$$

$$\eta H_y = -\frac{\beta}{k} A_n \exp(-j\beta x)\sin\{h(b - y)\} \tag{2c}$$

where $h = n\pi/2b$.

By using above set of equations, patterns of magnetic field lines are obtained as shown in Figure 2 which shows that magnetic field is tangential to the wall of the guide located at $hy = 1.5$.

In order to examine the field behavior within PEC waveguide in xy-plane, here, $\theta_1 = \pi/6$ so that $\beta/k = \cos(\pi/6)$, $h/k = \sin(\pi/6)$ are taken for the plot. In Figure 2, magnetic field lines become parallel to the walls of waveguide which shows that the walls are PEC, and the mode is transverse electric. Here, plot is considered for region $0 \le y \le b$. It may be noted that symmetry exists for region $-b \le y \le 0$.

Now, suppose that transverse magnetic (TM) ($H_x = 0$) mode is propagating inside the waveguide. Field inside the waveguide is written as linear combination of two uniform plane waves. The total electric and magnetic field inside the waveguide may be written as,

$$E_x = \frac{h}{k} A_n \exp(-j\beta x) \sin\{h(b-y)\} \tag{3a}$$

$$E_y = \frac{\beta}{k} A_n \exp(-j\beta x) \cos\{h(b-y)\} \tag{3b}$$

$$\eta H_z = A_n \exp(-j\beta x) \cos\{h(b-y)\} \tag{3c}$$

By using Eq. (4), patterns of electric field are obtained as shown in Figure 3. It is obvious to note that electric field is normal to the wall of the guide located at $hy = 1.5$.

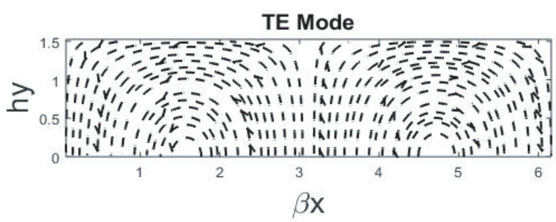

Figure 2. Pattern of magnetic filed in PEC waveguide.

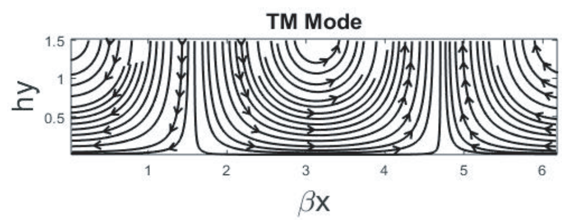

Figure 3. Pattern of electric field in PEC waveguide.

In Figure 3, electric field lines are perpendicular to the walls of waveguide which shows that the walls are PEC, and the mode is transverse magnetic. Here, again plot is considered for region $0 \le y \le +b$. The field lines repeat themselves for every change of 2π (rad) in βx. In the remaining part of the discussion NID space will be taken into account, and the result given in the Section 2 will be used for comparison.

3. NID-PEC WAVEGUIDE

Consider a parallel plate PEC waveguide as shown in Figure 1. Two interfaces of the waveguide are located at $y = \pm b$. The medium inside the waveguide is described by the constitutive parameters ϵ and μ. It has also been assumed that medium inside the waveguide is NID in direction normal to the walls of the waveguide.

Suppose that transverse electric (TE) ($E_x = 0$) mode is propagating inside the waveguide. Field inside the waveguide is written as linear combination of two waves propagating towards and away from each interface. These two waves are written below.

$$E_{1z} = \frac{A_n}{2} \exp(-j\beta x)(hy)^n H_n^{(1)}(hy)$$

$$E_{2z} = \frac{A_n}{2} R \exp(-j\beta x)(hy)^n H_n^{(2)}(hy)$$

E_{1z} is propagating in the $+ve$ y direction whereas E_{2z} is propagating in the $-ve$ y direction. The total electric field can be written as linear combination of above two waves, i.e.,

$$E_z = E_{1z} + E_{2z}$$

$$E_z = \frac{A_n}{2} \exp(-j\beta x) \left[(hy)^n H_n^{(1)}(hy) + R(hy)^n H_n^{(2)}(hy)\right] \tag{4a}$$

where $H_n^{(1)}(.)$ and $H_n^{(2)}(.)$ are the Hankel functions of the first and second kinds with the nth order, $n = \frac{3-D}{2}$, $1 < D \leq 2$, and R is the unknown.

Corresponding magnetic field may be obtained using,

$$\mathbf{H} = \frac{j}{\omega\mu}\nabla_D \times \mathbf{E}$$

$$\mathbf{H} = \frac{j}{\omega\mu}\left[\left\{\frac{\partial}{\partial y}E_z + \frac{(\alpha-1)}{2y}E_z - \frac{\partial}{\partial z}E_y\right\}\hat{x} - \left\{\frac{\partial}{\partial x}E_z\right\}\hat{y} + \left\{\frac{\partial}{\partial x}E_y\right\}\hat{z}\right]$$

Substituting $\alpha = D - 1$, above equation takes the following form,

$$\mathbf{H} = \frac{j}{\omega\mu}\left[\left\{\frac{\partial}{\partial y}E_z + \frac{(D-2)}{2y}E_z - \frac{\partial}{\partial z}E_y\right\}\hat{x} - \left\{\frac{\partial}{\partial x}E_z\right\}\hat{y} + \left\{\frac{\partial}{\partial x}E_y\right\}\hat{z}\right]$$

As electric field contains only z-component,

$$\eta\mathbf{H} = \frac{j}{k}\left\{\left(\frac{\partial}{\partial y}E_z + \frac{1}{2}\frac{D-2}{y}E_z\right)\hat{x} - \left(\frac{\partial}{\partial x}E_z\right)\hat{y}\right\}$$

where $k = \omega\sqrt{\mu\epsilon}$ is the wave number, and $\eta = \sqrt{\mu/\epsilon}$ is the intrinsic impedance of the medium. Components of the magnetic field are,

$$\eta H_x = j\frac{h}{k}\frac{A_n}{2}\exp(-j\beta x)\left[(hy)^n H_{nh}^{(1)}(hy) + R(hy)^n H_{nh}^{(2)}(hy)\right] \tag{4b}$$

$$\eta H_y = -\frac{\beta}{k}\frac{A_n}{2}\exp(-j\beta x)\left[(hy)^n H_n^{(1)}(hy) + R(hy)^n H_n^{(2)}(hy)\right] \tag{4c}$$

where

$$H_{nh}^{(1)}(hy) = H_{n-1}^{(1)}(hy) + \frac{1}{2}\frac{D-2}{hy}H_n^{(1)}(hy)$$

$$H_{nh}^{(2)}(hy) = H_{n-1}^{(2)}(hy) + \frac{1}{2}\frac{D-2}{hy}H_n^{(2)}(hy)$$

Applying boundary condition, unknown is obtained as,

$$R = -\frac{H_n^{(1)}(hb)}{H_n^{(2)}(hb)} \tag{5}$$

Field expressions for electric and magnetic fields are,

$$\mathbf{E}_z = \frac{A_n}{2}\exp(-j\beta x)\left[(hy)^n H_n^{(1)}(hy) - \frac{H_n^{(1)}(hb)}{H_n^{(2)}(hb)}(hy)^n H_n^{(2)}(hy)\right] \tag{6a}$$

$$\eta H_x = j\frac{h}{k}\frac{A_n}{2}\exp(-j\beta x)\left[(hy)^n H_{nh}^{(1)}(hy) - \frac{H_n^{(1)}(hb)}{H_n^{(2)}(hb)}(hy)^n H_{nh}^{(2)}(hy)\right] \tag{6b}$$

$$\eta H_y = -\frac{\beta}{k}\frac{A_n}{2}\exp(-j\beta x)\left[(hy)^n H_n^{(1)}(hy) - \frac{H_n^{(1)}(hb)}{H_n^{(2)}(hb)}(hy)^n H_n^{(2)}(hy)\right] \tag{6c}$$

Using above set of equations, patterns of magnetic field for different values of D are obtained as shown in Figure 4.

In order to examine the field behaviour within NID-PEC waveguide for different values of D in xy-plane, here, $\theta_1 = \pi/6$ so that $\beta/k = \cos(\pi/6)$, $h/k = \sin(\pi/6)$ are taken for the plots. In all these plots, for each value of D, it is obvious that magnetic field lines become parallel to the walls of waveguide. Here, all these plots are given for region $0 \leq y \leq +b$.

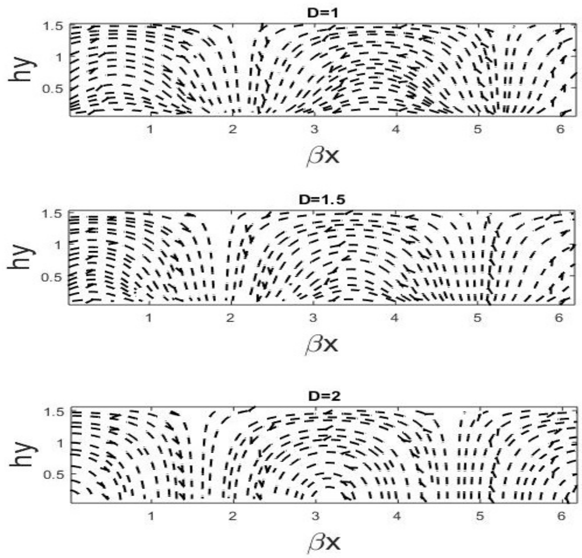

Figure 4. Patterns of magnetic field for different values of D.

4. PLANAR SLAB NID WAVEGUIDE

Consider a planar slab waveguide in unbounded dielectric medium as shown in Figure 5. The medium inside the slab is NID in direction normal to the interfaces of the slab. Two interfaces of the slab are located at $y = \pm b$. The medium inside the slab is described by the constitutive parameters ϵ_1 and μ. Host medium is dielectric, and its constitutive parameters are represented by ϵ_2 and μ. Suppose that a TE wave ($E_x = 0$) is propagating inside the slab. The whole geometry has been divided into three regions so that the unknown fields can be written using the solution of homogeneous Helmholtz's equation. Region $y > b$ is titled as region I, $-b < y < b$ titled as region II, and $y < -b$ titled as region III. Non-integer dimension of medium is designated for region II is D.

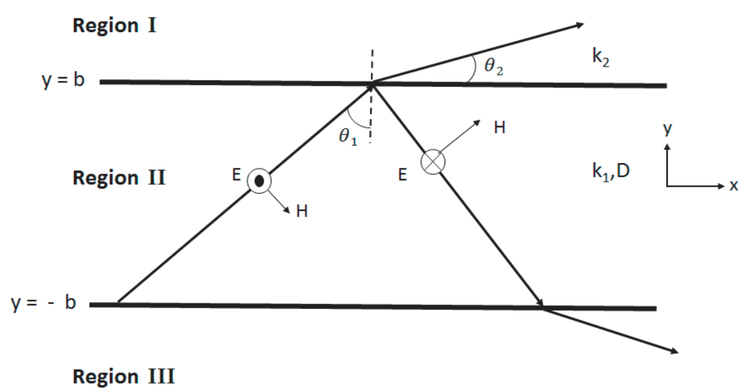

Figure 5. NID tunnel in a dielectric medium.

Field inside the slab is written as linear combination of two uniform plane waves. The total electric and magnetic fields are given below

$$E_{2z} = \frac{A_n}{2}\exp(-j\beta_1 x)\left[(h_1 y)^n H_n^{(1)}(h_1 y) + R(h_1 y)^n H_n^{(2)}(h_1 y)\right] \tag{7a}$$

It may be noted that $\mathbf{k}_{1\pm} = \beta_1\hat{x} \mp h_1\hat{y}$ are two waves vectors. Corresponding magnetic field may be obtained using the following Maxwell equation

$$\nabla_D \times \mathbf{E}_2 = -j\omega\mu_1 \mathbf{H}_2$$

Using above equation, the following may be obtained as

$$\eta_1 H_{2x} = j\frac{h_1}{k_1}\frac{A_n}{2}\exp(-j\beta_1 x)\left[(h_1 y)^n H_{nh}^{(1)}(h_1 y) + R(h_1 y)^n H_{nh}^{(2)}(h_1 y)\right] \tag{7b}$$

$$\eta_1 H_{2y} = -\frac{\beta_1}{k_1}\frac{A_n}{2}\exp(-j\beta_1 x)\left[(h_1 y)^n H_{n}^{(1)}(h_1 y) + R(h_1 y)^n H_{n}^{(2)}(h_1 y)\right] \tag{7c}$$

Field in region I is written below

$$\mathbf{E}_{1z} = T_1\frac{A_n}{2}\exp(-j\beta_2 x - jh_2 y) \tag{8a}$$

Corresponding magnetic field may be obtained as

$$\eta_2 \mathbf{H}_1 = -T_1\frac{A_n}{2}\frac{1}{k_2}\left[h_2\hat{x} - \beta_2\hat{y}\right]\exp(-j\beta_2 x - jh_2 y) \tag{8b}$$

Field in region III is written below

$$\mathbf{E}_{3z} = T_2\frac{A_n}{2}\exp(-j\beta_1 x + jh_2 y) \tag{9a}$$

Corresponding magnetic field may be obtained as

$$\eta_2 \mathbf{H}_3 = T_2\frac{A_n}{2}\frac{1}{k_2}\left[h_2\hat{x} + \beta_2\hat{y}\right]\exp(-j\beta_2 x + jh_2 y) \tag{9b}$$

The unknown coefficients in the above expressions can be determined using the boundary conditions. At $y = \pm b$, tangential components of electric and magnetic field must be continuous. Let field make an angle θ_1 normal to the planar interfaces of the slab. The refraction angle θ_2 can be determined by using the following Ibn-Sahl's law

$$k_1 \sin\theta_1 = k_2 \sin\theta_2$$

Application of boundary conditions yields

$$R = -\left[\frac{\eta_1\cos\theta_2 H_n^{(1)}(h_1 b) + j\eta_2\cos\theta_1 H_{nh}^{(1)}(h_1 b)}{\eta_1\cos\theta_2 H_n^{(2)}(h_1 b) + j\eta_2\cos\theta_1 H_{nh}^{(2)}(h_1 b)}\right] \tag{10}$$

Plots represent patterns of magnetic field, for different values of ϵ_2.

5. SPECIAL CASES

5.1. Case 1:

For $\epsilon_2 \to \infty$, planar slab NID waveguide converts into NID-PEC waveguide. The reflection coefficient of Eq. (10) is transformed as,

$$R = -\frac{H_n^{(1)}(h_1 b)}{H_n^{(2)}(h_1 b)}$$

Corresponding field expressions take the following forms,

$$\mathbf{E}_{2z} = \frac{A_n}{2}\exp(-j\beta_1 x)\left[(h_1 y)^n H_n^{(1)}(h_1 y) - \frac{H_n^{(1)}(h_1 b)}{H_n^{(2)}(h_1 b)}(h_1 y)^n H_n^{(2)}(h_1 y)\right] \tag{11a}$$

$$\eta_1 H_{2x} = -j\frac{h_1}{k_1}\frac{A_n}{2}\exp(-j\beta_1 x)\left[(h_1 y)^n H_{nh}^{(1)}(h_1 y) - \frac{H_n^{(1)}(h_1 b)}{H_n^{(2)}(h_1 b)}(h_1 y)^n H_{nh}^{(2)}(h_1 y)\right] \tag{11b}$$

$$\eta_1 H_{2y} = -\frac{\beta_1}{k_1}\frac{A_n}{2}\exp(-j\beta_1 x)\left[(h_1 y)^n H_n^{(1)}(h_1 y) - \frac{H_n^{(1)}(h_1 b)}{H_n^{(2)}(h_1 b)}(h_1 y)^n H_n^{(2)}(h_1 y)\right] \tag{11c}$$

Results are the same as that derived for NID-PEC waveguide.

5.2. Case 2:

For $\epsilon_2 \to \infty$, $D = 2$ yields $R = \exp(2jh_1 b)$.

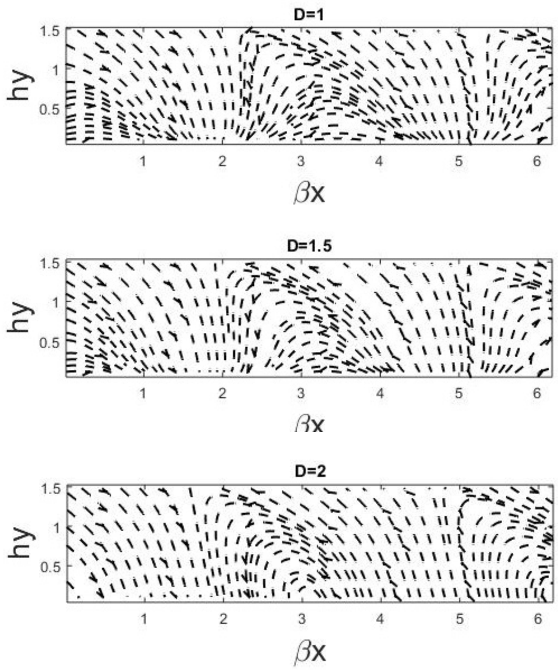

Figure 6. Patterns of magnetic field in NID tunnel ($\epsilon_2 = 2$).

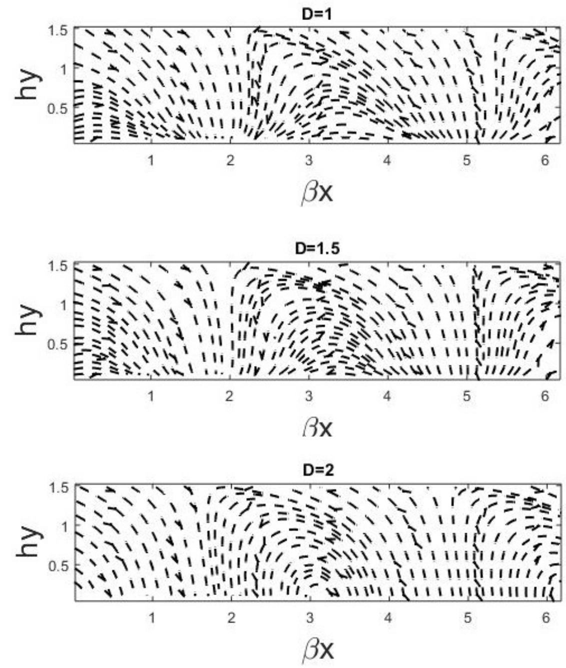

Figure 7. Patterns of magnetic field in NID tunnel ($\epsilon_2 = 4$).

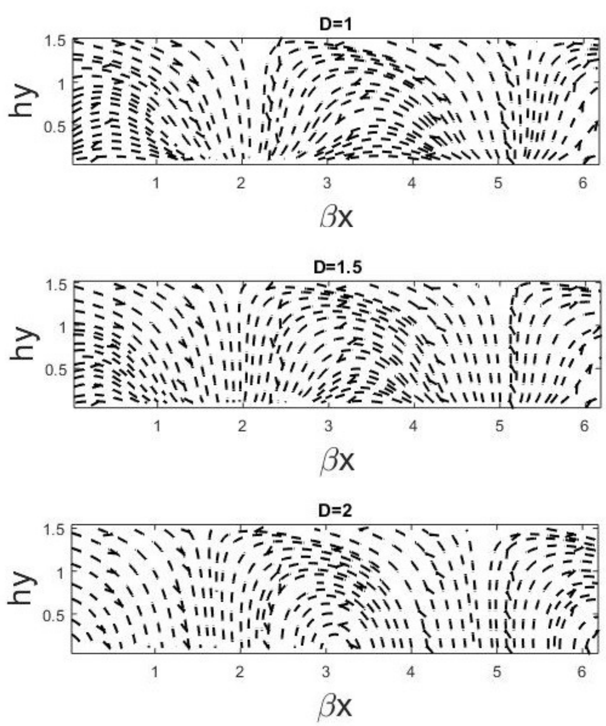

Figure 8. Patterns of magnetic field in NID tunnel ($\epsilon_2 = 8$).

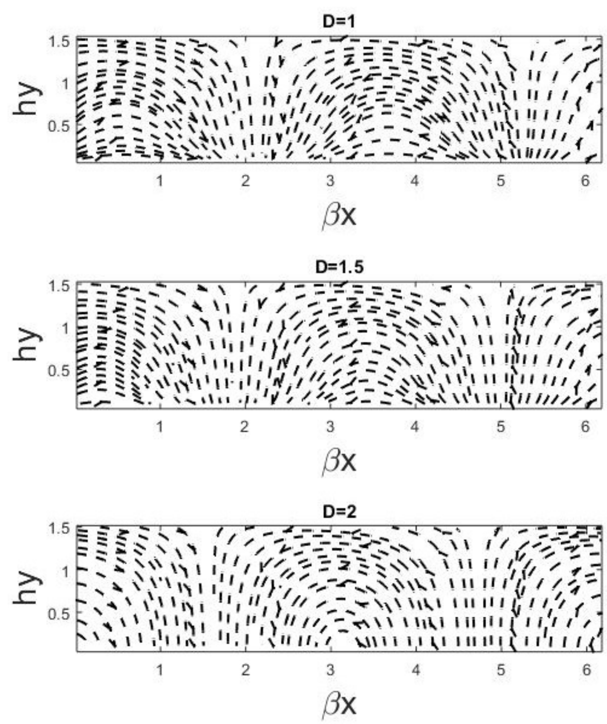

Figure 9. Patterns of magnetic field in NID tunnel ($\epsilon_2 = 100$).

This means that NID tunnel must behave as PEC waveguide.

In order to analyze the field behavior within NID tunnel, plots in xy plane are given taking different values of D and ϵ. It is obvious to note that as value of permittivity of the host medium increases, value of tangential component of the magnetic field increases at the walls of the tunnel. This behavior is depicted in Figure 6 to Figure 10. For very high values of the permittivity of host medium, the situation becomes the same as NID-PEC waveguide. In other words, normal component of magnetic field becomes zero at walls of the waveguide. So field lines are parallel to the walls. Keeping permittivity of the host medium constant, the variation in dimension has significant effects on the field pattern within the tunnel/waveguide.

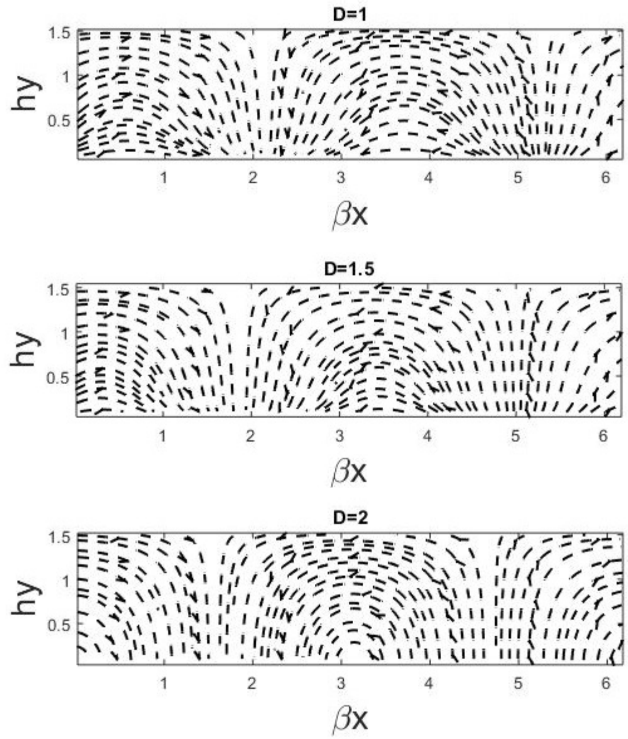

Figure 10. Patterns of magnetic field in NID tunnel ($\epsilon_2 = 1000$).

6. CONCLUSION

Field expressions for electromagnetic mode propagating inside NID-PEC waveguide and NID tunnel for different values of D are derived. The field inside the waveguide and tunnel are written in terms of plane waves, and unknowns are determined by applying boundary conditions. Derived expressions are reduced to classical result when integer dimension is taken. In all the NID plots, effects of variation in the values of D are very prominent.

APPENDIX A.

Laplacian operator for non-integer dimensional space having dimension D for cartesian coordinates is defined as [5]

$$\nabla_D^2 = \frac{\partial^2}{\partial x^2} + \frac{\alpha_1 - 1}{x}\frac{\partial}{\partial x} + \frac{\partial^2}{\partial y^2} + \frac{\alpha_2 - 1}{y}\frac{\partial}{\partial y} + \frac{\partial^2}{\partial z^2} + \frac{\alpha_3 - 1}{z}\frac{\partial}{\partial z} \tag{A1}$$

In NID space, curl operator for Cartesian coordinate is given below,

$$\begin{aligned}\nabla_D \times F(x,y,z) \;=\; & \left[\left(\frac{\partial}{\partial y}F_z + \frac{1}{2}\frac{\alpha_2-1}{y}F_z\right) - \left(\frac{\partial}{\partial z}F_y + \frac{1}{2}\frac{\alpha_3-1}{z}F_y\right)\right]\hat{x} \\ & + \left[\left(\frac{\partial}{\partial z}F_x + \frac{1}{2}\frac{\alpha_3-1}{z}F_x\right) - \left(\frac{\partial}{\partial x}F_z + \frac{1}{2}\frac{\alpha_1-1}{x}F_z\right)\right]\hat{y} \\ & + \left[\left(\frac{\partial}{\partial x}F_y + \frac{1}{2}\frac{\alpha_1-1}{x}F_y\right) - \left(\frac{\partial}{\partial y}F_x + \frac{1}{2}\frac{\alpha_2-1}{y}F_x\right)\right]\hat{z} \end{aligned} \quad \text{(A2)}$$

where $F(x,y,z)$ is a vector function.

Following identities have been used to get the expressions for integer dimensional cases [25].

$$H_{-n}^{(1)} = \exp(jn\pi)H_n^{(1)}(hy) \quad \text{(A3)}$$

$$H_{-n}^{(2)} = \exp(-jn\pi)H_n^{(2)}(hy) \quad \text{(A4)}$$

$$H_{\frac{1}{2}}^{(1)}(hy) = \sqrt{\frac{2}{\pi}}\frac{\exp\{j(hy-\frac{\pi}{2})\}}{\sqrt{hy}} \quad \text{(A5)}$$

$$H_{-\frac{1}{2}}^{(1)}(hy) = j\sqrt{\frac{2}{\pi}}\frac{\exp\{j(hy-\frac{\pi}{2})\}}{\sqrt{hy}} \quad \text{(A6)}$$

$$H_{\frac{1}{2}}^{(2)}(hy) = \sqrt{\frac{2}{\pi}}\frac{\exp\{-j(hy-\frac{\pi}{2})\}}{\sqrt{hy}} \quad \text{(A7)}$$

$$H_{-\frac{1}{2}}^{(2)}(hy) = -j\sqrt{\frac{2}{\pi}}\frac{\exp\{-j(hy-\frac{\pi}{2})\}}{\sqrt{hy}} \quad \text{(A8)}$$

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