# A Fourier Phase Mode Approach for Chebyshev Pattern Synthesis in Circular Antenna Array

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**Abstract**—In this article, a novel phase mode analysis for a circular antenna array is discussed. This proposition experiments on the synthesis of Dolph-Chebyshev pattern for circular geometry employing directional element $1 + \cos(\phi)$. Here, for pattern synthesis a modified uniform sampling method is proposed, and for investigation of continuous current excitation in a circular array, a Fourier phase-mode approach is proposed. The synthesis process permits generation of complex weights for each element to produce the Chebyshev pattern with a desired beamwidth or Side Lobe Level (SLL). The radius is a key factor for a circular geometry and also decides the pattern synthesis, which is determined by using the phase mode concept. Also, this article contributes to the formulation of a mathematical relationship between the number of phase modes ($P$) and the number of antenna elements in the array ($N$) such as $N = 2(P - 1)$.

## 1. INTRODUCTION

Nowadays, circular arrays have many practical applications in radar, sonar, mobile and wireless communications [1–6]. For the design of circular arrays, one has to adequately choose the number of antennas in the array, their positions along the circle, the circle's radius, and the feeding currents (amplitudes and phases) of the antenna elements. The primary design objective of antenna array geometry is to determine the positions of array elements that together produce a radiation pattern towards the desired pattern as closely as possible [7]. Due to circular and symmetrical nature of the circular array, it is convenient to analyze the excitation of a circular array in terms of its Fourier components or phase modes, for many practical applications. The new approach employs a transformation technique, first proposed by Davies [8]. Here the phase modes to be considered is equivalent to elements of a linear array, thus adding all the phase modes co-physically produces a beam in particular direction, equivalent to the boresight direction of the linear array. In practice by approximating the continuous excitation function for obtaining the desired pattern by a finite number of elements, the basic amplitude and phase mode concept does not change, and also if the aperture excitation function contains harmonics up to a maximum order $m$ and total of $P$, then the antenna array must contain at least $2P$ elements in order to reproduce all the spatial harmonics [9]. Dolph-Chebyshev arrays were first introduced by Dolph in 1946 [10]. The computation and synthesis of current distributions for Dolph-Chebyshev patterns for linear arrays were further developed by others [11–13]. The synthesis of a circular array with isotropic elements are documented in [14, 15, 23–26]. Some special cases of directive elements were discussed in [16, 17]. The development in [18] introduced the phase mode method pattern synthesis, which can deal with both isotropic elements and directive elements cases on pattern synthesis for a circular array. Most circular (and cylindrical) arrays are made up of directional elements since the pattern characteristics of circular and cylindrical arrays cannot be represented in terms of the product of an element pattern and an array factor, and it is especially important to consider

the array patterns with directional elements [1, 18]. [19, 20] reported the importance of directional elements in pattern synthesis of circular arrays, by including directional elements in pattern synthesis, thus the mutual coupling between the elements is reduced, and the element pattern, as well as elevation pattern of the circular array, narrows.

The organization of this article is as follows. The mathematical formulations of a circular array with the directional element are reported in Section 2. In Section 3, the numerical analysis and simulation results are reported. Some concluding key observations are summarized in Section 4.

## 2. MATHEMATICAL FORMULATION

A far-field expression $M(\phi)$ for circular array with directional element function $E_L$, weight function magnitude $W_n$ and phase $\alpha_n$ are given as [18, 21, 22]

$$M(\phi) = \sum_{n=1}^{N} W_n E_L(\phi - \phi_n) \exp(jkr \cos(\phi - \phi_n) + j\alpha_n) \tag{1}$$

where $N$ is the number of antennas in the array, $r$ the circular array radius, $\phi$ the azimuthal plane, $\phi_n$ the $n$th element's azimuth position, $j$ the imaginary unit, and $k = \frac{2\pi}{\lambda}$ the free-space wave number, where $\lambda$ is the wavelength. In Equation (1), phase $(\phi_n)$ is referenced to the center of the circle. As shown in Figure 1, the antenna elements in the array are spaced $\phi_n$ along the circle, with each element pointing in the radial direction. Therefore, the element function cannot be brought outside summation, since it is a function of the element position. In order to accommodate directional elements into the above development, the element pattern must be transformed into a set of Fourier series coefficients, $D_p$, as given in Figure 2.
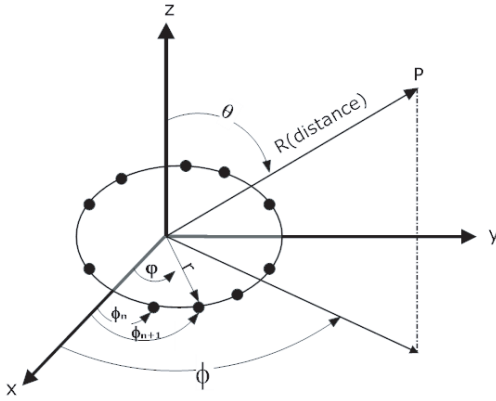


**Figure 1.** Circular antenna array.
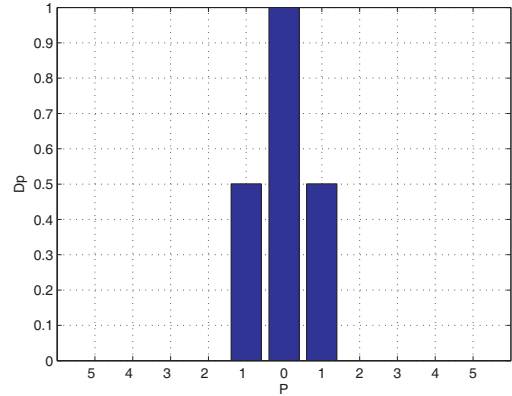


**Figure 2.** Directional element $1 + \cos(\phi)$ pattern in Fourier domain.

### 2.1. Phase Mode Analysis of Directional Elements

The far-field pattern of a circular array with the element pattern and weight function in continuous form can be expressed as [18, 25].

$$M(\phi) = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\varphi) E_L(\phi - \varphi) e^{jkr \cos(\phi - \varphi)} d\varphi \tag{2}$$

where $W(\varphi)$ and $E_L(\varphi)$ are the complex weight function and directional element function of each antenna element, respectively. Here, the notation $\phi$ designates the angle of the array pattern in the far field, centered at the center of the array. The symbol $\varphi$ designates the angle around the array and also centered at the center of the array.

Due to the circular nature of the array, the weight function is a periodic function with period $2\pi$. Thus one can expand $W(\varphi)$ into a Fourier series as follows [27]

$$W(\varphi) = \sum_{m=-\infty}^{\infty} C_m \exp(jm\varphi) \tag{3}$$

where:

$$C_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} W(\varphi)e^{-jm\varphi}d\varphi \tag{4}$$

The series expansions of the weight function coefficients $C_m$ are the phase modes of the excitation function. Thus, each $m$ in Equation (3) represents a mode number, and the phase varies $2\pi$ along the circumference of circular array for each phase mode. In order to accommodate directional elements into the above development, the element pattern must be transformed into a set of Fourier series coefficients, $D_p$, as shown in Equation (5).

$$E_L(\varphi) = \sum_{p=-\infty}^{\infty} D_p \exp(jp\varphi) \tag{5}$$

where:

$$D_p = \frac{1}{2\pi} \int_{-\pi}^{\pi} E_L(\varphi)e^{-jp\varphi}d\varphi \tag{6}$$

After simplification, the relation among weight function, element pattern and far-field pattern can be found as

$$M(\phi) = \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} C_m e^{jm\phi} D_p \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(m-p)(\phi-\varphi)} e^{jkr\cos(\phi-\varphi)}d\varphi \tag{7}$$

$$= \sum_{m=-\infty}^{\infty} \sum_{p=-\infty}^{\infty} C_m e^{jm\phi} D_p j^{(m-p)} J_{m-p}(kr)$$

where $J_{m-p}(kr)$ is the Bessel function of order $(m-p)$ and argument $kr$. On the other hand, the far-field pattern for a circular array can be expressed as

$$M(\phi) = \sum_{m=-\infty}^{\infty} A_m \exp(jm\phi) \tag{8}$$

Now by relating the excitation phase modes $C_m$ and far-field phase modes $A_m$ from Equations (7) and (8), we arrive at the following:

$$C_m = A_m \left( \sum_{p=-\infty}^{\infty} D_p j^{(m-p)} J_{m-p}(kr) \right)^{-1} \tag{9}$$

## 2.2. Selection of Radius of an Circular Array

In pattern synthesis of a circular array, the radius is a critical parameter since the near-field excitation coefficients $C_m$ are used to calculate the weight function $W(\varphi)$ which is very sensitive to small variations of $r$, and also the near-field excitation coefficients $C_m$ (phase modes) are influenced by the Bessel function $J_{m-p}(kr)$. So in this paper, a suitable radius $r$ is determined for the synthesis of the radiation pattern. First, consider the desired radiation pattern as Chebyshev pattern with constraint such as side lobe level (SLL) is $-25$ dB. For this, the mathematical far-field expression is given in Equation (8). After applying phase mode analysis, the far-field expression in terms of phase modes including the element function is given in Equation (7).

Now the radius is evaluated by considering the error between the desired radiation pattern near SLL (dB) value, i.e., $-25$ dB and that near SLL (dB) values obtained from Equation (7) for different $r$ values. For $r = 0.8555\lambda$, the difference between near SLL (dB) value of desired pattern and the
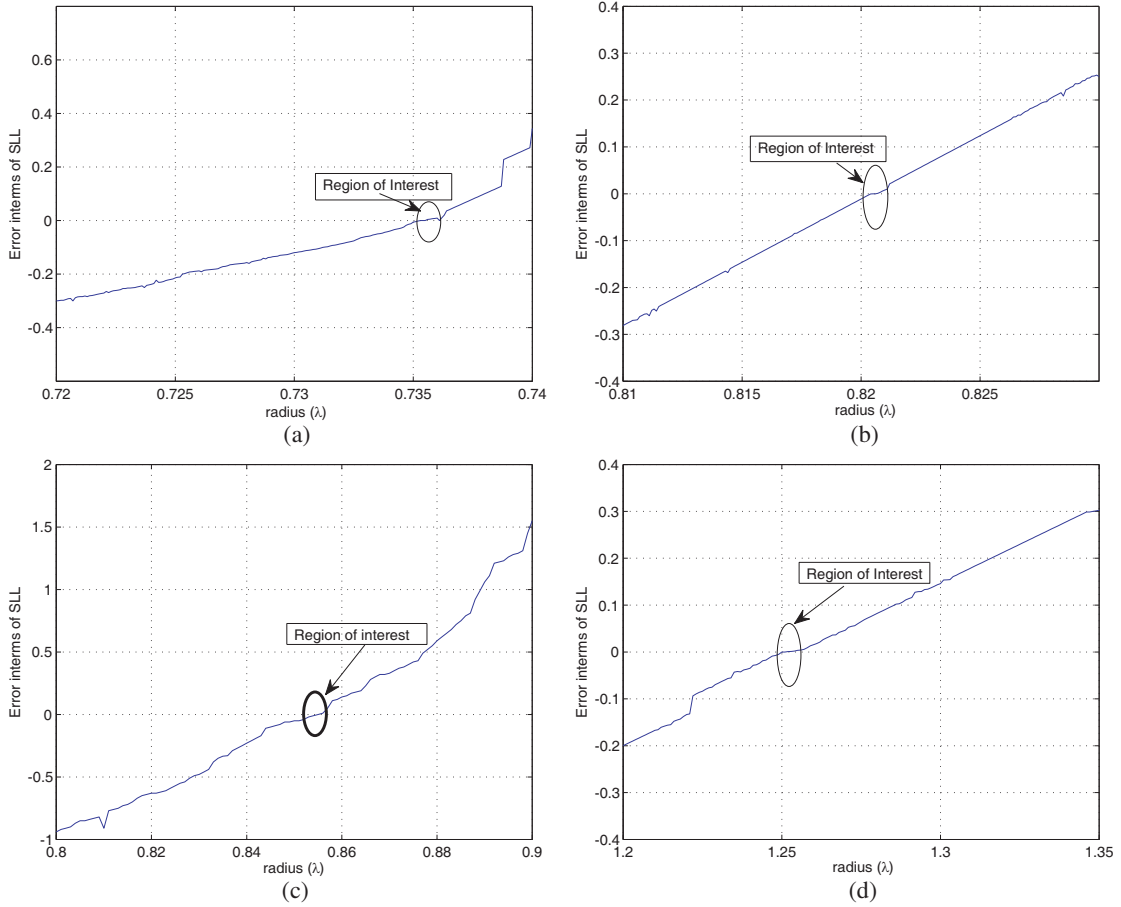
**Figure 3.** Graphical representation of evaluating radius for different phase modes. (a) 7 phase modes. (b) 8 phase modes. (c) 9 phase modes. (d) 10 phase modes.

computed pattern from Equation (7) is approximately $0\,\mathrm{dB}$. Similarly for $r = 0.88\lambda$, the difference is $0.66\,\mathrm{dB}$, and for $r = 0.82\lambda$ the difference is $-0.89\,\mathrm{dB}$. Therefore, in this way, the suitable value of radius $r = 0.8555\lambda$ is determined with the minimum error in terms of near SLL (dB) values for $P = 9$ phase modes. In a similar way for 7, 8, 10 phase modes of circular array, the radius is evaluated, and respective graphical representations are shown in Figure 3. This way of approach conceptually equates the far-field excitation coefficients $A_m$ to the near-field excitation coefficients $C_m$.

### 2.3. Proposed Methodology of Uniform Sampling Method for Pattern Synthesis

Step 1. Choose desired chebyshev pattern from synthesized linear antenna array [18].

Step 2. Apply phase mode analysis discussed in Section 2.1 for circular array with directional element find the $C_m$ and evaluate the continuous current distribution from Equation (3).

Step 3. Divide the continuous interval of azimuth plane $(-\pi$ to $\pi)$ into $-\pi : (2 * pi/N) : \pi$ for obtaining the angular element positions. where $N$ is the number of antenna elements in the circular array.

Step 4. From obtained element positions $\phi_n$, sample the continuous current distribution magnitude $|W(\phi_n)|$ and phase $\arg(\phi_n)$ respectively to obtain the element excitations.

## 3. NUMERICAL ANALYSIS AND SIMULATION RESULTS

This section surveys the proposed analysis effectively on different phase modes such as even and odd phase modes. For odd phase modes $P = 2m + 1$ where $m$ is the highest positive harmonic, for example, 7 phase modes, the highest positive mode number is $m = 3$, and for even phase modes $P = 2(m + 1)$ where the variation of the mode number will be $-(m + 1) : 1 : m$ or $m : 1 : m + 1$. The simulations are performed on the i5 processor with 4GB RAM and MATLAB R2013a.

### 3.1. Case 1: 7 Phase modes ($P = 7$)

As the first case, $P = 7$ phase modes of a circular array are considered, which are the excitations from the synthesized linear array [18]. As discussed in Section 2.1, the radius of a circular array for 7 phase modes is determined as $0.7359\lambda$. And according to [18], in order to avoid aliasing of the phase-mode spectrum and to radiate the useful unambiguous mode spectrum, the requirement of a number of elements ($N$) based on the phase modes ($P$) and chosen radius ($r$) is given as

$$P \leq 2kr + 1 \leq N \tag{10}$$

where $P$ is the number of phase modes and $N$ the number of antenna elements in the array. In practice, by approximating the continuous excitation function for obtaining the desired pattern by a finite number
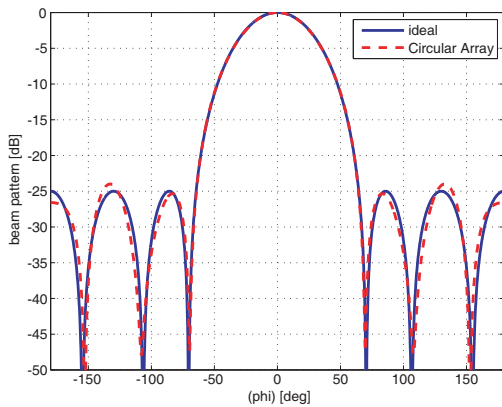


**Figure 4.** Radiation pattern for circular array with 7 phase modes, 11 antenna elements.
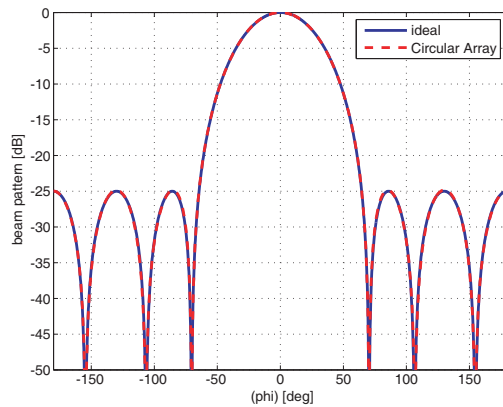


**Figure 5.** Radiation pattern for circular array with 7 phase modes, 12 antenna elements.
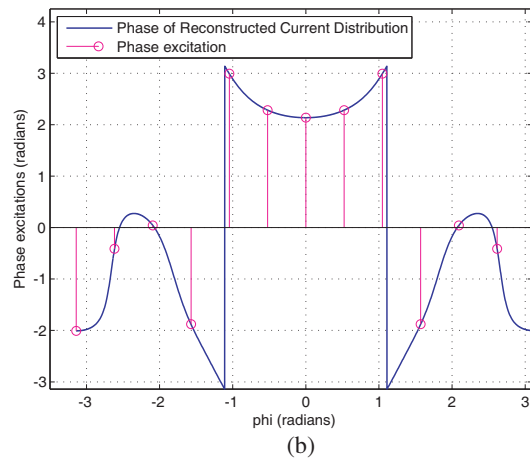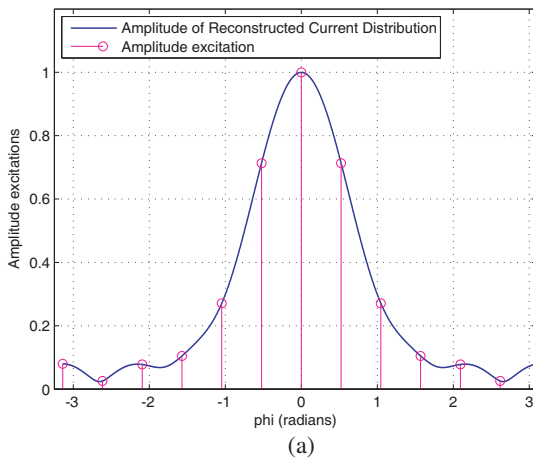


**Figure 6.** Representation of amplitudes and phases at angular positions for $N = 12$ and $P = 7$. (a) Amplitude. (b) Phase.

of elements, if the aperture excitation function contains the total of phase modes $P$, then the antenna array must contain at least $2P$ elements in order to reproduce the same desired pattern [9]. So in this, the requirement is $7 \leq 10.2428 \leq (11, 12, 13, 14)$. So for 7 phase modes and 11 elements, the radiation pattern is computed according to the proposed method as shown in Figure 4. As we noticed from Figure 4, the pattern computed from the 7 phase modes 11 elements has an improved side-lobe level (SLL) by an amount of $-0.35$ dB compared with the desired radiation pattern SLL of $-25$ dB. So for obtaining the relevant design, the number of elements is increased to 12 instead of 11, and the corresponding radiation pattern is as shown in Figure 5. The obtained radiation for a circular array with 7 phase modes and 12 elements is much more similar to the desired radiation pattern. So for this, the respective magnitude and phase plots are shown in Figure 6 according to the proposed method.

### 3.2. Case 2: 8 Phase Modes ($P = 8$)

Here, 8 phase modes for the synthesis of a circular array with Chebyshev pattern are considered. The radius of an circular array for 8 phase modes is determined as $0.8208\lambda$, and the possible requirement of elements to synthesize the radiation pattern based on the standard limit given in Equation (12) is $8 \leq 11.3092 \leq (12, 13, 14, 15, 16)$. The radiation plot for 8 phase modes with 12 elements is shown in Figure 7. As we notice from Figure 7, the pattern computed from the circular array with 8 phase modes and 12 elements has an improved SLL by an amount of $-1.01$ dB compared with the desired radiation pattern SLL of $-25$ dB. So for obtaining the relevant design, the number of elements is increased to 13 instead of 12, and the corresponding radiation pattern is shown in Figure 8. But from Figure 8 it is observed that the radiation pattern obtained for 13 elements and 8 phase modes does not perfectly match
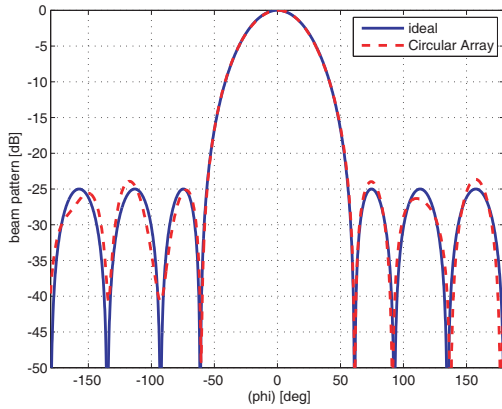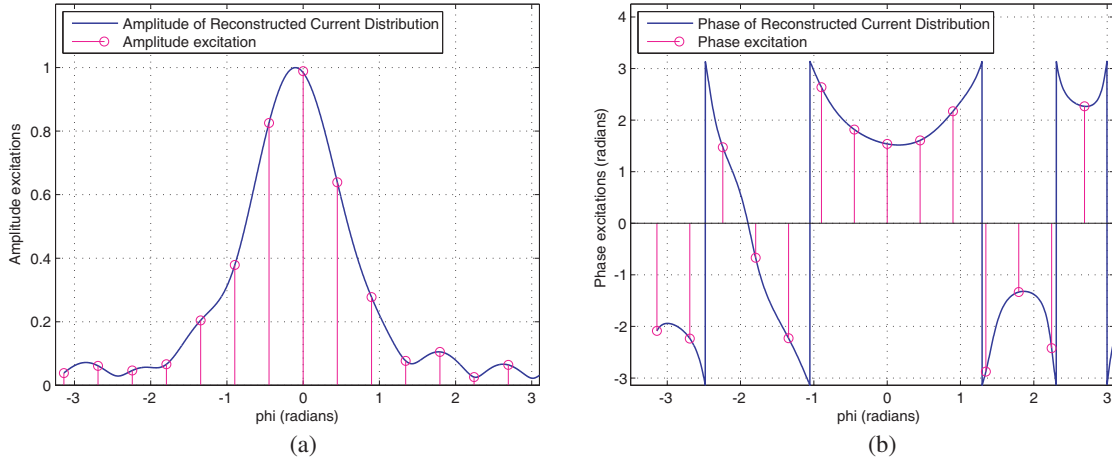


**Figure 7.** Radiation pattern for circular array with 8 phase modes,12 antenna elements.
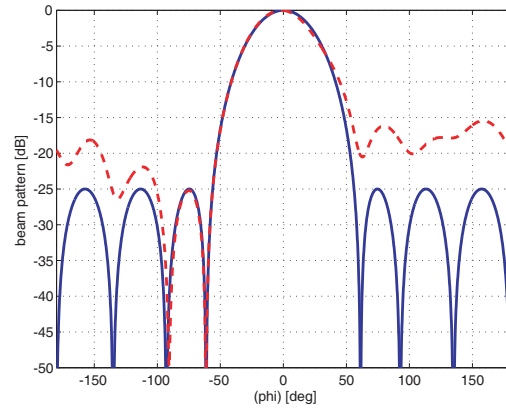


**Figure 8.** Radiation pattern for circular array with 8 phase modes, 13 antenna elements.
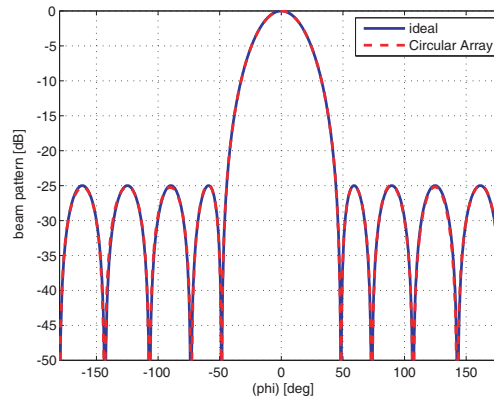


**Figure 9.** Radiation pattern for circular array with 8 phase modes, 14 antenna elements.
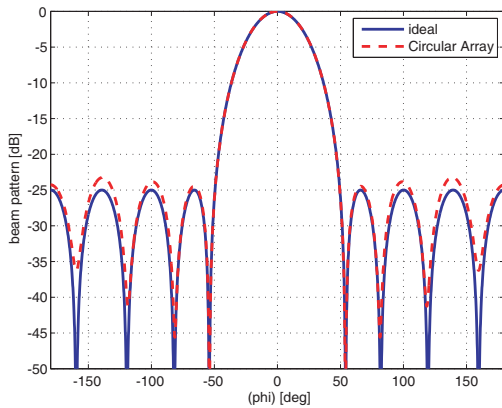
**Figure 10.** Representation of amplitudes and phases at angular positions for $N = 14$ and $P = 8$. (a) Amplitude. (b) Phase.

the design goal and deviates by a value of $-8.75$ dB. The reason behind is that for this combination, i.e., 8 phase modes, 13 elements and $r = 0.8208\lambda$, the number of distortion terms influences the radiation pattern. Figure 9 shows the radiation plot for the circular array after we increase the number of elements to 14. From Figure 9 it is observed that the radiation plot is very close to the desired radiation pattern. For this, the respective magnitude and phase plots are shown in Figure 10. From the above case studies of odd (7) and even (8) phase modes, it is evident that the number of elements by the proposed analysis is 12 and 14, respectively, i.e., less than 2 elements in each case according to the minimum requirement of elements as discussed in [9]. So for better perception, the proposed analysis is extended to $P = 9$ and $P = 10$ as a separate case i.e., case 3 and case 4.

### 3.3. Case 3: 9 Phase Modes ($P = 9$)

For a circular array with 9 phase modes, the suitable radius is determined as $r = 0.8555\lambda$, and the standard limit for minimum requirement of elements is $9 \leq 11.74 \leq (12, 13, 14, 15, 16, 17, 18)$. Here directly consider a circular array of 9 phase modes and 15 elements in order to evaluate the requirement of elements according to the proposed analysis, i.e., $N = 2(P - 1)$. The radiation plot for the circular array with 9 phase modes and 15 elements is shown in Figure 11. The pattern computed from the circular array with 9 phase modes and 15 elements has an improved SLL by an amount of $-0.52$ dB



**Figure 11.** Radiation pattern for circular array with 9 phase modes, 15 antenna elements.
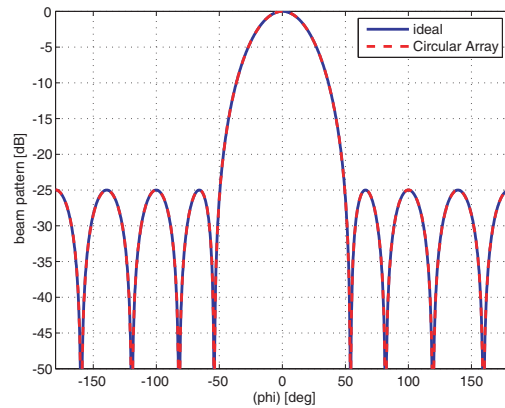
**Figure 12.** Radiation pattern for circular array with 9 phase modes, 16 antenna elements.
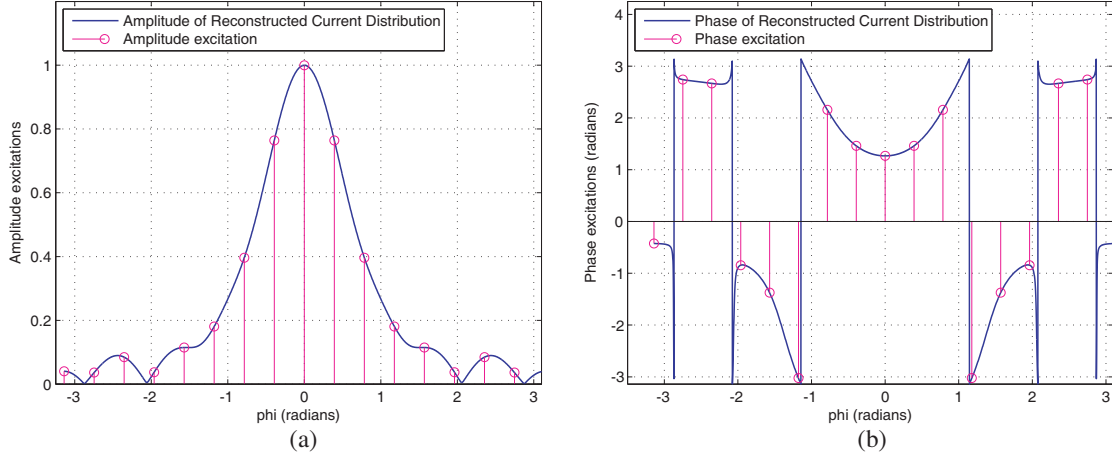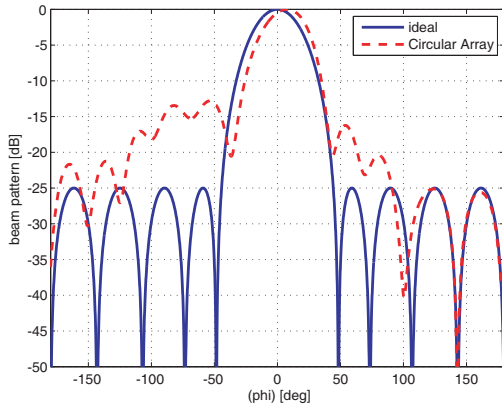
**Figure 13.** Representation of amplitudes and phases at angular positions for $N = 16$ and $P = 9$. (a) Amplitude. (b) Phase.

compared to the desired radiation pattern SLL. So for obtaining the relevant design, the number of elements is increased to 16 instead of 15, and the corresponding radiation pattern is shown in Figure 12. Now from Figure 12, it is noticed that the radiation pattern obtained for 16 elements and 9 phase modes is similar to the desired Chebyshev pattern. For this, the respective magnitude and phase plots are shown in Figure 13.

### 3.4. Case 4: 10 Phase Modes ($P = 10$)

For the circular array with 10 phase modes according to the proposed analysis, the suitable radius is determined as $r = 1.255\lambda$, and the possible requirement of elements according to standard limit is $10 \leq 16.7628 \leq (17, 18, 19, 20)$. So the radiation plot for the circular array with 10 phase modes and 17 elements is shown in Figure 14. It is observed from Figure 14 that the SLL deviates by an amount of $-8.75$ dB compared to the design goal of $-25$ dB. So for obtaining the relevant design, the number of elements is increased to 18 instead of 17, and the corresponding radiation pattern is as shown in Figure 15. From Figure 15 we observed that the radiation pattern obtained for 18 elements is very close to the desired radiation pattern. For this, the respective magnitude and phase plots are shown in Figure 16. So from the above case studies, i.e., case 3 odd (9) phase modes and case 4 even (10) phase modes, the number of required elements according to proposed analysis is 16 and 18 elements,
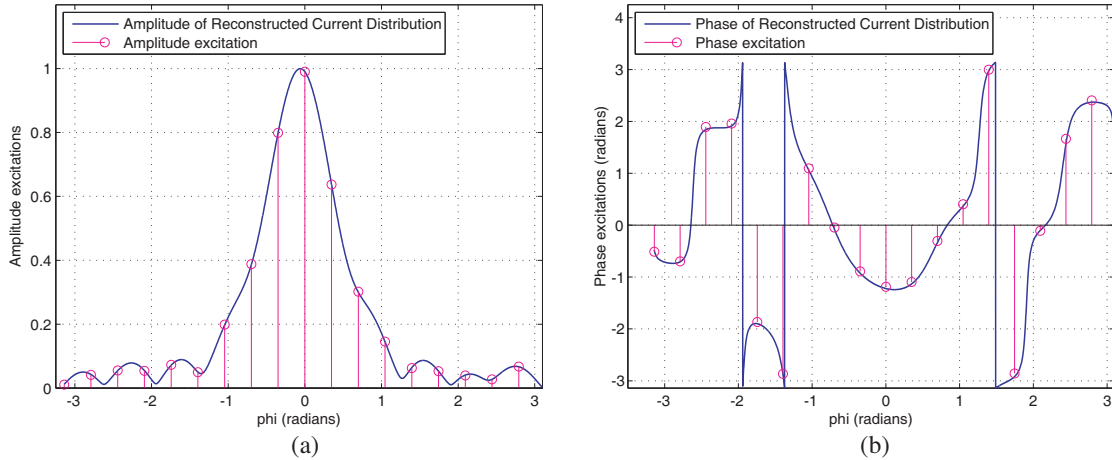


**Figure 14.** Radiation pattern for circular array with 10 phase modes and 17 antenna elements.
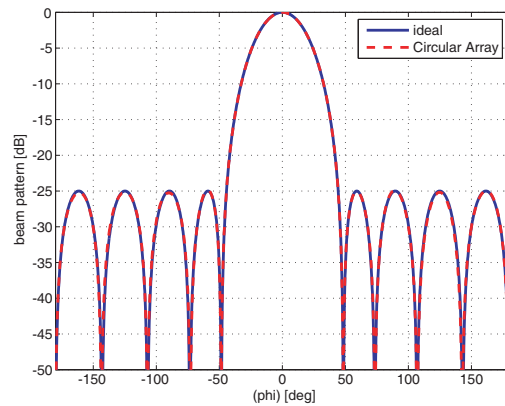


**Figure 15.** Radiation for circular array with 10 phase modes, 18 antenna elements.

**Figure 16.** Representation of amplitudes and phases at angular positions for $N = 18$ and $P = 10$. (a) Amplitude. (b) Phase.

respectively. Therefore, from all cases, it is observed that to obtain the desired pattern, the minimum number of elements required is $N = 2(P - 1)$, where $N$ is the number of antenna elements in the array and $P$ the number of phase modes. All these numerical values are tabulated in Table 1.

**Table 1.** Comparison table for different array sizes.

| S. No | $P$ | Radius $(r)$ | $N = 2P$ [9] | Proposed Analysis $N = 2(P - 1)$ |
|-------|-----|--------------|--------------|----------------------------------|
| 1 | 7 | 0.7359 | 14 | 12 |
| 2 | 8 | 0.8208 | 16 | 14 |
| 3 | 9 | 0.8555 | 18 | 16 |
| 4 | 10 | 1.2555 | 20 | 18 |

## 4. CONCLUSION

This paper gives an inference to the selection of the number of antenna elements $(N)$, number of phase modes $(P)$ and choice of suitable radius $(r)$ for a circular antenna array employing directional element $1 + \cos(\phi)$. The radiation properties of the circular array are clearly studied through the phase modes and are very useful for the proposed analysis. According to the proposed analysis for both even and odd phase modes, the requirement of minimum number of antenna elements for obtaining the desired pattern is formulated as $N = 2(P - 1)$, where $P$ is the number of phase modes and $N$ the number of antenna elements in the array. The present communication is focused on the directional element $1 + \cos(\phi)$ with the SLL constraint. In future research, this work can be extended for various directional elements with multiple constraints.

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