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Electromagnetic Scattering from a Zero-Thickness PEC Disk: A Note on the Helmholtz-Galerkin Analytically Regularizing Procedure

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Abstract—Recently, a new analytically regularizing procedure, based on Helmholtz decomposition and Galerkin method, has been proposed to analyze the electromagnetic scattering from a zero-thickness perfectly electrically conducting disk. The convergence of the discretization scheme is guaranteed and of exponential type, i.e., few expansion functions are needed to achieve highly accurate solutions. However, it leads to the numerical evaluation of improper integrals of asymptotically oscillating and slowly decaying functions. Asymptotic acceleration techniques allow to obtain faster decaying integrands without overcoming the problem of the oscillating nature of the integrands themselves, i.e., the convergence of the integrals becomes slower and slower as the accuracy required for the solution is higher. In this paper, by means of algebraic manipulations and a suitable integration procedure in the complex plane, an alternative expression for the scattering matrix coefficients involving only fast converging proper integrals is devised. As shown in the numerical results section, the proposed technique is very effective and drastically outperforms the classical analytical asymptotic acceleration technique.

1. INTRODUCTION

Electric field integral equation (EFIE) is frequently used in the analysis of the electromagnetic scattering from perfectly electrically conducting (PEC) surfaces. Since no closed form solution is in general available, one needs to apply a discretization scheme in order to find an approximate solution. However, for such first-kind weakly-singular or hyper-singular integral equations, the convergence of discretization schemes cannot be guaranteed, and the sequence of condition numbers of truncated system is divergent due to the unboundedness of the integral operator or of its inverse [1]. In order to overcome these problems, a general approach, detailed in [2], has been extensively applied to the analysis of propagation, radiation and scattering problems [3–16], combining analytical regularization and discretization of the integral operator in a single step. By means of Galerkin method with a complete set of basis functions that makes the most singular part of the integral operator invertible with a continuous two-side inverse, the integral equation at hand is recast as a matrix equation at which Fredholm alternative can be applied [17].

In a recent paper [18], the analysis of the electromagnetic scattering from a zero-thickness PEC disk has been carried out by means of a new analytically regularizing procedure based on Helmholtz decomposition and Galerkin method. An EFIE in the vector Hankel transform domain for the surface curl-free and divergence-free contributions of the surface current density has been devised. A second-kind Fredholm infinite matrix-operator equation has been obtained by selecting a complete set of orthogonal eigenfunctions of the most singular part of the integral operator as expansion basis. It is interesting to note that such functions reconstruct the physical behaviour of the surface current density at the centre and the edges of the disk, thus leading to a convergence of exponential type, and

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admit closed-form spectral domain counterparts in terms of Bessel functions of first kind, so that the convolution integrals are always reduced to algebraic products. However, the obtained matrix coefficients are improper integrals of asymptotically oscillating and slowly decaying functions to be numerically evaluated. In order to overcome this problem, in [18] the classical analytical asymptotic acceleration technique proposed in [19–21] has been used. It consists of the extraction from the kernels of their asymptotic behaviour, while the slowly converging integrals of the extracted parts are expressed in closed form. However, such a technique allows to obtain faster decaying integrands without overcoming the most important problem of their oscillating nature. For this reason, the convergence of the accelerated integrals becomes slower and slower as the accuracy required for the solution is higher.

This paper is aimed at overcoming the bottleneck of the technique proposed in [18]. By means of algebraic manipulations and a suitable integration procedure in the complex plane, an alternative expression for the scattering matrix coefficients involving only fast converging proper integrals is devised. As shown in the Numerical Results section, the proposed technique is very effective and drastically outperforms the analytical asymptotic acceleration technique used in [18].

2. PROPOSED SOLUTION

The integrals to be numerically evaluated are [18]

$$\left(\bar{\mathbf{M}}_{CC}^{(|n|)}\right)_{k,h} = \left(\bar{\mathbf{M}}_{CC}^{(|n|)}\right)_{h,k} = -\frac{\sqrt{(|n|+2k+5/2)\,(|n|+2h+5/2)}}{\omega\varepsilon_0}\bar{I}_{k,h}^{(|n|)}, \tag{1a}$$

$$\left(\bar{\mathbf{M}}_{DD}^{(|n|)}\right)_{k,h} = \left(\bar{\mathbf{M}}_{DD}^{(|n|)}\right)_{h,k} = -\omega\mu_0\sqrt{(|n|+2k+3/2)\,(|n|+2h+3/2)}I_{k,h}^{(|n|)}, \tag{1b}$$

where

$$\bar{I}_{k,h}^{(|n|)} = \int_0^{+\infty} J_{|n|+2k+5/2}(aw)\,J_{|n|+2h+5/2}(aw)\,\frac{\sqrt{k_0^2-w^2}}{w^2}dw, \tag{2a}$$

$$I_{k,h}^{(|n|)} = \int_0^{+\infty} J_{|n|+2k+3/2}(aw)\,J_{|n|+2h+3/2}(aw)\,\frac{dw}{\sqrt{k_0^2-w^2}}, \tag{2b}$$

a is the radius of the disk, k_0 the wavenumber in vacuum, $J_\nu(\cdot)$ the Bessel function of first kind and order ν [22], n the general cylindrical harmonic, and k and h denote the general test function and basis function, respectively.

The accurate and efficient evaluation of the improper integrals in Eq. (2) is a key point. This is due to the oscillating nature and the slow asymptotic decay of their integrands. In [18], in order to accelerate the asymptotic decay of such integrands, suitable asymptotic contributions have been extracted from the kernels and the integrals of the extracted contributions expressed in closed form, obtaining the following alternative expressions for the integrals in Eq. (2)

$$\begin{aligned}\bar{I}_{k,h}^{(|n|)} =& \int_0^{+\infty} J_{|n|+2k+5/2}(aw)\,J_{|n|+2h+5/2}(aw)\left(\sqrt{k_0^2-w^2}-j\frac{k_0}{\sqrt{\pi}}\sum_{q=0}^{\overline{Q}}\frac{\Gamma(q+1/2)\,(w/k_0)^{1-2q}}{(2q-1)\,q!}\right)\frac{dw}{w^2}\\ &+j\frac{1}{2\pi}\sum_{q=0}^{\overline{Q}}\frac{(ak_0)^{2q}\,\Gamma^2(q+1/2)\,\Gamma(|n|+k+h+5/2-q)}{(2q-1)\,\Gamma(k-h+q+1)\,\Gamma(-k+h+q+1)\,\Gamma(|n|+k+h+7/2+q)},\end{aligned} \tag{3a}$$

$$\begin{aligned}I_{k,h}^{(|n|)} =& \int_0^{+\infty} J_{|n|+2k+3/2}(aw)\,J_{|n|+2h+3/2}(aw)\left(\frac{1}{\sqrt{k_0^2-w^2}}-j\frac{1}{k_0\sqrt{\pi}}\sum_{q=0}^{Q}\frac{\Gamma(q+1/2)\,(w/k_0)^{-1-2q}}{q!}\right)dw\\ &+j\frac{1}{2\pi}\sum_{q=0}^{Q}\frac{(ak_0)^{2q}\,\Gamma^2(q+1/2)\,\Gamma(|n|+k+h+3/2-q)}{\Gamma(k-h+q+1)\,\Gamma(-k+h+q+1)\,\Gamma(|n|+k+h+5/2+q)},\end{aligned} \tag{3b}$$

where $0 \leq \bar{Q} < |n| + k + h + 5/2$, $0 \leq Q < |n| + k + h + 3/2$, and $\Gamma(\cdot)$ denotes the Gamma function [22]. Unfortunately, such a technique does not change the oscillating nature of the integrands. This means that, higher accurate solutions are associated to slower converging integrals.

A different technique, completely overcoming the problems detailed above, will be presented in the following.

By means of the recurrence formula for the Bessel functions [22]

$$\frac{2\nu}{z} J_\nu(z) = J_{\nu-1}(z) + J_{\nu+1}(z), \tag{4}$$

it is simple to obtain

$$\bar{I}_{k,h}^{(|n|)} = \frac{k_0^2 a^2 \left(I_{k,h}^{(|n|)} + I_{k+1,h}^{(|n|)} + I_{k,h+1}^{(|n|)} + I_{k+1,h+1}^{(|n|)} \right)}{4(|n| + 2k + 5/2)(|n| + 2h + 5/2)} - I_{k,h}^{(|n|+1)}, \tag{5}$$

hence, the numerical evaluation of the matrix coefficients is reduced to the numerical evaluation of the integrals in Eq. (2b).

For the sake of symmetry, it can be supposed $k \geq h$. Now, the functions

$$F_{k,h}^{(|n|,l)}(az) = \frac{1}{\sqrt{k_0^2 - z^2}} J_{|n|+2k+3/2}(az) H_{|n|+2h+3/2}^{(l)}(az), \tag{6}$$

with $l \in \{1, 2\}$, where $H_\nu^{(l)}(\cdot) = J_\nu(\cdot) + j(-1)^{\nu+1} Y_\nu(\cdot)$ is the Hankel function of the lth kind and order ν and $Y_\nu(\cdot)$ is the Bessel function of second kind and order ν, are analytical in the regions of the complex plane $z = w + j\bar{w}$ delimited by the contours C_l sketched in Figure 1.

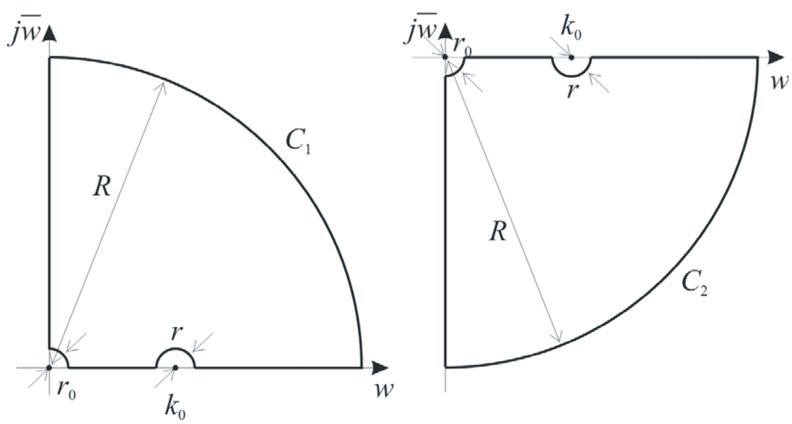

Figure 1. Integration contours in the complex plane.

Hence, by means of Cauchy's integral theorem it is possible to write

$$\lim_{\substack{R \to +\infty \\ r_0,\, r \to 0}} \oint_{C_l} F_{k,h}^{(|n|,l)}(az)\, dz = 0. \tag{7}$$

Starting from the behaviour for small and large arguments of the Bessel functions of the first kind and the Hankel functions [22], i.e.,

$$J_\nu(z) \overset{z\to 0}{\sim} (z/2)^\nu / \Gamma(\nu + 1) \text{ for } \nu \neq -q \text{ with } q \text{ integer}, \tag{8a}$$

$$j\pi(-1)^l H_\nu^{(l)}(z) \overset{z\to 0}{\sim} \begin{cases} 2\ln z & \text{for } \nu = 0 \\ -(2/z)^\nu \Gamma(\nu) & \text{for } \Re\{\nu\} > 0 \end{cases}, \tag{8b}$$

$$J_\nu(z) \overset{|z|\to+\infty}{\sim} \sqrt{\frac{2}{\pi z}} \cos\left(z - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \text{ for } -\pi < \arg(z) < \pi, \tag{8c}$$

$$H_\nu^{(l)}(z) \overset{|z|\to+\infty}{\sim} \sqrt{\frac{2}{\pi z}} e^{(-1)^{l-1} j(z - \nu\pi/2 - \pi/4)} \text{ for } -\pi < \arg(z) < \pi, \tag{8d}$$

it is simple to conclude that the integrands in Eq. (7) have no singularity in $z = 0$ being $k \geq h$, while they decay asymptotically as $1/z^2$ for $l = 1$ and $0 \leq \arg(z) < \pi$, and for $l = 2$ and $-\pi < \arg(z) \leq 0$.

Therefore, by means of Jordan's lemma, it is simple to rewrite formula (7), for $l = 1$ and $l = 2$, respectively, as follows

$$\int_0^{+\infty} F_{k,h}^{(|n|,1)}(aw)\,dw = j\int_0^{+\infty} F_{k,h}^{(|n|,1)}(ja\bar{w})\,d\bar{w}, \tag{9a}$$

$$\int_0^{k_0} F_{k,h}^{(|n|,2)}(aw)\,dw - \int_{k_0}^{+\infty} F_{k,h}^{(|n|,2)}(aw)\,dw = j\int_0^{+\infty} F_{k,h}^{(|n|,1)}(ja\bar{w})\,d\bar{w}, \tag{9b}$$

where the relations [22]

$$J_\nu\left(ze^{jq\pi}\right) = e^{jq\nu\pi} J_\nu(z) \text{ with } q \text{ integer} \tag{10a}$$

$$H_\nu^{(2)}\left(ze^{-j\pi}\right) = -e^{j\nu\pi} H_\nu^{(1)}(z) \tag{10b}$$

have been used.

By taking the difference between Eqs. (9a) and (9b), and making the substitution $w = k_0 \sin t$, it is simple to conclude that

$$I_{k,h}^{(|n|)} = \int_0^{k_0} F_{k,h}^{(|n|,2)}(aw)\,dw = \int_0^{\pi/2} J_{|n|+2k+3/2}(ak_0 \sin t)\, H_{|n|+2h+3/2}^{(2)}(ak_0 \sin t)\,dw \tag{11}$$

for $k \geq h$, which are proper integrals of bounded continuous functions.

3. NUMERICAL RESULTS

This section shows the accuracy and efficiency of the presented technique. The simulations are performed on a laptop equipped with an Intel Core 2 Duo CPU T9600 2.8-GHz 3-GB RAM, running Windows XP and the integrals evaluated by means of a Gauss-Legendre quadrature routine.

The absolute value of the ρ-component of the surface current density in the position $\rho/a = 0.63$, $\varphi = 0$ deg on a disk with $k_0 a = 5$, for an impinging TM polarized plane wave with $|\underline{H}| = 1$ A/m, $\theta_i = 60$ deg and $\varphi_i = 0$ deg, obtained by using both formulas (3) and (11), is reported in Tables 1 and 2

Table 1. $|J_\rho|$ with varying the number of expansion functions used (N) and the relative accuracy (R) in the numerical evaluation of the integrals. The integrals in Eq. (3) are evaluated by extracting only the first order asymptotic behaviour of the integrands. $k_0 a = 5$, $\rho/a = 0.63$, $\varphi = 0$ deg., TM incidence with $|\underline{H}| = 1$ A/m, $\theta_i = 60$ deg. and $\varphi_i = 0$ deg., 17 cylindrical harmonics.

N	Formulas (3)			Formula (11)
	$R = 10^{-4}$	$R = 10^{-8}$	$R = 10^{-12}$	$R = 10^{-4}, 10^{-8}, 10^{-12}$
3	1.465654273544891	1.465686798125570	1.465686933804895	1.465686945736443
4	1.450861632719635	1.450886927996987	1.450887077271100	1.450887089102934
5	1.437604865025232	1.437630774338152	1.437630927568240	1.437630939374279
6	1.437424654408773	1.437450633393541	1.437450786640072	1.437450798447578
7	1.437455424674373	1.437481406588271	1.437481559831062	1.437481571638675
8	1.437455979674769	1.437481961742863	1.437482114985662	1.437482126793269
9	1.437455963063291	1.437481945141001	1.437482098383801	1.437482110191411
10	1.437455962717336	1.437481944795007	1.437482098037807	1.437482109845413

by using 17 cylindrical harmonics (the maximum number of cylindrical harmonics to be used has been estimated as in [23]), and as the number of expansion functions used (N) and the relative accuracy (R) in the numerical evaluation of the integrals change. In Table 1, the integrals in Eq. (3) are evaluated by extracting only the first order asymptotic behaviour of the integrands, while the first order and second order asymptotic behaviours of the integrands are pulled out in order to obtain the results in Table 2. As can be seen, the results obtained by using the representation in Eq. (11) are independent of R, and the values obtained by using Eq. (3) tend to the ones obtained by means of Eq. (11) as the accuracy required for the solution increases. Moreover, by comparing the results in Tables 1 and 2, it is interesting to note that the accuracy of the solution is substantially independent of the number of asymptotic terms extracted, which, conversely, significantly affects the computation time (as will be shown in the following).

Table 2. $|J_\rho|$ with varying the number of expansion functions used (N) and the relative accuracy (R) in the numerical evaluation of the integrals. The integrals in Eq. (3) are evaluated by extracting the first order and the second order asymptotic behaviour of the integrands. $k_0a = 5$, $\rho/a = 0.63$, $\varphi = 0$ deg., TM incidence with $|\underline{H}| = 1$ A/m, $\theta_i = 60$ deg. and $\varphi_i = 0$ deg., 17 cylindrical harmonics.

N	Formulas (3)			Formula (11)
	$R = 10^{-4}$	$R = 10^{-8}$	$R = 10^{-12}$	$R = 10^{-4}, 10^{-8}, 10^{-12}$
3	1.465564828374628	1.465686888239401	1.465686934691235	1.465686945736443
4	1.450771704572309	1.450887027812345	1.450887077125728	1.450887089102934
5	1.437515443753201	1.437630872450123	1.437630928450345	1.437630939374279
6	1.437335251045673	1.437450731040232	1.437450787518923	1.437450798447578
7	1.437366022842356	1.437481504920134	1.437481559708562	1.437481571638675
8	1.437366577729281	1.437482059872345	1.437482114831655	1.437482126793269
9	1.437366561068543	1.437482043473452	1.437482098223401	1.437482110191411
10	1.437366560910234	1.437482042819726	1.437482097867398	1.437482109845413

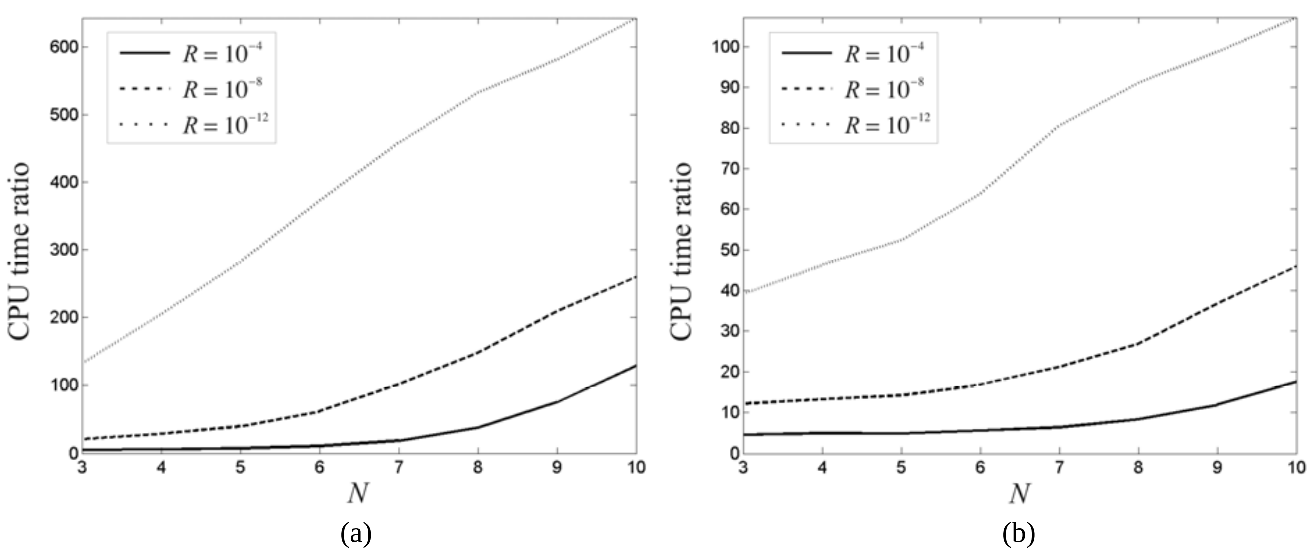

Figure 2. Ratio between the CPU time needed to reconstruct the solution as obtained by using (3) with respect to (11) with varying N and for different values of R. The integrals in (3) are evaluated by extracting (a) only the first order or (b) the first order and the second order asymptotic behaviour of the integrand. $k_0a = 5$, $\rho/a = 0.63$, $\varphi = 0$ deg., TM incidence with $|\underline{H}| = 1$ A/m, $\theta_i = 60$ deg. and $\varphi_i = 0$ deg., 17 cylindrical harmonics.

In Figure 2, the ratio between the CPU time needed to reconstruct the solution as obtained by using Eq. (3) with respect to Eq. (11) is reported with varying N for different values of R. In the first case, the integrals in Eq. (3) are evaluated by extracting only the first order asymptotic behaviour of the integrands, while in the second case, the first order and second order asymptotic behaviours of the integrands in Eq. (3) are pulling out. The second case examined is about 6 times faster than the first one. Despite that, the representation in Eq. (11) drastically outperforms the one in Eq. (3) in both the examined cases as the number of expansion functions used is higher and the accuracy required for the solution is as higher.

4. CONCLUSIONS

In a recent paper, the analysis of the electromagnetic scattering from a zero-thickness PEC disk has been addressed by means of a guaranteed-convergence method leading to the numerical evaluation of improper integrals of asymptotically oscillating and slowly decaying functions. In this paper, a new expression for such a kind of integrals in terms of fast converging proper integrals is devised. As shown in the Numerical Results section, the proposed technique is very effective and drastically outperforms the classical analytical asymptotic acceleration technique.

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