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Helmholtz Equation in Transverse Circular Representation

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Abstract—The use of transverse circular representation in circular cylinder coordinate system provides an alternative approach to the solutions for vector Helmholtz partial differential equations (VH-PDE) of electromagnetics. After separation, VH-PDE for electric (magnetic) field splits into a set of three ordinary differential (Bessel) equations for two opposite transverse circular polarizations (TCP) and the axial component. The approach is suitable for solving the problem of cylindrical waveguides and cavities starting from transverse fields. The coupling between TCP fields via the axial component affects nonrecirocal propagation in waveguides. The procedure is illustrated on a dielectric waveguide. It may be extended to the media with circular eigen polarizations including those displaying magnetooptical Faraday effect or optical activity.

1. INTRODUCTION

In circular cylinder coordinates, solutions to the vector Helmholtz partial differential equation in a homogeneous medium (i) characterized at the angular frequency ω by scalar magnetic permeability $\mu^{(i)}$ and electric permittivity $\varepsilon^{(i)}$ [1–5]

$$\left(\nabla^2 + \omega^2 \mu^{(i)} \varepsilon^{(i)}\right) \mathbf{E}^{(\mathbf{i})} = 0, \tag{1}$$

$$\left(\nabla^2 + \omega^2 \mu^{(i)} \varepsilon^{(i)}\right) \mathbf{H}^{(\mathbf{i})} = 0, \tag{2}$$

employ the decomposition of electric, $\mathbf{E}^{(\mathbf{i})}$ and magnetic $\mathbf{H}^{(\mathbf{i})}$ fields into the components parallel to the unit vectors $\hat{\varrho}$, $\hat{\varphi}$, and $\hat{z}$, i.e.,

$$\begin{aligned}\mathbf{E}^{(i)} &= E_\varrho^{(i)} \hat{\varrho} + E_\varphi^{(i)} \hat{\varphi} + E_z^{(i)} \hat{z},\\ \mathbf{H}^{(i)} &= H_\varrho^{(i)} \hat{\varrho} + H_\varphi^{(i)} \hat{\varphi} + H_z^{(i)} \hat{z},\end{aligned}$$

with the vector Laplacians of the form

$$\nabla^2 \mathbf{E}^{(i)} = \hat{\varrho} \left(\nabla^2 E_\varrho^{(i)} - \frac{2}{\varrho^2} \frac{\partial E_\varphi^{(i)}}{\partial \varphi} - \frac{E_\varrho^{(i)}}{\varrho^2} \right) + \hat{\varphi} \left(\nabla^2 E_\varphi^{(i)} + \frac{2}{\varrho^2} \frac{\partial E_\varrho^{(i)}}{\partial \varphi} - \frac{E_\varphi^{(i)}}{\varrho^2} \right) + \hat{z} \nabla^2 E_z, \tag{3}$$

$$\nabla^2 \mathbf{H}^{(i)} = \hat{\varrho} \left(\nabla^2 H_\varrho^{(i)} - \frac{2}{\varrho^2} \frac{\partial H_\varphi^{(i)}}{\partial \varphi} - \frac{H_\varrho^{(i)}}{\varrho^2} \right) + \hat{\varphi} \left(\nabla^2 H_\varphi^{(i)} + \frac{2}{\varrho^2} \frac{\partial H_\varrho^{(i)}}{\partial \varphi} - \frac{H_\varphi^{(i)}}{\varrho^2} \right) + \hat{z} \nabla^2 H_z^{(i)}. \tag{4}$$

Here ϱ and φ denote the transverse cylindrical coordinates and

$$\nabla^2 = \frac{\partial^2}{\partial \varrho^2} + \frac{1}{\varrho} \frac{\partial}{\partial \varrho} + \frac{1}{\varrho^2} \frac{\partial^2}{\partial \varphi^2} + \frac{\partial^2}{\partial z^2}.$$

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For the time, t and z dependence characterized by the complex factor $\exp[\mathrm{j}(\omega t - \beta z)]$, the Maxwell equations provide the transverse field components expressed in terms of $E_z^{(i)}$ and $H_z^{(i)}$

$$E_\varrho^{(i)} = -\mathrm{j}\frac{1}{\omega^2\varepsilon^{(i)}\mu^{(i)} - \beta^2}\left(\beta\frac{\partial E_z^{(i)}}{\partial\varrho} + \omega\mu^{(i)}\frac{1}{\varrho}\frac{\partial H_z^{(i)}}{\partial\varphi}\right), \tag{5}$$

$$E_\varphi^{(i)} = -\mathrm{j}\frac{1}{\omega^2\varepsilon^{(i)}\mu^{(i)} - \beta^2}\left(\beta\frac{1}{\varrho}\frac{\partial E_z^{(i)}}{\partial\varphi} - \omega\mu^{(i)}\frac{\partial H_z^{(i)}}{\partial\varrho}\right), \tag{6}$$

$$H_\varrho^{(i)} = v - \mathrm{j}\frac{1}{\omega^2\varepsilon^{(i)}\mu^{(i)} - \beta^2}\left(\beta\frac{\partial H_z^{(i)}}{\partial\varrho} - \omega\varepsilon^{(i)}\frac{1}{\varrho}\frac{\partial E_z^{(i)}}{\partial\varphi}\right), \tag{7}$$

$$H_\varphi^{(i)} = -\mathrm{j}\frac{1}{\omega^2\varepsilon^{(i)}\mu^{(i)} - \beta^2}\left(\beta\frac{1}{\varrho}\frac{\partial H_z^{(i)}}{\partial\varphi} + \omega\varepsilon^{(i)}\frac{\partial E_z^{(i)}}{\partial\varrho}\right), \tag{8}$$

where β represents the z component of the propagation vector.

The separation of Eqs. (1) and (2) is achieved with the products

$$\mathbf{E}_\nu^{(i)} = \mathbf{E}_0^{(i)}(\varrho)\,\mathrm{e}^{\mathrm{j}\nu\varphi}\exp\left[\mathrm{j}(\omega t - \beta z)\right], \tag{9}$$

$$\mathbf{H}_\nu^{(i)} = \mathbf{H}_0^{(i)}(\varrho)\,\mathrm{e}^{\mathrm{j}\nu\varphi}\exp\left[\mathrm{j}(\omega t - \beta z)\right], \tag{10}$$

where ν is an integer. Eqs. (1) and (2) then lead to a set of three equations for $\mathbf{E}_\nu^{(i)}$ ($\mathbf{H}_\nu^{(i)}$) where the $\hat{\varrho}$ and $\hat{\varphi}$ components are functions of both $E_\varrho^{(i)}$ ($H_\varrho^{(i)}$) and $E_\varphi^{(i)}$ ($H_\varphi^{(i)}$) and only the $\hat{z}$ component is a function of $E_z^{(i)}$ ($H_z^{(i)}$) alone. The solution procedure starts from the ordinary differential (Bessel) equation for $E_z^{(i)}(\varrho)$ or $H_z^{(i)}(\varrho)$ and the transverse components are deduced from Eqs. (5)–(8) [2–8].

In practice, there are situations of interest where $|E_z^{(i)}|$ and $|H_z^{(i)}|$ are much smaller than the transverse field components $|E_\varrho^{(i)}|$ ($|H_\varrho^{(i)}|$) and $|E_\varphi^{(i)}|$ ($|H_\varphi^{(i)}|$) [3, 9, 10]. From a formal point of view, this presents no problem in the exact analysis but may be a less convenient approach in the development of approximations.

The present work employs an alternative procedure which provides solutions to the vector Helmholtz partial differential equations (VH-PDE) in terms of transverse circularly polarized (TCP) field components coupled via the axial field component. After separation, VH-PDE for electric (magnetic) field splits into a set of three ordinary differential (Bessel) equations. The coupling between the TCP components plays role in cylindrical waveguides displaying nonreciprocal propagation or optical activity [7]. It also presents interest in optical angular momentum studies [8]. Its effect (often undesirable) can be controlled by a proper choice of waveguide parameters. In Section 2, the Helmholtz equations are expressed with the transverse fields decomposed in circularly polarized components. The solutions for the cases $\nu = 0$ and $\nu = \pm 1$ are given in Section 3. Section 4 applies the procedure to the analysis of dielectric waveguides of circular cross-sections with homogeneous core and cladding. In the last section, Section 5, the procedure is summarized and illustrated on the case of monomode waveguides where the effect of material parameters on TCP amplitudes is of a primary interest as it includes conditions for weak guiding regime.

2. TRANSVERSE CIRCULAR POLARIZATIONS

With the circularly polarized (CP) unit vectors $\hat{\varrho}_\pm$ and unit vector $\hat{z}$, the field vectors in Eqs. (9) and (10) (with the time factor $\mathrm{e}^{\mathrm{j}\omega t}$ omitted) can be decomposed to

$$\mathbf{E}_\nu^{(i)}(\varrho,\varphi,z) = \left[\hat{\varrho}_+\mathrm{e}^{-\mathrm{j}\varphi}E_{\nu+}^{(i)}(\varrho,\varphi) + \hat{\varrho}_-\mathrm{e}^{\mathrm{j}\varphi}E_{\nu-}^{(i)}(\varrho,\varphi) + \hat{z}E_{\nu,z}^{(i)}(\varrho,\varphi)\right]\mathrm{e}^{-\mathrm{j}\beta z}, \tag{11}$$

$$\mathbf{H}_\nu^{(i)}(\varrho,\varphi,z) = \left[\hat{\varrho}_+\mathrm{e}^{-\mathrm{j}\varphi}H_{\nu+}^{(i)}(\varrho,\varphi) + \hat{\varrho}_-\mathrm{e}^{\mathrm{j}\varphi}H_{\nu-}^{(i)}(\varrho,\varphi) + \hat{z}H_{\nu,z}^{(i)}(\varrho,\varphi)\right]\mathrm{e}^{-\mathrm{j}\beta z}. \tag{12}$$

The circularly polarized unit vectors $\hat{\varrho}_\pm$ are related to the cylindrical unit vectors $\hat{\varrho}$ and $\hat{\varphi}$ and to the Cartesian unit vectors, $\hat{x}$ and $\hat{y}$ by

$$\hat{\varrho}_\pm = 2^{-1/2} \left(\hat{\varrho} \pm \mathrm{j}\hat{\varphi}\right) \mathrm{e}^{\pm \mathrm{j}\varphi} = 2^{-1/2} \left(\hat{x} \pm \mathrm{j}\hat{y}\right).$$

The fields decomposed into $\hat{\varrho}$, $\hat{\varphi}$, and $\hat{z}$ in the classical solution become

$$\begin{aligned} \mathbf{E}_\nu^{(i)}(\varrho,\varphi,z) &= \Big\{2^{-1/2}\hat{\varrho}\left[E_{\nu+}^{(i)}(\varrho,\varphi) + E_{\nu-}^{(i)}(\varrho,\varphi)\right] \\ &\quad + 2^{-1/2}\mathrm{j}\hat{\varphi}\left[E_{\nu+}^{(i)}(\varrho,\varphi) - E_{\nu-}^{(i)}(\varrho,\varphi)\right] + \hat{z}E_{\nu,z}^{(i)}(\varrho,\varphi)\Big\}\mathrm{e}^{-\mathrm{j}\beta z}, \end{aligned} \tag{13}$$

$$\begin{aligned} \mathbf{H}_\nu^{(i)}(\varrho,\varphi,z) &= \Big\{2^{-1/2}\hat{\varrho}\left[H_{\nu+}^{(i)}(\varrho,\varphi) + H_{\nu-}^{(i)}(\varrho,\varphi)\right] \\ &\quad + 2^{-1/2}\mathrm{j}\hat{\varphi}\left[H_{\nu+}^{(i)}(\varrho,\varphi) - H_{\nu-}^{(i)}(\varrho,\varphi)\right] + \hat{z}H_{\nu,z}^{(i)}(\varrho,\varphi)\Big\}\mathrm{e}^{-\mathrm{j}\beta z}. \end{aligned} \tag{14}$$

The decomposition to $\hat{x}$, $\hat{y}$, and $\hat{z}$ applied in the weak guiding approximation [3, 4, 9, 10] gives

$$\begin{aligned} \mathbf{E}_\nu^{(i)}(\varrho,\varphi,z) &= \Big\{2^{-1/2}\hat{x}\left[\mathrm{e}^{-\mathrm{j}\varphi}E_{\nu+}^{(i)}(\varrho,\varphi) + \mathrm{e}^{\mathrm{j}\varphi}E_{\nu-}^{(i)}(\varrho,\varphi)\right] \\ &\quad + 2^{-1/2}\mathrm{j}\hat{y}\left[\mathrm{e}^{-\mathrm{j}\varphi}E_{\nu+}^{(i)}(\varrho,\varphi) - \mathrm{e}^{\mathrm{j}\varphi}E_{\nu-}^{(i)}(\varrho,\varphi)\right] + \hat{z}E_{\nu,z}^{(i)}(\varrho,\varphi)\Big\}\mathrm{e}^{-\mathrm{j}\beta z}, \end{aligned} \tag{15}$$

$$\begin{aligned} \mathbf{H}_\nu^{(i)}(\varrho,\varphi,z) &= \Big\{2^{-1/2}\hat{x}\left[\mathrm{e}^{-\mathrm{j}\varphi}H_{\nu+}^{(i)}(\varrho,\varphi) + \mathrm{e}^{\mathrm{j}\varphi}H_{\nu-}^{(i)}(\varrho,\varphi)\right] \\ &\quad + 2^{-1/2}\mathrm{j}\hat{y}\left[\mathrm{e}^{-\mathrm{j}\varphi}H_{\nu+}^{(i)}(\varrho,\varphi) - \mathrm{e}^{\mathrm{j}\varphi}H_{\nu-}^{(i)}(\varrho,\varphi)\right] + \hat{z}H_{\nu,z}^{(i)}(\varrho,\varphi)\Big\}\mathrm{e}^{-\mathrm{j}\beta z}. \end{aligned} \tag{16}$$

The substitution of the vectors $\mathbf{E}_\nu^{(i)}$ and $\mathbf{H}_\nu^{(i)}$ given in Eqs. (11) and (12) into Eqs. (1) and (2) reduces them to sets of ordinary differential (Bessel) equations separately for the electric field components $E_{\nu+}^{(i)}$, $E_{\nu-}^{(i)}$, and $E_{\nu,z}^{(i)}$,

$$\begin{aligned} &\hat{\varrho}_+\mathrm{e}^{-\mathrm{j}\varphi}\left\{\frac{\mathrm{d}^2E_{\nu+}^{(i)}}{\mathrm{d}\varrho^2} + \frac{1}{\varrho}\frac{\mathrm{d}E_{\nu+}^{(i)}}{\mathrm{d}\varrho} + \left[\left(\omega^2\mu^{(i)}\varepsilon^{(i)} - \beta^2\right) - \frac{1}{\varrho^2}(\nu-1)^2\right]E_{\nu+}^{(i)}\right\} \\ &+\hat{\varrho}_-\mathrm{e}^{\mathrm{j}\varphi}\left\{\frac{\mathrm{d}^2E_{\nu-}^{(i)}}{\mathrm{d}\varrho^2} + \frac{1}{\varrho}\frac{\mathrm{d}E_{\nu-}^{(i)}}{\mathrm{d}\varrho} + \left[\left(\omega^2\mu^{(i)}\varepsilon^{(i)} - \beta^2\right) - \frac{1}{\varrho^2}(\nu+1)^2\right]E_{\nu-}^{(i)}\right\} \\ &+\hat{z}\left\{\frac{\mathrm{d}^2E_{\nu,z}^{(i)}}{\mathrm{d}\varrho^2} + \frac{1}{\varrho}\frac{\mathrm{d}E_{\nu,z}^{(i)}}{\mathrm{d}\varrho} + \left[\left(\omega^2\mu^{(i)}\varepsilon^{(i)} - \beta^2\right) - \frac{1}{\varrho^2}\nu^2\right]E_{\nu,z}^{(i)}\right\} = 0, \end{aligned} \tag{17}$$

and separately for the magnetic field components $H_{\nu+}^{(i)}$, $H_{\nu-}^{(i)}$, and $H_{\nu,z}^{(i)}$,

$$\begin{aligned} &\hat{\varrho}_+\mathrm{e}^{-\mathrm{j}\varphi}\left\{\frac{\mathrm{d}^2H_{\nu+}^{(i)}}{\mathrm{d}\varrho^2} + \frac{1}{\varrho}\frac{\mathrm{d}H_{\nu+}^{(i)}}{\mathrm{d}\varrho} + \left[\left(\omega^2\mu^{(i)}\varepsilon^{(i)} - \beta^2\right) - \frac{1}{\varrho^2}(\nu-1)^2\right]H_{\nu+}^{(i)}\right\} \\ &+\hat{\varrho}_-\mathrm{e}^{\mathrm{j}\varphi}\left\{\frac{\mathrm{d}^2H_{\nu-}^{(i)}}{\mathrm{d}\varrho^2} + \frac{1}{\varrho}\frac{\mathrm{d}H_{\nu-}^{(i)}}{\mathrm{d}\varrho} + \left[\left(\omega^2\mu^{(i)}\varepsilon^{(i)} - \beta^2\right) - \frac{1}{\varrho^2}(\nu+1)^2\right]H_{\nu-}^{(i)}\right\} \\ &+\hat{z}\left\{\frac{\mathrm{d}^2H_{\nu,z}^{(i)}}{\mathrm{d}\varrho^2} + \frac{1}{\varrho}\frac{\mathrm{d}H_{\nu,z}^{(i)}}{\mathrm{d}\varrho} + \left[\left(\omega^2\mu^{(i)}\varepsilon^{(i)} - \beta^2\right) - \frac{1}{\varrho^2}\nu^2\right]H_{\nu,z}^{(i)}\right\} = 0. \end{aligned} \tag{18}$$

Each of the Bessel equations has two linearly independent solutions. In the problem of cylindrically multilayered media, both solutions must be considered [11]. Here the procedure will be demonstrated on a circular cylindrical dielectric waveguide formed by a core $\varrho \leq a$ and a cladding $\varrho \geq a$ with the solution proportional to the Bessel function of a real argument $\mathcal{J}_\nu(\kappa\rho)$ in the region $\varrho \leq a$ and with the solution proportional to the modified Hankel function of the first kind of an argument imaginary pure

$\mathcal{H}_\nu^{(1)}(\mathrm{j}\gamma\varrho)$ in the region $\varrho \geq a$. Then the solutions can be expressed using with the solutions proportional to the amplitudes $A_{\nu\pm}^{(i)}$ and $A_{\nu,z}^{(i)}$ for the electric fields

$$\begin{aligned} E_{\nu+}^{(i)} &= A_{\nu+}^{(i)} \mathcal{Z}_{\nu-1}^{(i)} \left(\kappa^{(i)} \varrho\right) \mathrm{e}^{\mathrm{j}\nu\varphi}, \\ E_{\nu-}^{(i)} &= A_{\nu-}^{(i)} \mathcal{Z}_{\nu+1}^{(i)} \left(\kappa^{(i)} \varrho\right) \mathrm{e}^{\mathrm{j}\nu\varphi}, \\ E_{\nu,z}^{(i)} &= A_{\nu,z}^{(i)} \mathcal{Z}_{\nu}^{(i)} \left(\kappa^{(i)} \varrho\right) \mathrm{e}^{\mathrm{j}\nu\varphi}, \end{aligned} \tag{19}$$

and the amplitudes $B_{\nu\pm}^{(i)}$ and $B_{\nu,z}^{(i)}$ for the magnetic fields

$$\begin{aligned} H_{\nu+}^{(i)} &= B_{\nu+}^{(i)} \mathcal{Z}_{\nu-1}^{(i)} \left(\kappa^{(i)} \varrho\right) \mathrm{e}^{\mathrm{j}\nu\varphi}, \\ H_{\nu-}^{(i)} &= B_{\nu-}^{(i)} \mathcal{Z}_{\nu+1}^{(i)} \left(\kappa^{(i)} \varrho\right) \mathrm{e}^{\mathrm{j}\nu\varphi}, \\ H_{\nu,z}^{(i)} &= B_{\nu,z}^{(i)} \mathcal{Z}_{\nu}^{(i)} \left(\kappa^{(i)} \varrho\right) \mathrm{e}^{\mathrm{j}\nu\varphi}. \end{aligned} \tag{20}$$

Here the transverse component of propagation vector, $\kappa^{(i)}$, follows from the definition

$$\kappa^{(i)2} = \omega^2 \mu^{(i)} \varepsilon^{(i)} - \beta^2. \tag{21}$$

and $\mathcal{Z}_\nu^{(i)}(\kappa^{(i)}\varrho)$ denotes a cylindrical function of the order ν.

The solutions given by Eqs. (19) and (20) must be consistent with the Maxwell equations, i.e., with the Faraday law

$$\begin{aligned} &\hat{\varrho}_+ \mathrm{e}^{-\mathrm{j}\varphi} \left[\mathrm{j}\beta E_{\nu+}^{(i)} + 2^{-1/2} \left(\frac{\mathrm{d}}{\mathrm{d}\varrho} + \frac{\nu}{\varrho}\right) E_{\nu,z}^{(i)}\right] \\ &+\hat{\varrho}_- \mathrm{e}^{\mathrm{j}\varphi} \left[-\mathrm{j}\beta E_{\nu-}^{(i)} - 2^{-1/2} \left(\frac{\mathrm{d}}{\mathrm{d}\varrho} - \frac{\nu}{\varrho}\right) E_{\nu,z}^{(i)}\right] \\ &+2^{-1/2} \hat{z} \left[\frac{1}{\varrho} E_{\nu+}^{(i)} + \left(\frac{\mathrm{d}}{\mathrm{d}\varrho} - \frac{\nu}{\varrho}\right) E_{\nu+}^{(i)} - \frac{1}{\varrho} E_{\nu-}^{(i)} - \left(\frac{\mathrm{d}}{\mathrm{d}\varrho} + \frac{\nu}{\varrho}\right) E_{\nu-}^{(i)}\right] \\ = \; & -\omega\mu^{(i)} \left(\mathrm{e}^{-\mathrm{j}\varphi} \hat{\varrho}_+ H_{\nu+}^{(i)} + \mathrm{e}^{\mathrm{j}\varphi} \hat{\varrho}_- H_{\nu-}^{(i)} + \hat{z} H_{\nu,z}^{(i)}\right), \end{aligned} \tag{22}$$

and with the Ampère law

$$\begin{aligned} &\hat{\varrho}_+ \mathrm{e}^{-\mathrm{j}\varphi} \left[\mathrm{j}\beta H_{\nu+}^{(i)} + 2^{-1/2} \left(\frac{\mathrm{d}}{\mathrm{d}\varrho} + \frac{\nu}{\varrho}\right) H_{\nu,z}^{(i)}\right] \\ &+\hat{\varrho}_- \mathrm{e}^{\mathrm{j}\varphi} \left[-\mathrm{j}\beta H_{\nu-}^{(i)} - 2^{-1/2} \left(\frac{\mathrm{d}}{\mathrm{d}\varrho} - \frac{\nu}{\varrho}\right) H_{\nu,z}^{(i)}\right] \\ &+2^{-1/2} \hat{z} \left[\frac{1}{\varrho} H_{\nu+}^{(i)} + \left(\frac{\mathrm{d}}{\mathrm{d}\varrho} - \frac{\nu}{\varrho}\right) H_{\nu+}^{(i)} - \frac{1}{\varrho} H_{\nu-}^{(i)} - \left(\frac{\mathrm{d}}{\mathrm{d}\varrho} + \frac{\nu}{\varrho}\right) H_{\nu-}^{(i)}\right] \\ = \; & \omega\varepsilon^{(i)} \left(\mathrm{e}^{-\mathrm{j}\varphi} \hat{\varrho}_+ E_{\nu+}^{(i)} + \mathrm{e}^{\mathrm{j}\varphi} \hat{\varrho}_- E_{\nu-}^{(i)} + \hat{z} E_{\nu,z}^{(i)}\right). \end{aligned} \tag{23}$$

Equations (22) and (23) are related by the duality transformation which requires $\mathbf{E} \to \pm\mathbf{H}$, $\mathbf{H} \to \mp\mathbf{E}$, and $\varepsilon^{(i)} \leftrightarrow \mu^{(i)}$. The substitutions of Eqs. (19) and (20) into Eqs. (22) and (23) provide for the Faraday law

$$\begin{aligned} H_{\nu+}^{(i)} &= -\frac{1}{\omega\mu^{(i)}} \left(\mathrm{j}\beta A_{\nu+}^{(i)} + 2^{-1/2}\kappa^{(i)} A_{\nu,z}^{(i)}\right) \mathcal{Z}_{\nu-1}^{(i)} \mathrm{e}^{\mathrm{j}\nu\varphi}, \\ H_{\nu-}^{(i)} &= \frac{1}{\omega\mu^{(i)}} \left(\mathrm{j}\beta A_{\nu-}^{(i)} - 2^{-1/2}\kappa^{(i)} A_{\nu,z}^{(i)}\right) \mathcal{Z}_{\nu+1}^{(i)} \mathrm{e}^{\mathrm{j}\nu\varphi}, \\ H_{\nu,z}^{(i)} &= \frac{2^{1/2}}{\omega\mu^{(i)}} \kappa^{(i)} \left(A_{\nu+}^{(i)} + A_{\nu-}^{(i)}\right) \mathcal{Z}_{\nu}^{(i)} \mathrm{e}^{\mathrm{j}\nu\varphi}, \end{aligned} \tag{24}$$

and for the Ampère law

$$\begin{aligned} E_{\nu+}^{(i)} &= \frac{1}{\omega\varepsilon^{(i)}}\left(\mathrm{j}\beta B_{\nu+}^{(i)} + 2^{-1/2}\kappa^{(i)} B_{\nu,z}^{(i)}\right)\mathcal{Z}_{\nu-1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}, \\ E_{\nu-}^{(i)} &= -\frac{1}{\omega\varepsilon^{(i)}}\left(\mathrm{j}\beta B_{\nu-}^{(i)} - 2^{-1/2}\kappa^{(i)} B_{\nu,z}^{(i)}\right)\mathcal{Z}_{\nu+1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}, \\ E_{\nu,z}^{(i)} &= -\frac{2^{1/2}}{\omega\varepsilon^{(i)}}\kappa^{(i)}\left(B_{\nu+}^{(i)} + B_{\nu-}^{(i)}\right)\mathcal{Z}_{\nu}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}. \end{aligned} \tag{25}$$

The relations among the amplitudes follow from Eqs. (19) and (20) and Eqs. (24) and (25)

$$A_{\nu+}^{(i)} = \frac{1}{\omega\varepsilon^{(i)}}\left(\mathrm{j}\beta B_{\nu+}^{(i)} + 2^{-1/2}\kappa^{(i)} B_{\nu,z}^{(i)}\right), \tag{26}$$

$$A_{\nu-}^{(i)} = -\frac{1}{\omega\varepsilon^{(i)}}\left(\mathrm{j}\beta B_{\nu-}^{(i)} - 2^{-1/2}\kappa^{(i)} B_{\nu,z}^{(i)}\right), \tag{27}$$

$$A_{\nu,z}^{(i)} = \frac{-2^{-1/2}\kappa^{(i)}}{\varepsilon^{(i)}\omega}\left(B_{\nu+}^{(i)} + B_{\nu-}^{(i)}\right) = 2^{-1/2}\mathrm{j}\frac{\kappa^{(i)}}{\beta}\left(A_{\nu+}^{(i)} - A_{\nu-}^{(i)}\right), \tag{28}$$

$$B_{\nu+}^{(i)} = -\frac{1}{\omega\mu^{(i)}}\left(\mathrm{j}\beta A_{\nu+}^{(i)} + 2^{-1/2}\kappa^{(i)} A_{\nu,z}^{(i)}\right), \tag{29}$$

$$B_{\nu-}^{(i)} = \frac{1}{\omega\mu^{(i)}}\left(\mathrm{j}\beta A_{\nu-}^{(i)} - 2^{-1/2}\kappa^{(i)} A_{\nu,z}^{(i)}\right), \tag{30}$$

$$B_{\nu,z}^{(i)} = \frac{+2^{-1/2}\kappa^{(i)}}{\mu^{(i)}\omega}\left(A_{\nu+}^{(i)} + A_{\nu-}^{(i)}\right) = 2^{-1/2}\mathrm{j}\frac{\kappa^{(i)}}{\beta}\left(B_{\nu+}^{(i)} - B_{\nu-}^{(i)}\right). \tag{31}$$

The six amplitudes can be expressed in terms of two of them, e.g., using the axial $A_{\nu,z}^{(i)}$ and $B_{\nu,z}^{(i)}$ [2, 4, 5]. Here, the preferred selections employ the transverse amplitudes, i.e., $A_{\nu\pm}^{(i)}$ and $B_{\nu\pm}^{(i)}$ or $A_{\nu+}^{(i)}$ and $A_{\nu-}^{(i)}$, etc.. Substitutions of the solutions from Eqs. (19) and (20) into Eqs. (11) and (12) provide

$$\mathbf{E}_{\nu}^{(i)}(\varrho,\varphi,z) = \left[\mathrm{e}^{-\mathrm{j}\varphi}\hat{\varrho}_{+}A_{\nu+}^{(i)}\mathcal{Z}_{\nu-1}^{(i)} + \mathrm{e}^{\mathrm{j}\varphi}\hat{\varrho}_{-}A_{\nu-}^{(i)}\mathcal{Z}_{\nu+1}^{(i)} + \hat{z}A_{\nu,z}^{(i)}\mathcal{Z}_{\nu}^{(i)}\right]\mathrm{e}^{\mathrm{j}\nu\varphi}\mathrm{e}^{-\mathrm{j}\beta z}, \tag{32}$$

$$\mathbf{H}_{\nu}^{(i)}(\varrho,\varphi,z) = \left[\mathrm{e}^{-\mathrm{j}\varphi}\hat{\varrho}_{+}B_{\nu+}^{(i)}\mathcal{Z}_{\nu-1}^{(i)} + \mathrm{e}^{\mathrm{j}\varphi}\hat{\varrho}_{-}B_{\nu-}^{(i)}\mathcal{Z}_{\nu+1}^{(i)} + \hat{z}B_{\nu,z}^{(i)}\mathcal{Z}_{\nu}^{(i)}\right]\mathrm{e}^{\mathrm{j}\nu\varphi}\mathrm{e}^{-\mathrm{j}\beta z}. \tag{33}$$

The amplitudes $A_{\nu,z}^{(i)}$ and $B_{\nu,z}^{(i)}$ can be replaced using Eqs. (28) and (31). This gives, with the factor $\mathrm{e}^{-\mathrm{j}\beta z}$ omitted,

$$\mathbf{E}_{\nu}^{(i)}(\varrho,\varphi) = \left[\mathrm{e}^{-\mathrm{j}\varphi}\hat{\varrho}_{+}A_{\nu+}^{(i)}\mathcal{Z}_{\nu-1}^{(i)} + \mathrm{e}^{\mathrm{j}\varphi}\hat{\varrho}_{-}A_{\nu-}^{(i)}\mathcal{Z}_{\nu+1}^{(i)} + 2^{-1/2}\mathrm{j}\hat{z}\frac{\kappa^{(i)}}{\beta}\left(A_{\nu+}^{(i)} - A_{\nu-}^{(i)}\right)\mathcal{Z}_{\nu}^{(i)}\right]\mathrm{e}^{\mathrm{j}\nu\varphi}, \tag{34}$$

$$\mathbf{H}_{\nu}^{(i)}(\varrho,\varphi) = \left[\mathrm{e}^{-\mathrm{j}\varphi}\hat{\varrho}_{+}B_{\nu+}^{(i)}\mathcal{Z}_{\nu-1}^{(i)} + \mathrm{e}^{\mathrm{j}\varphi}\hat{\varrho}_{-}B_{\nu-}^{(i)}\mathcal{Z}_{\nu+1}^{(i)} + 2^{-1/2}\mathrm{j}\hat{z}\frac{\kappa^{(i)}}{\beta}\left(B_{\nu+}^{(i)} - B_{\nu-}^{(i)}\right)\mathcal{Z}_{\nu}^{(i)}\right]\mathrm{e}^{\mathrm{j}\nu\varphi}. \tag{35}$$

Note that the field components $A_{\nu,z}^{(i)}$ ($B_{\nu,z}^{(i)}$) parallel to $\hat{z}$ depend on the CP amplitudes $A_{\nu\pm}^{(i)}$ ($B_{\nu\pm}^{(i)}$) parallel to $\hat{\varrho}_{\pm}$. Alternatively, the transverse (CP, i.e., perpendicular to $\hat{z}$) field components can be expressed in terms of the amplitudes $A_{\nu,z0}^{(i)}$ and $B_{\nu,z0}^{(i)}$ for the axial (parallel to $\hat{z}$) components given in Eqs. (19) and (20)

$$\begin{aligned} E_{\nu+}^{(i)} &= \frac{2^{-1/2}}{\kappa^{(i)}}\left(\omega\mu^{(i)}B_{\nu,z}^{(i)} - \mathrm{j}\beta A_{\nu,z}^{(i)}\right)\mathcal{Z}_{\nu-1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}, \\ H_{\nu+}^{(i)} &= -\frac{2^{-1/2}}{\kappa^{(i)}}\left(\omega\varepsilon^{(i)}A_{\nu,z}^{(i)} + \mathrm{j}\beta B_{\nu,z}^{(i)}\right)\mathcal{Z}_{\nu-1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}, \\ E_{\nu-}^{(i)} &= \frac{2^{-1/2}}{\kappa^{(i)}}\left(\omega\mu^{(i)}B_{\nu,z}^{(i)} + \mathrm{j}\beta A_{\nu,z}^{(i)}\right)\mathcal{Z}_{\nu+1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}, \\ H_{\nu-}^{(i)} &= -\frac{2^{-1/2}}{\kappa^{(i)}}\left(\omega\varepsilon^{(i)}A_{\nu,z}^{(i)} - \mathrm{j}\beta B_{\nu,z}^{(i)}\right)\mathcal{Z}_{\nu+1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}. \end{aligned}$$

3. SPECIAL CASES

In this section, the solutions will be given for the modal fields of the lowest orders 0 and ±1 which determine the range of monomode operation in a circular cylindrical waveguide with a uniform core and a uniform cladding [2, 5].

3.1. TE and TM waves

For $\nu = 0$, the Helmholtz equations consists of Bessel equations for $E_{0,z}^{(i)}$ and $H_{0,z}^{(i)}$ which are of the same form with the solutions proportional to $\mathcal{Z}_0^{(i)}$. Also, the Bessel equations for $E_{0,\pm 1}^{(i)}$ and $H_{\pm 1}^{(i)}$ are of the same form. Their solutions are proportional to $\mathcal{Z}_{\pm 1}^{(i)}$. According to Eq. (28) the electric field component $E_{0,z}^{(i)}$ vanishes for $-B_{0+}^{(i)} = B_{0-}^{(i)} \equiv B_{\mathrm{TE}}^{(i)}$ and $A_{0+}^{(i)} = A_{0-}^{(i)} \equiv A_{\mathrm{TE}}^{(i)}$, where $B_{\mathrm{TE}}^{(i)} = \frac{\mathrm{j}\beta}{\omega\mu^{(i)}} A_{\mathrm{TE}}^{(i)}$. The solution corresponds to the transverse electric (TE) wave

$$\mathbf{E}_{\mathrm{TE}}^{(i)} = \left(\hat{\varrho}_+ \mathrm{e}^{-\mathrm{j}\varphi} - \hat{\varrho}_- \mathrm{e}^{\mathrm{j}\varphi}\right) A_{\mathrm{TE}}^{(i)} \mathcal{Z}_1^{(i)} = \mathrm{j} 2^{1/2} \hat{\varphi} A_{\mathrm{TE}}^{(i)} \mathcal{Z}_1^{(i)}, \tag{36}$$

$$\begin{aligned}\mathbf{H}_{\mathrm{TE}}^{(i)} &= \frac{\mathrm{j}\beta}{\omega\mu^{(i)}} \left(\hat{\varrho}_+ \mathrm{e}^{-\mathrm{j}\varphi} + \hat{\varrho}_- \mathrm{e}^{\mathrm{j}\varphi}\right) A_{\mathrm{TE}}^{(i)} \mathcal{Z}_1^{(i)} + 2^{1/2} \hat{z} \frac{\kappa^{(i)}}{\omega\mu^{(i)}} A_{\mathrm{TE}}^{(i)} \mathcal{Z}_0^{(i)} \\ &= \frac{2^{1/2}\mathrm{j}}{\omega\mu^{(i)}} \left(\hat{\varrho}\beta \mathcal{Z}_1^{(i)} - \mathrm{j}\hat{z}\kappa^{(i)} \mathcal{Z}_0^{(i)}\right) A_{\mathrm{TE}}^{(i)}.\end{aligned} \tag{37}$$

According to Eq. (30), the magnetic field component $H_{0,z}^{(i)}$ vanishes for $-A_{0+}^{(i)} = A_{0-}^{(i)} \equiv A_{\mathrm{TM}}^{(i)}$ and $B_{0+}^{(i)} = B_{0-}^{(i)} \equiv B_{\mathrm{TM}}^{(i)}$, where $A_{\mathrm{TM}}^{(i)} = -\frac{\mathrm{j}\beta}{\omega\varepsilon^{(i)}} B_{\mathrm{TM}}^{(i)}$. The solution corresponds to the transverse magnetic (TM) wave

$$\mathbf{H}_{\mathrm{TM}}^{(i)} = -\left(\hat{\varrho}_+ \mathrm{e}^{-\mathrm{j}\varphi} - \hat{\varrho}_- \mathrm{e}^{\mathrm{j}\varphi}\right) B_{\mathrm{TM}}^{(i)} \mathcal{Z}_1^{(i)} = -\mathrm{j} 2^{1/2} \hat{\varphi} B_{\mathrm{TM}}^{(i)} \mathcal{Z}_1^{(i)}, \tag{38}$$

$$\begin{aligned}\mathbf{E}_{\mathrm{TM}}^{(i)} &= -\frac{\mathrm{j}\beta}{\omega\varepsilon^{(i)}} \left(\hat{\varrho}_+ \mathrm{e}^{-\mathrm{j}\varphi} + \hat{\varrho}_- \mathrm{e}^{\mathrm{j}\varphi}\right) B_{\mathrm{TM}}^{(i)} \mathcal{Z}_1^{(i)} - 2^{1/2} \hat{z} \frac{\kappa^{(i)}}{\omega\varepsilon(i)} B_{\mathrm{TM}}^{(i)} \mathcal{Z}_0^{(i)} \\ &= -\frac{2^{1/2}\mathrm{j}}{\omega\varepsilon^{(i)}} \left(\hat{\varrho}\beta \mathcal{Z}_1^{(i)} - \mathrm{j}\hat{z}\kappa^{(i)} \mathcal{Z}_0^{(i)}\right) B_{\mathrm{TM}}^{(i)}.\end{aligned} \tag{39}$$

3.2. Solutions for $\nu = \pm 1$

For the special case of $\nu = \pm 1$, Eqs. (34) and (35) provide

$$\mathbf{E}_{+1}^{(i)}(\varrho,\varphi) = \hat{\varrho}_+ A_{+1+}^{(i)} \mathcal{Z}_0^{(i)} + \hat{\varrho}_- A_{+1-}^{(i)} \mathcal{Z}_2^{(i)} \mathrm{e}^{2\mathrm{j}\varphi} + \hat{z} A_{+1,z}^{(i)} \mathcal{Z}_1^{(i)} \mathrm{e}^{\mathrm{j}\varphi}, \tag{40}$$

$$\mathbf{H}_{+1}^{(i)}(\varrho,\varphi) = \hat{\varrho}_+ B_{+1+}^{(i)} \mathcal{Z}_0^{(i)} + \hat{\varrho}_- B_{+1-}^{(i)} \mathcal{Z}_2^{(i)} \mathrm{e}^{2\mathrm{j}\varphi} + \hat{z} B_{+1,z}^{(i)} \mathcal{Z}_1^{(i)} \mathrm{e}^{\mathrm{j}\varphi}, \tag{41}$$

$$\mathbf{E}_{-1}^{(i)}(\varrho,\varphi) = \hat{\varrho}_+ A_{-1+}^{(i)} \mathcal{Z}_2^{(i)} \mathrm{e}^{-2\mathrm{j}\varphi} + \hat{\varrho}_- A_{-1-}^{(i)} \mathcal{Z}_0^{(i)} - \hat{z} A_{-1,z}^{(i)} \mathcal{Z}_1^{(i)} \mathrm{e}^{-\mathrm{j}\varphi}, \tag{42}$$

$$\mathbf{H}_{-1}^{(i)}(\varrho,\varphi) = \hat{\varrho}_+ B_{-1+}^{(i)} \mathcal{Z}_2^{(i)} \mathrm{e}^{-2\mathrm{j}\varphi} + \hat{\varrho}_- B_{-1-}^{(i)} \mathcal{Z}_0^{(i)} - \hat{z} B_{-1,z}^{(i)} \mathcal{Z}_1^{(i)} \mathrm{e}^{-\mathrm{j}\varphi}. \tag{43}$$

The use has been made of the parity relation [1]

$$\mathcal{Z}_{\nu}^{(i)} = (-1)^{\nu} \mathcal{Z}_{\nu}^{(i)}.$$

In the z components, the amplitudes $A_{\pm 1,z}^{(i)}$ and $A_{\pm 1,z}^{(i)}$ can again be replaced according to Eqs. (28) and (31), i.e.,

$$\begin{aligned}A_{\pm 1,z}^{(i)} &= 2^{-1/2}\mathrm{j}\frac{\kappa^{(i)}}{\beta}\left(A_{\pm 1+}^{(i)} - A_{\pm 1-}^{(i)}\right), \\ B_{\pm 1,z}^{(i)} &= 2^{-1/2}\mathrm{j}\frac{\kappa^{(i)}}{\beta}\left(B_{\pm 1+}^{(i)} - B_{\pm 1-}^{(i)}\right).\end{aligned}$$

4. WAVEGUIDE

For the purpose of illustration, Eqs. (17) and (18) will be now applied to the analysis of a simple cylindrically layered medium, a cylindrical dielectric waveguide. The waveguide is formed by the structure consisting of a homogeneous core in the region $0 \leq \varrho \leq a$ ($i = 1$) characterized by $\mu^{(1)}$ and $\varepsilon^{(1)}$ and a homogeneous cladding occupying the region $a \leq \varrho$ ($i = 2$). In the core, $\mathcal{Z}_\nu^{(1)}(\kappa^{(1)}\varrho) = \mathcal{J}_\nu(\kappa\varrho)$ represent Bessel functions of the first kind with the real argument $\kappa\varrho$. In the cladding, $\mathcal{Z}_\nu^{(2)}(\kappa^{(2)}\varrho) = \mathcal{H}_\nu^{(1)}(\mathrm{j}\gamma\varrho)$ represent modified Hankel functions of the first kind, with the argument imaginary pure $\mathrm{j}\gamma\varrho$ (see Section 2).

4.1. Boundary conditions

The continuity of the field components parallel to the interface $\varrho = a$ follows from the projections of fields to $\hat{\varphi}$ and $\hat{z}$. Projections to $\hat{\varphi}$ of the fields in Eqs. (34) and (35) are given by

$$\hat{\varphi}\cdot\mathbf{E}_\nu^{(i)}(\varrho,\varphi) = \hat{\varphi}\cdot\hat{\varrho}_+\mathrm{e}^{-\mathrm{j}\varphi}A_{\nu+}^{(i)}\mathcal{Z}_{\nu-1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi} + \hat{\varphi}\cdot\hat{\varrho}_-\mathrm{e}^{\mathrm{j}\varphi}A_{\nu-}^{(i)}\mathcal{Z}_{\nu+1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}, \tag{44}$$

$$\hat{\varphi}\cdot\mathbf{H}_\nu^{(i)}(\varrho,\varphi) = \hat{\varphi}\cdot\hat{\varrho}_+\mathrm{e}^{-\mathrm{j}\varphi}B_{\nu+}^{(i)}\mathcal{Z}_{\nu-1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi} + \hat{\varphi}\cdot\hat{\varrho}_-\mathrm{e}^{\mathrm{j}\varphi}B_{\nu-}^{(i)}\mathcal{Z}_{\nu+1}^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}, \tag{45}$$

where $\hat{\varphi}\cdot\hat{\varrho}_+\mathrm{e}^{-\mathrm{j}\varphi} = 2^{-1/2}\mathrm{j}$ and $\hat{\varphi}\cdot\hat{\varrho}_-\mathrm{e}^{\mathrm{j}\varphi} = -2^{-1/2}\mathrm{j}$. Projections to $\hat{z}$ of the fields in Eqs. (34) and (35) provide

$$\hat{z}\cdot\mathbf{E}_\nu^{(i)}(\varrho,\varphi) = 2^{-1/2}\mathrm{j}\frac{\kappa^{(i)}}{\beta}\left(A_{\nu+}^{(i)} - A_{\nu-}^{(i)}\right)\mathcal{Z}_\nu^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}, \tag{46}$$

$$\hat{z}\cdot\mathbf{H}_\nu^{(i)}(\varrho,\varphi) = 2^{-1/2}\mathrm{j}\frac{\kappa^{(i)}}{\beta}\left(B_{\nu+}^{(i)} - B_{\nu-}^{(i)}\right)\mathcal{Z}_\nu^{(i)}\mathrm{e}^{\mathrm{j}\nu\varphi}. \tag{47}$$

The amplitudes $B_{\nu\pm}^{(i)}$ can be replaced with $A_{\nu\pm}^{(i)}$ using Eqs. (26)–(31)

$$B_{\nu+}^{(i)} = \frac{-\mathrm{j}}{2\beta\omega\mu^{(i)}}\left[\left(2\beta^2+\kappa^{(i)2}\right)A_{\nu+}^{(i)} - \kappa^{(i)2}A_{\nu-}^{(i)}\right], \tag{48}$$

$$B_{\nu-}^{(i)} = \frac{\mathrm{j}}{2\beta\omega\mu^{(i)}}\left[\left(2\beta^2+\kappa^{(i)2}\right)A_{\nu-}^{(i)} - \kappa^{(i)2}A_{\nu+}^{(i)}\right]. \tag{49}$$

The inverse relations follow from the duality transformation which requires $A_{\nu\pm}^{(i)} \to B_{\nu\pm}^{(i)}$, $B_{\nu\pm}^{(i)} \to -A_{\nu\pm}^{(i)}$, and $\varepsilon^{(i)} \leftrightarrow \mu^{(i)}$, i.e.,

$$A_{\nu+}^{(i)} = \frac{\mathrm{j}}{2\beta\omega\varepsilon^{(i)}}\left[\left(2\beta^2+\kappa^{(i)2}\right)B_{\nu+}^{(i)} - \kappa^{(i)2}B_{\nu-}^{(i)}\right], \tag{50}$$

$$A_{\nu-}^{(i)} = \frac{-\mathrm{j}}{2\beta\omega\varepsilon^{(i)}}\left[\left(2\beta^2+\kappa^{(i)2}\right)B_{\nu-}^{(i)} - \kappa^{(i)2}B_{\nu+}^{(i)}\right], \tag{51}$$

Equations (48)–(51) show the coupling between the circularly polarized field components via the field z-component typical for the circular cylinder symmetry. For example, the electric field amplitude $A_{\nu+}^{(i)}$ with the circular polarization $\hat{\varrho}_+$ is coupled to the magnetic field amplitudes $B_{\nu+}^{(i)}$ and $B_{\nu-}^{(i)}$ with both $\hat{\varrho}_+$ and $\hat{\varrho}_-$.

Expressed in a matrix form, the boundary conditions become

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ \mathcal{Z}_{\nu-1}^{(1)} & -\mathcal{Z}_{\nu+1}^{(1)} & -\mathcal{Z}_{\nu-1}^{(2)} & \mathcal{Z}_{\nu+1}^{(2)} \\ \kappa^{(1)}\mathcal{Z}_\nu^{(1)} & -\kappa^{(1)}\mathcal{Z}_\nu^{(1)} & -\kappa^{(2)}\mathcal{Z}_\nu^{(2)} & \kappa^{(2)}\mathcal{Z}_\nu^{(2)} \\ \frac{\kappa^{(1)}}{\mu^{(1)}}\mathcal{Z}_\nu^{(1)} & \frac{\kappa^{(1)}}{\mu^{(1)}}\mathcal{Z}_\nu^{(1)} & -\frac{\kappa^{(2)}}{\mu^{(2)}}\mathcal{Z}_\nu^{(2)} & -\frac{\kappa^{(2)}}{\mu^{(2)}}\mathcal{Z}_\nu^{(2)} \end{pmatrix}\begin{pmatrix} A_{\nu+}^{(1)} \\ A_{\nu-}^{(1)} \\ A_{\nu+}^{(2)} \\ A_{\nu}^{(2)} \end{pmatrix} = 0, \tag{52}$$

where in the first row (continuity of H_φ)

$$\begin{aligned}
a_{11} &= \frac{1}{\mu^{(1)}}\left[-\left(2\beta^2+\kappa^{(1)2}\right)\mathcal{Z}^{(1)}_{\nu-1}+\kappa^{(1)2}\mathcal{Z}^{(1)}_{\nu+1}\right],\\
a_{12} &= \frac{1}{\mu^{(1)}}\left[\kappa^{(1)2}\mathcal{Z}^{(1)}_{\nu-1}-\left(2\beta^2+\kappa^{(1)2}\right)\mathcal{Z}^{(1)}_{\nu+1}\right],\\
a_{13} &= \frac{1}{\mu^{(2)}}\left[\left(2\beta^2+\kappa^{(2)2}\right)\mathcal{Z}^{(2)}_{\nu-1}-\kappa^{(2)2}\mathcal{Z}^{(2)}_{\nu+1}\right],\\
a_{14} &= \frac{1}{\mu^{(2)}}\left[-\kappa^{(2)2}\mathcal{Z}^{(2)}_{\nu-1}+\left(2\beta^2+\kappa^{(2)2}\right)\mathcal{Z}^{(2)}_{\nu+1}\right].
\end{aligned}$$

The second, third and fourth rows express the continuity of E_φ, E_z and H_z, respectively.

4.2. Characteristic equation

The condition of zero determinant of the 4×4 matrix in Eq. (52) leads to the characteristic equation (the guidance condition) for β [5]

$$\begin{aligned}
&\beta^2\left(\kappa^{(1)2}-\kappa^{(2)2}\right)^2\frac{\nu^2}{a^2}\\
&= \omega^2\kappa^{(1)4}\kappa^{(2)4}\left[\frac{\mu^{(1)}}{\kappa^{(1)}}\frac{\mathcal{Z}^{(1)'}_\nu\left(\kappa^{(1)}a\right)}{\mathcal{Z}^{(1)}_\nu\left(\kappa^{(1)}a\right)}-\frac{\mu^{(2)}}{\kappa^{(2)}}\frac{\mathcal{Z}^{(2)'}_\nu\left(\kappa^{(2)}a\right)}{\mathcal{Z}^{(2)}_\nu\left(\kappa^{(2)}a\right)}\right]\left[\frac{\varepsilon^{(1)}}{\kappa^{(1)}}\frac{\mathcal{Z}^{(1)'}_\nu\left(\kappa^{(1)}a\right)}{\mathcal{Z}^{(1)}_\nu\left(\kappa^{(1)}a\right)}-\frac{\varepsilon^{(2)}}{\kappa^{(2)}}\frac{\mathcal{Z}^{(2)'}_\nu\left(\kappa^{(2)}a\right)}{\mathcal{Z}^{(2)}_\nu\left(\kappa^{(2)}a\right)}\right]. \quad (53)
\end{aligned}$$

Here $\kappa^{(i)2}=\omega^2\varepsilon^{(i)}\mu^{(i)}-\beta^2$ $(i=1,2)$ and $\kappa^{(1)2}-\kappa^{(2)2}=\omega^2(\varepsilon^{(1)}\mu^{(1)}-\varepsilon^{(2)}\mu^{(2)})$. The derivatives at the interface $\varrho=a$, i.e., $[\frac{\mathrm{d}\mathcal{Z}^{(i)}_\nu(\kappa^{(i)}\varrho)}{\mathrm{d}(\kappa^{(i)}\varrho)}]_{\varrho=a}$, are denoted by $\mathcal{Z}^{(i)'}_\nu(\kappa^{(i)}a)$. The use has been made of the recursion relations for cylindric functions [1]

$$\begin{aligned}
\mathcal{Z}^{(i)}_{\nu+1}\left(\kappa^{(i)}\varrho\right) &= \frac{\nu}{\kappa^{(i)}a}\mathcal{Z}^{(i)}_\nu\left(\kappa^{(i)}\varrho\right)-\mathcal{Z}^{(i)'}_\nu\left(\kappa^{(i)}\varrho\right),\\
\mathcal{Z}^{(i)}_{\nu-1}\left(\kappa^{(i)}\varrho\right) &= \frac{\nu}{\kappa^{(i)}a}\mathcal{Z}^{(i)}_\nu\left(\kappa^{(i)}\varrho\right)+\mathcal{Z}^{(i)'}_\nu\left(\kappa^{(i)}\varrho\right).
\end{aligned}$$

4.3. Amplitude Ratios

The relations among $A^{(i)}_{\nu\pm}$ follow from the equation system Eq. (52). From Eq. (52) with the first row removed, the amplitudes expressed in terms of $A^{(1)}_{\nu+}$ become

$$\frac{A^{(1)}_{\nu-}}{A^{(1)}_{\nu+}} = \frac{\mu^{(1)}\frac{\nu}{a}\frac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}-\left(\frac{\mu^{(1)}}{\kappa^{(1)}}\frac{\mathcal{Z}^{(1)'}_\nu}{\mathcal{Z}^{(1)}_\nu}-\frac{\mu^{(2)}}{\kappa^{(2)}}\frac{\mathcal{Z}^{(2)'}_\nu}{\mathcal{Z}^{(2)}_\nu}\right)}{\mu^{(1)}\frac{\nu}{a}\frac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}+\left(\frac{\mu^{(1)}}{\kappa^{(1)}}\frac{\mathcal{Z}^{(1)'}_\nu}{\mathcal{Z}^{(1)}_\nu}-\frac{\mu^{(2)}}{\kappa^{(2)}}\frac{\mathcal{Z}^{(2)'}_\nu}{\mathcal{Z}^{(2)}_\nu}\right)}, \quad (54)$$

$$\frac{A^{(2)}_{\nu+}}{A^{(1)}_{\nu+}} = \frac{\kappa^{(1)}\mathcal{Z}^{(1)}_\nu}{\kappa^{(2)}\mathcal{Z}^{(2)}_\nu}\frac{\mu^{(2)}\frac{\nu}{a}\frac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}+\left(\frac{\mu^{(1)}}{\kappa^{(1)}}\frac{\mathcal{Z}^{(1)'}_\nu}{\mathcal{Z}^{(1)}_\nu}-\frac{\mu^{(2)}}{\kappa^{(2)}}\frac{\mathcal{Z}^{(2)'}_\nu}{\mathcal{Z}^{(2)}_\nu}\right)}{\mu^{(1)}\frac{\nu}{a}\frac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}+\left(\frac{\mu^{(1)}}{\kappa^{(1)}}\frac{\mathcal{Z}^{(1)'}_\nu}{\mathcal{Z}^{(1)}_\nu}-\frac{\mu^{(2)}}{\kappa^{(2)}}\frac{\mathcal{Z}^{(2)'}_\nu}{\mathcal{Z}^{(2)}_\nu}\right)}, \quad (55)$$

$$\frac{A^{(2)}_{\nu-}}{A^{(1)}_{\nu+}} = \frac{\kappa^{(1)}\mathcal{Z}^{(1)}_\nu}{\kappa^{(2)}\mathcal{Z}^{(2)}_\nu}\frac{\mu^{(2)}\frac{\nu}{a}\frac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}-\left(\frac{\mu^{(1)}}{\kappa^{(1)}}\frac{\mathcal{Z}^{(1)'}_\nu}{\mathcal{Z}^{(1)}_\nu}-\frac{\mu^{(2)}}{\kappa^{(2)}}\frac{\mathcal{Z}^{(2)'}_\nu}{\mathcal{Z}^{(2)}_\nu}\right)}{\mu^{(1)}\frac{\nu}{a}\frac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}+\left(\frac{\mu^{(1)}}{\kappa^{(1)}}\frac{\mathcal{Z}^{(1)'}_\nu}{\mathcal{Z}^{(1)}_\nu}-\frac{\mu^{(2)}}{\kappa^{(2)}}\frac{\mathcal{Z}^{(2)'}_\nu}{\mathcal{Z}^{(2)}_\nu}\right)}. \quad (56)$$

Equivalently, these results can be deduced from Eq. (52), now with the second row removed

$$\frac{A_{\nu-}^{(1)}}{A_{\nu+}^{(1)}} = \frac{\dfrac{\beta^2}{\mu^{(1)}}\dfrac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}\dfrac{\nu}{a} - \omega^2\left(\dfrac{\varepsilon^{(1)}}{\kappa^{(1)}}\dfrac{\mathcal{Z}_\nu^{(1)\prime}}{\mathcal{Z}_\nu^{(1)}} - \dfrac{\varepsilon^{(2)}}{\kappa^{(2)}}\dfrac{\mathcal{Z}_\nu^{(2)\prime}}{\mathcal{Z}_\nu^{(2)}}\right)}{-\dfrac{\beta^2}{\mu^{(1)}}\dfrac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}\dfrac{\nu}{a} - \omega^2\left(\dfrac{\varepsilon^{(1)}}{\kappa^{(1)}}\dfrac{\mathcal{Z}_\nu^{(1)\prime}}{\mathcal{Z}_\nu^{(1)}} - \dfrac{\varepsilon^{(2)}}{\kappa^{(2)}}\dfrac{\mathcal{Z}_\nu^{(2)\prime}}{\mathcal{Z}_\nu^{(2)}}\right)}, \tag{57}$$

$$\frac{A_{\nu+}^{(2)}}{A_{\nu+}^{(1)}} = \frac{\dfrac{\kappa^{(1)}\mathcal{Z}_\nu^{(1)}}{\mu^{(1)}}}{\dfrac{\kappa^{(2)}\mathcal{Z}_\nu^{(2)}}{\mu^{(2)}}}\frac{\dfrac{\beta^2}{\mu^{(2)}}\dfrac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}\dfrac{\nu}{a} + \omega^2\left(\dfrac{\varepsilon^{(1)}}{\kappa^{(1)}}\dfrac{\mathcal{Z}_\nu^{(1)\prime}}{\mathcal{Z}_\nu^{(1)}} - \dfrac{\varepsilon^{(2)}}{\kappa^{(2)}}\dfrac{\mathcal{Z}_\nu^{(2)\prime}}{\mathcal{Z}_\nu^{(2)}}\right)}{\dfrac{\beta^2}{\mu^{(1)}}\dfrac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}\dfrac{\nu}{a} + \omega^2\left(\dfrac{\varepsilon^{(1)}}{\kappa^{(1)}}\dfrac{\mathcal{Z}_\nu^{(1)\prime}}{\mathcal{Z}_\nu^{(1)}} - \dfrac{\varepsilon^{(2)}}{\kappa^{(2)}}\dfrac{\mathcal{Z}_\nu^{(2)\prime}}{\mathcal{Z}_\nu^{(2)}}\right)}, \tag{58}$$

$$\frac{A_{\nu-}^{(2)}}{A_{\nu+}^{(1)}} = \frac{\dfrac{\kappa^{(1)}\mathcal{Z}_\nu^{(1)}}{\mu^{(1)}}}{\dfrac{\kappa^{(2)}\mathcal{Z}_\nu^{(2)}}{\mu^{(2)}}}\frac{\dfrac{\beta^2}{\mu^{(2)}}\dfrac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}\dfrac{\nu}{a} - \omega^2\left(\dfrac{\varepsilon^{(1)}}{\kappa^{(1)}}\dfrac{\mathcal{Z}_\nu^{(1)\prime}}{\mathcal{Z}_\nu^{(1)}} - \dfrac{\varepsilon^{(2)}}{\kappa^{(2)}}\dfrac{\mathcal{Z}_\nu^{(2)\prime}}{\mathcal{Z}_\nu^{(2)}}\right)}{-\dfrac{\beta^2}{\mu^{(1)}}\dfrac{\kappa^{(1)2}-\kappa^{(2)2}}{\kappa^{(1)2}\kappa^{(2)2}}\dfrac{\nu}{a} - \omega^2\left(\dfrac{\varepsilon^{(1)}}{\kappa^{(1)}}\dfrac{\mathcal{Z}_\nu^{(1)\prime}}{\mathcal{Z}_\nu^{(1)}} - \dfrac{\varepsilon^{(2)}}{\kappa^{(2)}}\dfrac{\mathcal{Z}_\nu^{(2)\prime}}{\mathcal{Z}_\nu^{(2)}}\right)}. \tag{59}$$

The corresponding ratios in Eqs. (54)–(56) and (57)–(59) are related by Eq. (53).

5. DISCUSSION

The use of transverse circular representation in circular cylinder coordinate system enables simplified solutions to the vector Helmholtz partial differential equation of electromagnetics. After separation, the equation for electric (magnetic) field splits into a set of three ordinary differential, i.e., Bessel equations for two opposite transverse circular polarizations, $\hat{\varrho}_+$ and $\hat{\varrho}_-$, and the axial $\hat{z}$ component. The approach is suitable for solving the problem of cylindrical waveguides and cavities starting from transverse field components. It is illustrated on a circular cylindrical dielectric waveguide. Its monomode operation is deduced from the solutions for $\nu = 0$ and $\nu = \pm 1$ given in Sections 3 and 4 and defined by the V-number range,

$$V = \omega\left(\mu^{(1)}\varepsilon^{(1)} - \mu^{(2)}\varepsilon^{(2)}\right)^{1/2} a \leq 2.405.$$

Table 1 provides the solutions to Eq. (40) at a fixed $V = 2.4028$ for several $\varepsilon^{(1)}/\varepsilon^{(2)}$ ratios and propagation constants in the cladding $k^{(2)} = \omega(\mu^{(2)}\varepsilon^{(2)})^{1/2}$ corresponding to the case $\nu = 1$, with restriction to $\mu^{(1)}/\mu^{(2)} = 1$. The model may be applied, e.g., to the evaluation of the trends in an optical fiber operating at the vacuum wavelength, $\lambda_{\text{vac}} = 1.55\,\mu\text{m}$ with the core radius $1\,\mu\text{m} < a < 36\,\mu\text{m}$. At the lowest $\varepsilon^{(1)}/\varepsilon^{(2)}$ ($\varepsilon^{(1)}/\varepsilon^{(2)} \approx 1$), the component propagating with $\hat{\varrho}_+$ dominates. This justifies the weak guiding approximation, i.e., $A_{+1-}^{(1)}/A_{+1+}^{(1)} \approx 0$. The same situation takes place for $\nu = -1$ and $\hat{\varrho}_-$

Table 1. Dependence of waveguide characteristics on permittivity ratio.

$\varepsilon^{(1)}/\varepsilon^{(2)}$	1.000143	1.00143	1.0143	1.1	1.143
$k^{(2)}a$	200.932	63.5615	20.0677	7.59855	6.34597
βa	200.94	63.5856	20.1437	7.79353	6.5762
κa	1.64605	1.64631	1.64885	1.66507	1.67288
γa	1.75042	1.75021	1.74782	1.73237	1.72484
$A_z^{(1)}/A_{+1+}^{(1)}$	0.00579243	0.0183079j	0.0578799j	0.151072j	0.179877j
$A_{+1-}^{(1)}/A_{+1+}^{(1)}$	−0.0000229546	−0.000236753	−0.00231887	−0.0153239	−0.0214017

where the ratio $A^{(1)}_{-1+}/A^{(1)}_{-1-} \approx 0$. The two solutions can be combined to construct the LP_{01} orthogonal modes linearly polarized parallel to $\hat{x}$ and $\hat{y}$ [3].

The coupling between fields with $\hat{\varrho}_+$ and $\hat{\varrho}_-$ presents interest in the development of approximations, in the design of nonreciprocal waveguides, in optical angular momentum studies, etc. [3, 8]. The approach may be applied to rectangular cylindrical waveguides in a way developed by Goell [6] and to the media with circular eigen polarizations displaying magnetooptical Faraday effect or optical activity [7, 12].

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