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An Optimized PLRC-FDTD Model of Wave Propagation in Anisotropic Magnetized Plasma

Jinchao Ding, Zhiqin Zhao*, Yue Yang, Yulang Liu, and Zai-Ping Nie

Abstract—Numerical dispersion is the main error source of the finite-difference time-domain (FDTD) method. In this paper, an optimized piecewise linear recursive convolution (PLRC) FDTD method with low numerical dispersion is presented first time for electromagnetic-wave propagation in anisotropic magnetized plasma. An optimized difference item which can achieve better approximation to the partial differential operator from transform domain is induced in this algorithm which decreases numerical dispersion. The item can be regarded as adding a correcting coefficient to conventional central difference format. And it is easy for programming and implementation. Numerical examples of electromagnetic pulse wave propagating in plasma demonstrate that the proposed optimized PLRC-FDTD method can not only reduce the numerical dispersion, but also improve precision, saving computational memory and computational time compared with the conventional PLRC-FDTD method. Same accuracy can be achieved when the spatial mesh size for the optimized PLRC-FDTD method is 2 times coarser as that in the conventional PLRC-FDTD method, corresponding to the computation time consumed in the optimized method is only 1/2 as that in the conventional one.

1. INTRODUCTION

The finite-difference time-domain (FDTD) method [1] has been widely used in solving electromagnetic problem for decades. Many methods with different characteristics have been derived [2–4]. Numerical dispersion is an important character of FDTD method. The numerical dispersion relations of some typical FDTD methods are compared in paper [5]. The advantage and disadvantage of each algorithm are also briefly discussed. The accuracy of an algorithm is closely related with numerical dispersion.

In the past two decades, numerous FDTD methods have been investigated in dispersive medium. The recursive convolution (RC) FDTD method [6] which has high numerical dispersion in most conditions is easy to be implemented and saves computational memory compared with current density J and electric field E (JE) convolution [7] method and auxiliary differential equation (ADE) [8] methods. The piecewise linear RC [9] method which has the advantage of high efficiency and small memory of RC-FDTD method is proposed to reduce the numerical dispersion of RC-FDTD method assuming that the field value of each grid has linear variation. The accuracy of the aforementioned efforts for expanding the FDTD method to frequency dependent materials is controlled by the choice of the second-order precision central difference format to achieve difference instead of differential. A fourth-order precision central difference format in time and space FDTD approach for propagation in collisionless plasma has been presented in [10] which can reduce numerical dispersion effectively. But the method presented in [10] is restricted to lossless dispersive media despite the accuracy and memory savings achieved. Then, a novel higher-order method for modeling lossy media and dispersive media has been presented in [11–13]. Some methods applied in anisotropic magnetized plasma have later been extended [14, 15]. But all the

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* Corresponding author: Zhiqin Zhao (zqzhao@uestc.edu.cn).
The authors are with the Department of Microwave Engineering, School of Electronic Engineering, University of Electronic Science and Technology of China (UESTC), Chengdu 611731, China.

algorithms applying in dispersion medium adopt central difference format to achieve difference instead of differential; therefore, the inevitable large numerical dispersion error is introduced.

An optimized difference item which is from transform domain to achieve the approximation of partial differential operator is presented in [16, 17]. It has been demonstrated that the optimized difference item can effectively reduce the numerical dispersion compared with conventional central difference item which is meaningful for model of wave propagation in dispersive medium using FDTD method [18].

In this paper, an optimized PLRC-FDTD is presented to reduce the numerical dispersion which puts the optimized difference item into conventional PLRC-FDTD method. The optimized PLRC-FDTD formulation is easy to complete and program for modeling electromagnetic wave propagation in anisotropic magnetized plasma. The reflection and transmission coefficients of a magnetized plasma slab are simulated and compared with those of analytical results and the results by using conventional PLRC-FDTD.

The reminder of the paper is organized as follows. The theory of the optimized difference scheme and optimized PLRC-FDTD method formulation is described in Section 2. Some numerical simulations are introduced in Section 3 to demonstrate the effectiveness of the proposed method. Conclusions are drawn in the final section.

2. METHODOLOGY

The iterative equations of electric fields for conventional PLRC-FDTD anisotropic magnetized plasma model are given by [15]. The curl in the space of magnetic fields involved in conventional PLRC-FDTD method for magnetized plasma can be expressed as

$$(\nabla \times H)_x^{n+1/2} = \frac{\partial H_z^{n+1/2}}{\partial y} - \frac{\partial H_y^{n+1/2}}{\partial z}, \tag{1}$$

$$(\nabla \times H)_y^{n+1/2} = \frac{\partial H_x^{n+1/2}}{\partial z} - \frac{\partial H_z^{n+1/2}}{\partial x}, \tag{2}$$

where H is the magnetic field.

The implementation of the difference items in Eqs. (1) and (2) affects the precision. An optimized central difference format was proposed to reduce numerical dispersion in [16, 17], where the second-order precision central difference format with the general form at point i in space is expressed as

$$\left.\frac{\partial f(t; x, y, z)}{\partial u}\right|_i \approx \frac{1}{\Delta u} d^u \left[f|_{u=(i+\frac{1}{2})\Delta u} - f|_{u=(i-\frac{1}{2})\Delta u} \right], \tag{3}$$

where Δu is the spatial step, $u = x, y, z$ for 3-D problems. d^u is an optimal coefficient. For conventional central difference format, the coefficient d^u is set to be 1.

In the following, the idea of using optimal coefficient d^u for PLRC-FDTD in anisotropic magnetized plasma model will be illustrated.

The Fourier transform of Eq. (3) can be written as

$$k_u \approx \frac{2}{\Delta u} d^u \sin\left(\frac{1}{2} k_u \Delta u\right), \tag{4}$$

where k_u is the theoretical wave number.

Let

$$k_{ue} = \frac{2}{\Delta u} d^u \sin\left(\frac{1}{2} k_u \Delta u\right). \tag{5}$$

k_{ue} is an approximation of k_u. And the error of this approximation is

$$e_u = \int_0^{k_{\max}\Delta u} \int_0^{\pi} \int_0^{2\pi} \{k_u \Delta u - k_{ue} \Delta u\}^2 (k\Delta u)^2 \sin\theta d(k\Delta u) d\theta d\phi, \tag{6}$$

where $k = \left(k_x^2 + k_y^2 + k_z^2\right)^{\frac{1}{2}}$, $k_x = k \sin\theta \cos\varphi$, $k_y = k \sin\theta \sin\varphi$, $k_z = k \cos\theta$, θ and φ are propagation angle, $k_{\max}$ is maximum value of k.

Setting the normalized wave number $\bar{k} = k\Delta u$, $\bar{k}_u = k_u \Delta u$ and Eq. (6) can be rewritten as

$$e_u = \int_0^{\bar{k}_{\max}} \int_0^{\pi} \int_0^{2\pi} \left\{ \bar{k}_u - 2d^u \sin\left(\frac{1}{2}\bar{k}_u\right) \right\}^2 (\bar{k})^2 \sin\theta d\bar{k} d\theta d\phi. \tag{7}$$

For the circumstance of one dimension, no component of the wave number k is involved, corresponding to $\bar{k} = \bar{k}_u$. Thus, Eq. (7) can be simplified as

$$e_u = \int_0^{\bar{k}_{\max}} \left\{ \bar{k} - 2d^u \sin\left(\frac{1}{2}\bar{k}\right) \right\}^2 d\bar{k}. \tag{8}$$

Taking derivation of the approximation error to determine the coefficient d^u,

$$\frac{\partial e_u}{\partial d^u} = 0. \tag{9}$$

The coefficient d^u can be induced in the spatial partial differential operator in Eqs. (1) and (2). Hence, Eqs. (1) and (2) can be modified as

$$d^x \cdot (\nabla \times H)_x^{n+1/2} = d^x \cdot \frac{\partial H_z^{n+1/2}}{\partial y} - d^x \cdot \frac{\partial H_y^{n+1/2}}{\partial z}, \tag{10}$$

$$d^y \cdot (\nabla \times H)_y^{n+1/2} = d^y \cdot \frac{\partial H_x^{n+1/2}}{\partial z} - d^y \cdot \frac{\partial H_z^{n+1/2}}{\partial x}. \tag{11}$$

Substituting Eqs. (10) and (11) into Eqs. (1) and (2) yield

$$\begin{aligned} E_x^{n+1} = & \frac{\left(1-\varsigma_1^0\right)\left(1+\chi_1^0-\varsigma_1^0\right)-\left(\chi_2^0-\varsigma_2^0\right)\varsigma_2^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} E_x^n + \frac{\left(1+\chi_1^0-\varsigma_1^0\right)\varsigma_2^0+\left(\chi_2^0-\varsigma_2^0\right)\left(1-\varsigma_1^0\right)}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} E_y^n \\ & + \frac{\chi_2^0-\varsigma_2^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} \psi_2^n + \frac{1+\chi_1^0-\varsigma_1^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} \psi_1^n \\ & + \frac{\chi_2^0-\varsigma_2^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} \left(d^y \cdot \frac{\Delta t}{\varepsilon_0} (\nabla \times H)_y^{n+1/2} \right) \\ & + \frac{1+\chi_1^0-\varsigma_1^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} \left(d^x \cdot \frac{\Delta t}{\varepsilon_0} (\nabla \times H)_x^{n+1/2} \right) \end{aligned} \tag{12}$$

$$\begin{aligned} E_y^{n+1} = & \frac{\left(1-\varsigma_1^0\right)\left(1+\chi_1^0-\varsigma_1^0\right)-\left(\chi_2^0-\varsigma_2^0\right)\varsigma_2^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} E_y^n - \frac{\left(1+\chi_1^0-\varsigma_1^0\right)\varsigma_2^0+\left(\chi_2^0-\varsigma_2^0\right)\left(1-\varsigma_1^0\right)}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} E_x^n \\ & - \frac{\chi_2^0-\varsigma_2^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} \psi_1^n + \frac{1+\chi_1^0-\varsigma_1^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} \psi_2^n \\ & - \frac{\chi_2^0-\varsigma_2^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} \left(d^x \cdot \frac{\Delta t}{\varepsilon_0} (\nabla \times H)_x^{n+1/2} \right) \\ & + \frac{1+\chi_1^0-\varsigma_1^0}{\left(1+\chi_1^0-\varsigma_1^0\right)^2+\left(\chi_2^0-\varsigma_2^0\right)^2} \left(d^y \cdot \frac{\Delta t}{\varepsilon_0} (\nabla \times H)_y^{n+1/2} \right). \end{aligned} \tag{13}$$

where χ_1^0, ς_1^0, χ_2^0, ς_2^0, ψ_1^n, ψ_2^n are intermediate variables which have the same definitions as those in [15], and E is the electric field.

Equations (12) and (13) include the optimal coefficient d^u, which relates normalized wavenumber. And this is the main difference from conventional PLRC-FDTD. Thus we call the PLRC-FDTD with this scheme, i.e., Eqs. (12) and (13), optimal PLRC-FDTD.

In order to demonstrate the effectiveness of optimized difference item for wave propagation in magnetized plasma, some numerical simulations will be conducted in the following section.

3. NUMERICAL SIMULATIONS

The maximum normalized phase velocity error defined in [19] is presented to analyze numerical dispersion error of the optimized difference format and conventional central difference format. Eq. (14) gives the definition of maximum normalized phase velocity error

$$V_{\max} = \max\left|\frac{v(\theta,\varphi)}{c} - 1\right|, \tag{14}$$

where $v(\theta, \varphi)$ is the phase velocity which can be derived from the numerical dispersion relation of FDTD method, and c is the speed of light in free space.

The formula of $v(\theta, \varphi)$ can be obtained by using

$$\left(\frac{v(\theta,\varphi)}{c}\right)^2 = \left(d^x \cdot N \cdot \sin\left(\frac{\pi \cdot \sin\theta\cos\varphi}{N}\right)/\pi\right)^2 + \left(d^y \cdot N \cdot \sin\left(\frac{\pi \cdot \sin\theta\sin\varphi}{N}\right)/\pi\right)^2 + \left(d^z \cdot N \cdot \sin\left(\frac{\pi \cdot \cos\theta}{N}\right)/\pi\right)^2, \tag{15}$$

where N is the grid sampling density which equals $\lambda/\Delta u$. d^u is set to be 1 for conventional central difference format.

From $N = \lambda/\Delta u$, we obtain

$$\bar{k}_{\max} = k_{\max}\Delta u = 2\pi/\lambda \cdot \Delta u = 2\pi/N. \tag{16}$$

Hence, for a given N, one can have a maximum normalized wavenumber $\bar{k}_{\max}$, then the coefficient d^u can be obtained by using Eqs. (8) and (9).

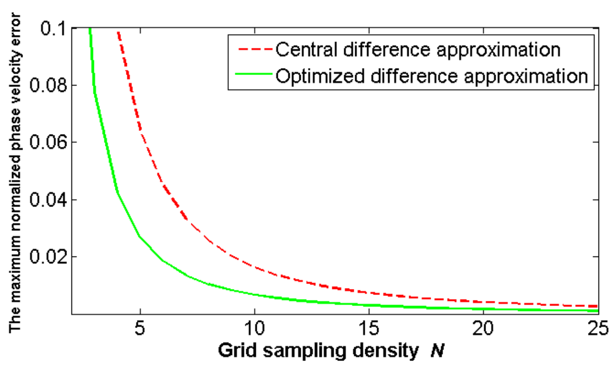

Figure 1. Comparison on the maximum normalized phase velocity error between central difference approximation and optimized difference approximation.

Figure 1 gives the comparison of maximum normalized phase velocity error $V_{\max}$ versus different Ns by using the central difference format and optimized difference format. The central difference approximation and optimized difference approximation are used for spatial discretization in conventional FDTD to verify the superiority of the optimized difference item. The dashed line is the error for the conventional central difference format. The solid line is the error for the optimized difference format. It can be seen that the error for conventional central difference format is much bigger than that of optimized difference format. When N is bigger than 25, both errors are very close. For smaller N, which implies coarser grid density, the optimized difference format takes the advantage.

In order to demonstrate the accuracy of the optimized PLRC-FDTD, one-dimensional problem of wave propagation in plasma slab is investigated. Reflection coefficients of the right circularly polarized (RCP) wave and left circularly polarized (LCP) wave through a magnetized plasma slab with a thickness of 1.2 mare simulated. The computational domain is 3.6 m long and the plasma slab is setat the region [1.2, 2.4] m. The direction of incident wave propagation is parallel to the external static magnetic field and normally incident on the magnetized plasma. The parameters of the magnetized plasma parameters are

$$\begin{aligned} \omega_p &= 2\pi \times 5 \times 10^9 \,\mathrm{rad/s}, \\ \omega_b &= 3 \times 10^{10} \,\mathrm{rad/s}, \quad , \\ v &= 2 \times 10^9 \,\mathrm{Hz}, \end{aligned} \tag{17}$$

where v is the electron collision frequency, ω_p the plasma frequency, and ω_b the electron cyclotron frequency.

A Gaussian-derivative pulsed plane wave is used, which is expressed as

$$g(t) = (t - 5\tau) \exp\left[-\frac{(t-5\tau)^2}{\tau^2}\right], \tag{18}$$

where $\tau = 15\Delta t$, Δt is the time step. The upper limit frequency of the incident wave is 20 GHz, corresponding to minimum wavelength $\lambda_{\min}$ of 0.015 m. The spatial discretemesh size δ used in the conventional PLRC-FDTD is 0.0015 m and 0.00075 m, corresponding to $N = \lambda_{\min}/\delta = 10, 20$, respectively. The time step $\Delta t = \delta/2c$s. The simulation results were obtained by running 6000 time steps. Reflection coefficients were obtained by using the time history of the electric fields. The boundaries are terminated with ten cells of perfectly matched layers (PML). For the optimized PLRC-FDTD, when δ is 0.0015 m, the corresponding N is 10.

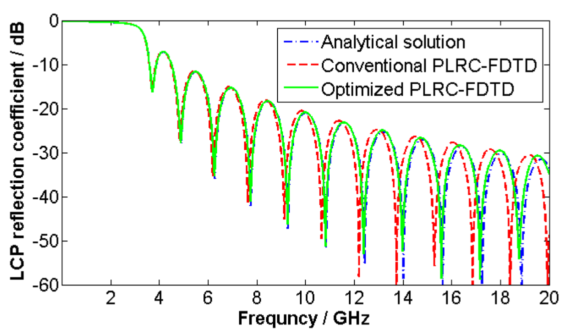

Figure 2. Reflection coefficients versus frequency for LCP wave by using PLRC-FDTD and optimized PLRC-FDTD ($\delta = 0.0015$ m).

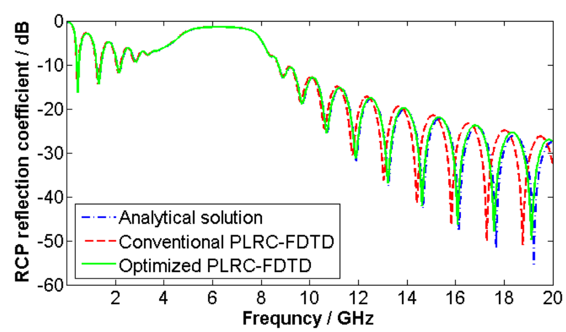

Figure 3. Reflection coefficients versus frequency for RCP wave by using PLRC-FDTD and optimized PLRC-FDTD (δ= 0.0015 m).

Figures 2 and 3 show the reflection coefficients versus frequencies of LCP and RCP waves by using the conventional PLRC-FDTD and the optimized PLRC-FDTD when δ= 0.0015 m. The dash-dotted lines are the analytical results which are used as references. The dashed lines are the results by using conventional PLRC-FDTD. The solid lines are for proposed optimized PLRC-FDTD. As displayed in the figures, when the frequency is below 10 GHz, the results of both methods have slight difference and both are very close to the analytical results. However, once the frequency goes higher, the result by using the optimized PLRC-FDTD is still almost overlapped with the analytical result. However, the difference between the results by using the conventional and the analytical results is getting bigger and bigger.

Figures 4 and 5 show the comparison of reflection coefficients versus frequency by using the optimized PLRC-FDTD method and the conventional PLRC-FDTD with different grid densities. For the optimized method, δ= 0.0015 m versus δ= 0.00075 m for the conventional one, which means that the grid size for the optimized method is two time coarser as that of the conventional method. The dash-dotted lines are the analytical results which are used as references. The dashed lines are the results by using conventional PLRC-FDTD. The solid lines are for the proposed optimized PLRC-FDTD. Compared with the results in Figure 4 and Figure 5, the agreements between the analytical results and the results by using the conventional PLRC-FDTD are getting better. However, the results by using

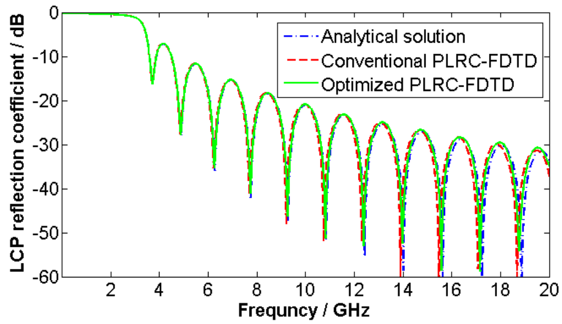

Figure 4. Comparison of reflection coefficients versus frequency for LCP wave by using the conventional PLRC-FDTD with $\delta = 0.00075\,\mathrm{m}$ and the optimized PLRC-FDTDwith $\delta = 0.0015\,\mathrm{m}$.

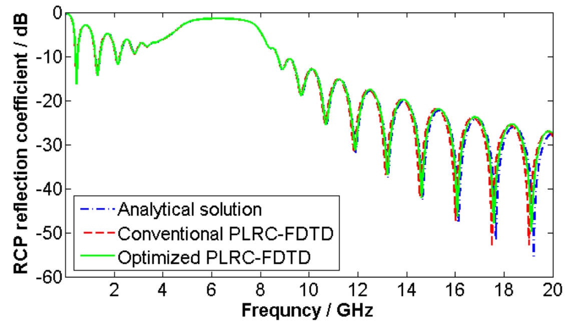

Figure 5. Comparison of reflection coefficients versus frequency for RCP wave by using the conventional PLRC-FDTD with $\delta = 0.00075\,\mathrm{m}$ and the optimized PLRC-FDTDwith $\delta = 0.0015\,\mathrm{m}$.

the optimized PLRC-FDTD are still slightly better. This means that the optimized PLRC-FDTD can achieve almost the same accuracy but with only half of the grids in the simulation.

Due to coarser grid density, less computational load is anticipated. Table 1 gives a comparison on the time consuming and memory occupation with same iteration steps between two methods. Both methods were carried out on a computer based on Intel(R) Core(TM) i3-4170 CPU @3.20 GHz. Obviously, the optimized PLRC-FDTD shows its advantage in computational time and memory. On account of some variables which are irrelevant with the grid size need to storage allocation, and this part of memory usage is universal. Hence, the memory improvement did not go close to half of the original. But the CPU time is more relevant with the grid size so that the CPU time was closely improved to be half of the original.

Table 1. Comparison of the time cost and memory occupation of of the two methods.

Method	δ	Steps	CPU time	Memory
PLRC-FDTD	0.00075 m	6000	15.612 s	1.35 Mb
Optimized PLRC-FDTD	0.0015 m	6000	8.348 s	0.89 Mb

4. CONCLUSIONS

In this paper, an optimization difference item is induced in PLRC-FDTD method, named as optimized PLRC-FDTD, which improves the accuracy in modelling electromagnetic-wave propagation in anisotropic magnetized plasma. The optimization difference item which can be regarded as adding a coefficient to conventional central difference format is easy for programming and implementation. Some simulation examples validate its ability in improving the simulation accuracy in a wide band. Simulation results show that the proposed method has even slightly better accuracy with only half of the grids for both LCP and RCP waves. All these results demonstrate that the proposed method has promising potential in the simulations of pulse waves propagating in magnetized plasma.

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