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Joint Azimuth and Elevation Angle Estimation Using Matrix Completion Method

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Abstract—Two-Dimensional Direction of Arrival (2D-DOA) estimation is increasingly important in recent years. In this paper, a new method is proposed to estimate the 2D-DOAs of multiple spatial sources using a three-parallel uniform linear array assuming that some of the sensors happened to be out-of-order. Firstly, a Matrix Completion (MC) algorithm is applied to recover the observed incomplete data, and then an improved joint azimuth and elevation angle estimation algorithm using the recovered data is proposed to obtain the correct parameter estimation. Finally, computer simulation results show that the proposed algorithm has a great performance improvement compared to those based on incomplete data in terms of Signal-to-Noise Ratio (SNR) and the sample rate of sensors.

1. INTRODUCTION

Two-Dimensional Direction of Arrival (2D-DOA) estimation is one of the most important topics in array signal processing thanks to its widely use in the fields of radar, sonar and wireless communication system. In the past decades, many effective 2D-DOA estimation methods such as 2D-MUSIC (Multiple Signal Classification) and 2D-ESPRIT (Estimation of Signal Parameters via Rotational Invariance Technique) have been proposed [1–9].

All the subspace-based methods for 2D-DOA estimation require a large amount of samples to compute the correlation matrix in order to obtain the signal subspace or noise subspace, and asymptotically achieve unbiased estimation. To reduce the computational load of eigen-decomposition in the subspace-based methods, the unitary 2D-ESPRIT algorithm [2] proposed by Zoltowski turns a complex correlation matrix into a real one. However, Zoltowski's method requires that the array has a centro-symmetric structure. In Chen et al.'s paper [10], they came up with a 2D-DOA estimation approach based on a three-parallel uniform linear array. Later, this has been improved by Wu et al.'s fast algorithm with a generalized propagation method [11].

In the overwhelming majority of cases, those 2D-DOA estimation methods can achieve a good result when the observed data of sensors array is complete. However, it is impossible to guarantee that all sensors of the array are properly functional in real engineering application. When the received data of an array are incomplete, most of the DOA estimation approaches fail to work well. Some DOA estimation methods are reported in the literature [12] that can cope with the incomplete array data. However, these methods require higher computational load, initialization, and training. In this paper, we use the Inexact Augmented Lagrange Multiplier algorithm (IALM) [13] to recover the incomplete data of a three-parallel linear array, and then a computationally efficient algorithm for 2-D DOA estimation is proposed by the recovered data, in which the subspace-based technique and the well-known propagation method are combined to obtain a closed-form parameter estimation without searching computation.

This paper is organized as follows. In Section 2, the signal model is introduced. In Section 3, we use an IALM algorithm for Matrix Completion to attain the missing part of the observed data, and

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then in Section 4, an improved azimuth and elevation angle estimation algorithm will be applied to the 2D-DOA estimation. The simulation results are presented in Section 5. At last, a conclusion is given in Section 6.

The notations $(\cdot)^T$, $(\cdot)^H$, $(\cdot)^+$ and $(\cdot)^{-1}$ represent transpose, conjugate transpose, pseudo-inverse and inverse. Also, diag$(\cdot)$ denotes the diagonalization operation of a vector; I_K is a $K \times K$ identity matrix; $E(\cdot)$ stands for the expectation operator; arg$(\cdot)$ means to get the phase angle of $(\cdot)$; $\|\cdot\|_*$ denotes the nuclear norm which is equal to the sum of the singular values of the matrix; $\|\cdot\|_F$ represents the Frobenius norm of the matrix. $\langle\cdot,\cdot\rangle$ returns the inner product.

2. SIGNAL MODEL

As shown in Figure 1, the array geometry is composed by a three parallel uniform linear arrays (ULAs) with the arrays named array X, array Y and array Z. Arrays Y and Z are made up of N sensors, but Array X has one more sensor than Array Y. We assume that the adjacent sensor distance of each array is d, which is equal to half of the wavelength of the incoming signal source, which means $d = \lambda/2$. There are P far-field narrow-band uncorrelated source signals impinging on the array and the ith source has the elevation angle θ_i and azimuth angle φ_i. Thus, in the noisy case [5], we can obtain the output data vector of the whole arrays at snapshot t as:

$$\mathbf{X}(t) = \mathbf{A}_x\mathbf{S}(t) + \mathbf{W}_x(t) \tag{1}$$

$$\mathbf{Y}(t) = \mathbf{A}_y\mathbf{\Omega}_y\mathbf{S}(t) + \mathbf{W}_y(t) \tag{2}$$

$$\mathbf{Z}(t) = \mathbf{A}_y\mathbf{\Omega}_z\mathbf{S}(t) + \mathbf{W}_z(t) \tag{3}$$

where $\mathbf{A}_x = [\mathbf{a}_x(\theta_1,\varphi_1),\ldots,\mathbf{a}_x(\theta_i,\varphi_i),\ldots,\mathbf{a}_x(\theta_P,\varphi_P),]$, and $\mathbf{a}_x(\theta_i,\varphi_i) = [1,\ldots,e^{-j2\pi d\sin\theta_i\sin\varphi_i/\lambda},\ldots,e^{-j2\pi Nd\sin\theta_i\sin\varphi_i/\lambda}]^T$. Thus $\mathbf{A}_y$ contains the first N rows of $\mathbf{A}_x$. $\mathbf{W}_x(t)$, $\mathbf{W}_y(t)$ and $\mathbf{W}_z(t)$ are independent additive Gaussian white noise vectors (AGWN). $\mathbf{\Omega}_y = \text{diag}[e^{-j2\pi d\cos\theta_1/\lambda}, e^{-j2\pi d\cos\theta_2/\lambda},\ldots,e^{-j2\pi d\cos\theta_P/\lambda}]$ and $\mathbf{\Omega}_z = \text{diag}[e^{-j2\pi d\sin\theta_1\cos\varphi_1/\lambda}, e^{-j2\pi d\sin\theta_2\cos\varphi_2/\lambda},\ldots,e^{-j2\pi d\sin\theta_P\cos\varphi_P/\lambda}]$. By giving a definition that $\mathbf{W} = [\mathbf{X}(t)^T\ \mathbf{Y}(t)^T\ \mathbf{Z}(t)^T]^T$, (1)–(3) can be expressed as below:

$$\mathbf{W} = \begin{bmatrix}\mathbf{X}\\ \mathbf{Y}\\ \mathbf{Z}\end{bmatrix} = \begin{bmatrix}\mathbf{A}_x\\ \mathbf{A}_y\mathbf{\Omega}_y\\ \mathbf{A}_y\mathbf{\Omega}_z\end{bmatrix}\mathbf{S}(t) + \mathbf{Q} \tag{4}$$

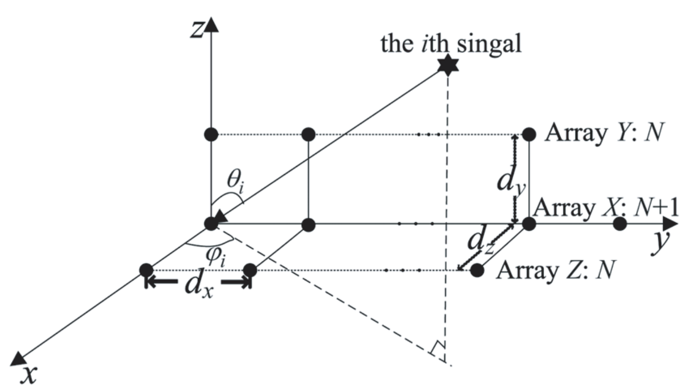

Figure 1. Array model.

In short, we write Eq. (4) as $\mathbf{W} = \mathbf{AS} + \mathbf{Q}$, where $\mathbf{A} = [\mathbf{A}_x^T\ (\mathbf{A}_y\mathbf{\Omega}_y)^T\ (\mathbf{A}_z\mathbf{\Omega}_z)^T]^T$,$\mathbf{S}(t) = [\mathbf{S}_1(t),\ldots,\mathbf{S}_P(t)]^T$ and $\mathbf{Q} = [\mathbf{W}_x^T\ \mathbf{W}_y^T\ \mathbf{W}_z^T]^T$ denotes the noise vector of array output.

3. MATRIX COMPLETION

In order to recover the missing part of data received at the array, we regard it as a Matrix Completion problem [14]. Assume that we have the available technology to locate the broken sensors [15]. Data which need to be recovered have low rank P obviously. Candes and Recht's research [16] shows that the recovered data are related to the following optimization problem:

$$\min_{\mathbf{W}} \|\mathbf{W}\|_*, \textit{subject to } \mathbf{W}_{ij} = \mathbf{D}_{ij}, \forall(i,j) \in \mathbf{\Omega}, \tag{5}$$

where $\mathbf{\Omega}$ is the set of indices of functional sensors, and $\mathbf{D}$ represents the data received from the array. The most famous algorithm to solve this problem is the singular value thresholding (SVT) method [17]. In this paper, we introduce an algorithm known as the Inexact Augmented Lagrange Multiplier (IALM) algorithm [13] to deal with it. Eq. (5) can be expressed as:

$$\min_{\mathbf{W}} \|\mathbf{W}\|_*, \textit{subject to } \mathbf{W} + \mathbf{E} = \mathbf{D},\ \pi_{\mathbf{\Omega}}(\mathbf{E}) = 0, \tag{6}$$

where $\pi_{\mathbf{\Omega}} : \mathbb{R}^{m\times n} \to \mathbb{R}^{m\times n}$ is a projection operator which keeps the elements in $\mathbf{\Omega}$ unchanged and other elements not in $\mathbf{\Omega}$ turns into zeros. $\mathbf{E}$ is the error matrix. The partial augmented Lagrangian function of Eq. (6) is:

$$L(\mathbf{W}, \mathbf{E}, \mathbf{Y}, \mu) = \|\mathbf{W}\|_* + \langle \mathbf{Y}, \mathbf{D} - \mathbf{W} - \mathbf{E}\rangle + \frac{\mu}{2} \|\mathbf{D} - \mathbf{W} - \mathbf{E}\|_F^2 . \tag{7}$$

Then we can have the Inexact ALM algorithm for the MC problem by updating $\mathbf{E}$ under the condition that $\pi_{\mathbf{\Omega}}(\mathbf{E}) = 0$ when minimizing $L(\mathbf{W}, \mathbf{E}, \mathbf{Y}, \mu)$. Above all, we have Algorithm 1 as follows:

Algorithm 1 Inexact ALM algorithm for Matrix Completion

Input: $\mathbf{D}_{ij}, (i,j) \in \mathbf{\Omega}, \mathbf{D} \in R^{m\times n}$
Output: $(\hat{\mathbf{W}}_k, \mathbf{E}_k)$
$\mathbf{Y}_0 = 0; \mathbf{E}_0 = 0; \mu_0 > 0; \rho > 1; k = 0.$
while not converged **do**
  $(\mathbf{U}, \mathbf{S}, \mathbf{V}) = svd(\mathbf{D} - \mathbf{E}_k + \mu_k^{-1}\mathbf{Y}_k);$
  $\hat{\mathbf{W}}_{k+1} = \mathbf{U} S_{\mu_k^{-1}}[\mathbf{S}]\mathbf{V}^T;$
  $\mathbf{E}_{k+1} = \pi_{\mathbf{\Omega}}(\mathbf{D} - \hat{\mathbf{W}}_{k+1} + \mu_k^{-1}\mathbf{Y}_k);$
  $\mathbf{Y}_{k+1} = \mathbf{Y}_k + \mu_k(\mathbf{D} - \hat{\mathbf{W}}_{k+1} - \mathbf{E}_{k+1});$
  $\mu_{k+1} = \rho\mu_k;$
  $k = k + 1;$
end while
return $(\hat{\mathbf{W}}_k, \mathbf{E}_k)$.

Because of the appropriate choice of $\mathbf{E}_k$, $\mathbf{Y} = 0$ is always established in the iteration, and it means that the values of the unknown part retain zeros. The iteration reaches its stopping criteria as follows:

$$\left\|\mathbf{D} - \hat{\mathbf{W}}_k - \mathbf{E}_k\right\|_F / \|\mathbf{D}\|_F < \varepsilon_1 \textit{ and } dist(\vartheta \left\|\hat{\mathbf{W}}_k\right\|_*, \mathbf{S}) / \|\mathbf{D}\|_F < \varepsilon_2, \tag{8}$$

The output matrix $\hat{\mathbf{W}}$ is what we need. With the recovered matrix $\hat{\mathbf{W}}$, we can come to the following DOA estimation part.

4. 2D-DOA

In this section, a computationally efficient azimuth and elevation angle estimation algorithm for a three-parallel uniform linear arrays [10] is applied to obtain a parameter estimation without searching computation. The algorithm, based on the propagator method (PM), can automatically pair the azimuth and elevation angles. In Section 2, we have obtained the incomplete output matrix $\mathbf{W}$ of the array. Then in Section 3, we get the recovered matrix $\hat{\mathbf{W}}$ which is equal to the complete output data of the array through the above procedure of matrix completion. We can partition A firstly as follows:

$$\mathbf{A} = \left[\mathbf{A}_1^T\ \mathbf{A}_2^T\right]^T, \tag{9}$$

where $\mathbf{A}_1$ is a $P \times P$ matrix, and $\mathbf{A}_2$ is a $(3N+1-P) \times P$ matrix. Then we can obtain a $P \times (3N+1-P)$ propagator as:

$$\mathbf{P}^H \mathbf{A}_1 = \mathbf{A}_2, \tag{10}$$

We introduce a matrix $\mathbf{P}_e$ defined as $\mathbf{P}_e = [\mathbf{I}_P^T\ \mathbf{P}]^H$. In the noiseless case, $\mathbf{P}_e \mathbf{A}_1 = \mathbf{A}_2$. Then we can partition $\mathbf{P}_e$ as:

$$\mathbf{P}_e = \left[\mathbf{P}_x^T\ \mathbf{P}_y^T\ \mathbf{P}_y^T\right]^T, \tag{11}$$

Combining Eqs. (9) and (11), we have $[\mathbf{P}_x^T\ \ \mathbf{P}_y^T\ \ \mathbf{P}_y^T]^T \mathbf{A}_1 = [\mathbf{A}_x^T\ \ (\mathbf{A}_y \mathbf{\Omega}_y)^T\ \ (\mathbf{A}_z \mathbf{\Omega}_z)^T]^T$. By introducing $\mathbf{P}_{x1}$ which has the first N rows of $\mathbf{P}_x$, we can obtain:

$$\mathbf{P}_{x1} \mathbf{A}_1 = \mathbf{A}_y \tag{12}$$

$$\mathbf{P}_z \mathbf{A}_1 = \mathbf{A}_y \mathbf{\Omega}_z \tag{13}$$

Then we can get $\mathbf{P}_{x1}^+ \mathbf{P}_z = \mathbf{A}_1 \mathbf{\Omega}_z \mathbf{A}_1^{-1}$. The eigenvalues $\beta_i (i = 1, 2, \ldots, P)$ of $\mathbf{P}_{x1}^+ \mathbf{P}_z$ can be obtained by doing the EVD on this equation. $\mathbf{\Omega}_y$ and $\mathbf{\Omega}_x$ can be obtained in the same way, where $\mathbf{\Omega}_x = \mathrm{diag}[e^{-j2\pi d \sin\theta_1 \cos\varphi_1/\lambda}, e^{-j2\pi d \sin\theta_2 \cos\varphi_2/\lambda}, \ldots, e^{-j2\pi d \sin\theta_P \cos\varphi_P/\lambda}]$.

Above all, the 2D-DOA estimation algorithm is described as follows:

Algorithm 2 Modified 2D-DOA estimation algorithm for three-parallel uniform linear arrays

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Input: Matrix $\mathbf{W}$, and then a recoverded data $\hat{\mathbf{W}}$ is obtained by the IALM algorithm and the incomplete data $\mathbf{W}$.

Output: $(\hat{\varphi}_i, \hat{\theta}_i)$

1: Compute the covariance matrix of $\hat{\mathbf{W}}$ by this formula: $\mathbf{R}_{\hat{W}} = E[\hat{\mathbf{W}} \hat{\mathbf{W}}^H]$.

2: The partition of $\mathbf{R}_{\hat{W}}$ can be written as: $\mathbf{R}_{\hat{W}} = [\mathbf{R}_{\hat{W}1}\ \mathbf{R}_{\hat{W}2}]$, where $\mathbf{R}_{\hat{W}1} \in C^{(3N+1)\times(3N+1-P)}$. Thus the estimation of $\hat{\mathbf{P}}$, which is a $(3N+1-P) \times P$ propagator matrix, can be written as: $\hat{\mathbf{P}} = (\mathbf{R}_{\hat{W}1}^H \mathbf{R}_{\hat{W}1})^{-1} \mathbf{R}_{\hat{W}1}^H \mathbf{R}_{\hat{W}2}$.

3: Then we extended propagator matrix $\mathbf{P}_e = [\mathbf{I}_P^H\ \hat{\mathbf{P}}]^H$.

4: Partition P_e as $\mathbf{P}_e = [\mathbf{P}_x^T\ \mathbf{P}_y^T\ \mathbf{P}_z^T]^T$, where $\mathbf{P}_x^T \in C^{(N+1)\times P}, \mathbf{P}_y^T \in C^{N\times P}, \mathbf{P}_z^T \in C^{N\times P}$. Then importing a matrix $\mathbf{P}_{x1}$ which has the first N rows of $\mathbf{P}_x$, we can get a matrix $\mathbf{\Psi}_z$ by defining $\mathbf{\Psi}_z = \mathbf{P}_{x1}^+ \mathbf{P}_z$. By performing EVD on $\mathbf{\Psi}_z$, we can obtain the eigenvectors $A_1^{'}$ and the eigenvalues $\hat{\beta}_i$ of $\mathbf{\Psi}_z$.

5: By importing a matrix $\mathbf{P}_{e1}$ written as $\mathbf{P}_{e1} = [\mathbf{P}_{x1}^T\ \mathbf{P}_y^T]^T$, we can get a matrix $\mathbf{B}$ in which $\mathbf{B} = \mathbf{P}_{e1} \mathbf{A}_1^{'}$. Then we construct two new matrixes $\mathbf{B}_1$ and $\mathbf{B}_2$. $\mathbf{B}_1$ has the first N rows of $\mathbf{B}$, and $\mathbf{B}_2$ has the rest rows of $\mathbf{B}$. Matrix $\hat{\mathbf{\Omega}}_y$ is defined by $\hat{\mathbf{\Omega}}_y = \mathbf{B}_1^+ \mathbf{B}_2$. Then we can get $\hat{\alpha}_i$ from the ith diagonal element of $\hat{\mathbf{\Omega}}_y$.

6: Now we import a series of matrixes $\mathbf{P}_{x2}, \mathbf{P}_{y1}, \mathbf{P}_{y2}, \mathbf{P}_{z1}, \mathbf{P}_{z2}$, where $\mathbf{P}_{x2}$ has the last N rows of $\mathbf{P}_x$; $\mathbf{P}_{y1}$ has the first $N-1$ rows of $\mathbf{P}_y$; $\mathbf{P}_{y2}$ has the last $N-1$ rows of $\mathbf{P}_y$; $\mathbf{P}_{z1}$ has the first $N-1$ rows of $\mathbf{P}_z$; $\mathbf{P}_{z2}$ has the last $N-1$ rows of $\mathbf{P}_z$. Then we construct two matrixes $\mathbf{C}_1$ and $\mathbf{C}_2$ by defining $\mathbf{C}_1 = [\mathbf{P}_{x1}^T\ \mathbf{P}_{y1}^T\ \mathbf{P}_{z1}^T]^T \mathbf{A}_1^{'}$ and $\mathbf{C}_2 = [\mathbf{P}_{x2}^T\ \mathbf{P}_{y2}^T\ \mathbf{P}_{z2}^T]^T \mathbf{A}_1^{'}$. $\hat{\mathbf{\Omega}}_x$ is obtained by performing $\hat{\mathbf{\Omega}}_x = \mathbf{C}_1^+ \mathbf{C}_2$. We can get $\hat{\gamma}_i$ when we put EVD on $\hat{\mathbf{\Omega}}_x$.

7: We can attain the estimate of $\hat{\varphi}_i$ and $\hat{\theta}_i$ from the following equations:

$$\hat{\varphi}_i = \arctan\left[\frac{\arg(\hat{\gamma}_i)}{\arg(\hat{\beta}_i)}\right]$$

$$\hat{\theta}_i = \arctan\left[\frac{\arg(\hat{\beta}_i)}{\arg(\hat{\alpha}_i)\cos(\hat{\varphi}_i)}\right]$$

8: **return** $(\hat{\varphi}_i, \hat{\theta}_i)$.

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5. SIMULATION RESULTS

In this section, several simulations are given to show the improvement of performance of the proposed method. We assume three uncorrelated signal sources with $(\varphi_1, \theta_1) = (20°, 10°)$, $(\varphi_2, \theta_2) = (40°, 30°)$ and $(\varphi_3, \theta_3) = (60°, 50°)$ impinging on the received array. Arrays Y and Z have 15 sensors, which means that array X has 16 sensors according to the array structure. We assume that all sensors have the same probability to break down. Also we have the ability to know which sensor is broken and get their locations. After all, we take 200 snapshots for each test. By introducing a new parameter sampling rate p, which equals the percentage of working sensors in all, we can describe the damage to the array. The Mean Square Error (MSE) is defined as:

$$\text{MSE}_{\theta_i} = \sqrt{E[(\theta_i - \hat{\theta}_i)^2]}; \tag{14}$$

$$\text{MSE}_{\varphi_i} = \sqrt{E[(\varphi_i - \hat{\varphi}_i)^2]}; \tag{15}$$

In the first test, we set the sampling rate p at 0.7, which means that only 70% sensor is functional in this array. And then let SNR change from 0 dB to 30 dB. Figure 2 and Figure 3 show that our proposed method has a better performance in the whole range of SNRs, and it works much better at higher SNR than that at lower SNR.

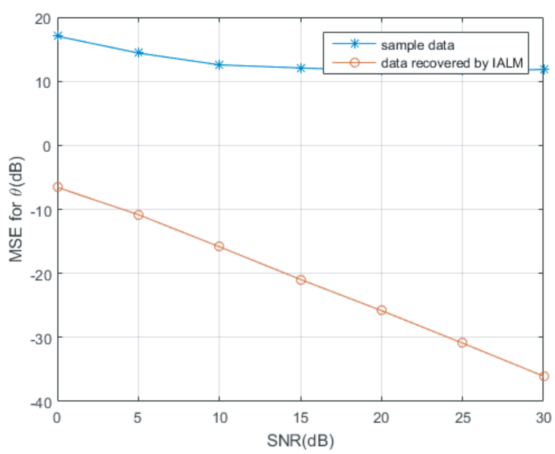

Figure 2. MSE for θ (dB).

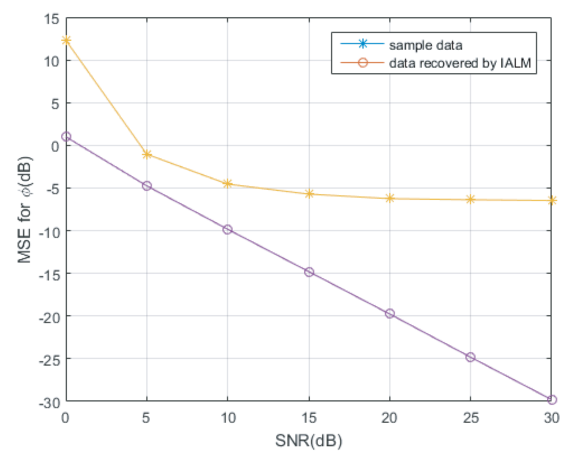

Figure 3. MSE for φ (dB).

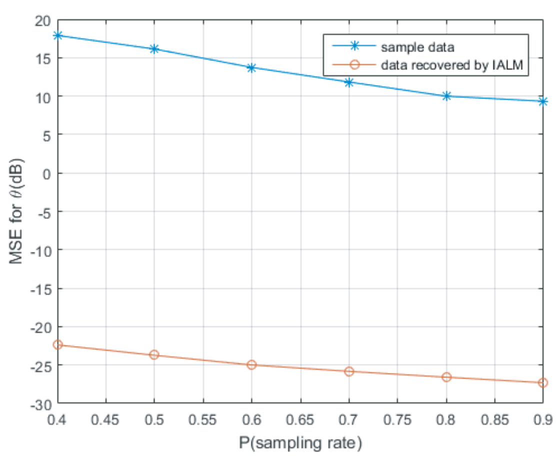

Figure 4. MSE for θ (dB).

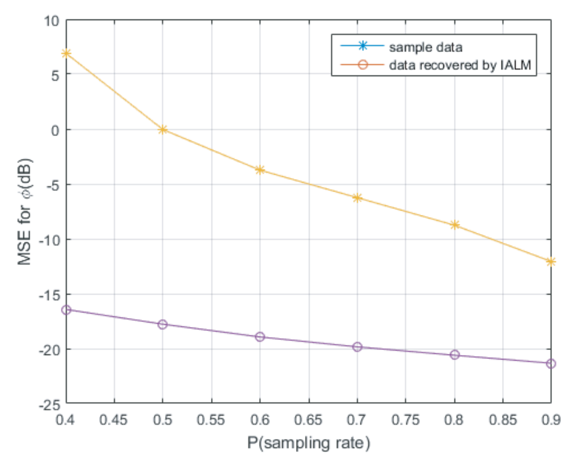

Figure 5. MSE for φ (dB).

In the second test, we set the SNR as 20 dB. Then let the sampling rate p change from 0.4 to 0.9. The result of DOA estimation is shown from Figure 4 and Figure 5. It is seen that this proposed method remains stable at a low sampling rate, which means that the proposed method can work well when most sensors of the array are functioned at a reasonable SNR.

We should notice that this method also suits for other 2D-DOA methods if the shape of the array changes. In the third test, we change array into a Uniform Circular Array (UCA) and use the UCA-ESPRIT algorithm to do the DOA estimation. Then we set the sampling rate p on 0.6 and let SNR change from 0 dB to 30 dB. It is shown in Figure 6 that this proposed method can still work if the shape of array configuration is changed.

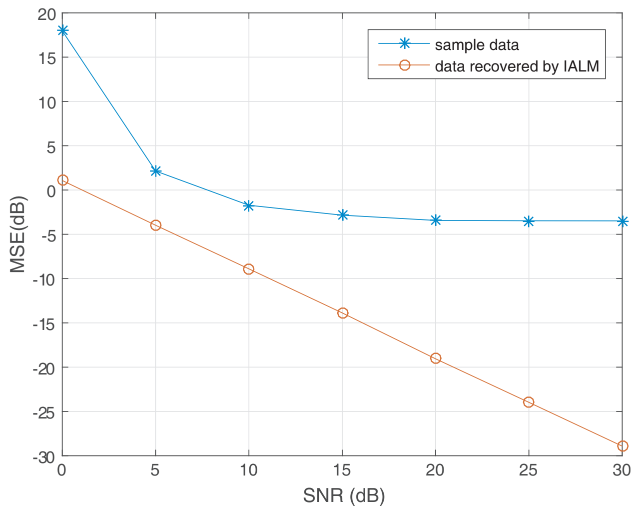

Figure 6. MSE for UCA array (dB).

6. CONCLUSION

In this paper, we propose a computationally efficient method for 2D-DOA estimation with a three-parallel array, which is aimed at solving the problem that when an incomplete data is received from an array, using a faster Matrix Completion method-IALM, we can get a correct 2D-DOA estimation result with the recovered data of the array. The simulation results show that the proposed algorithm has an improved performance compared to the conventional method when only a small number of sensors in the array are still working. Moreover, we can use different array configurations in practical application, and this method can still work.

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