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A Fast Explicit FETD Method Based on Compressed Sensing

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Abstract—Linear equations must be solved at each time step as the explicit finite element time-domain (FETD) method is used to solve time dependent Maxwell curl equations, which leads to a huge amount of computational cost in a long period time simulation. A new scheme to accelerate the iteration solution for matrix equation is proposed based on compressed sensing (CS), in which a low rank measurement matrix is established by randomly extracting rows from mass matrix. Meanwhile, to reduce the number of measurements required, a sparse transform is constructed with the help of prior knowledge offered by the solution results of previous time steps. Numerical results of homogeneous cavity and inhomogeneous cavity are discussed to validate the effectiveness and accuracy of the proposed approach.

1. INTRODUCTION

Finite element time-domain method (FETD) is an efficient tool for solving electromagnetic scattering problems, since it combines the advantages of time-domain techniques with the versatile spatial discretization options of the finite element method (FEM) [1]. It is easy to use FETD to handle multi-scale geometry and acquire information over a wide frequency band. Sorts of FETD methods have been proposed in recent years, an explicit method that directly solving Maxwell curl equations utilizes the electric field E and magnetic flux intensity B as simultaneous state variables has been mentioned in [2, 3]. This mixed method can also be considered as a generalization of the finite-difference time-domain (FDTD) method for unstructured grids. Due to its potential effect to simulate free space conveniently by introducing perfectly matched layer [4] and conserve energy over long period time in conjugation with symplectic method [5], more attention has been devoted to it. However, the computation of interpolation coefficients of global variables E and B has to solve two matrix equations at each time step in this approach, which makes the calculation extremely expensive in a long period time simulation. Although reference [6] offered an improved scheme that only one matrix equation is required to be solved, this defect still limits its development and application.

Compressed sensing (CS) [7], as a current research focus in signal processing, has been introduced into biological engineering, communication engineering, image processing and electromagnetic field [8], etc. In CS theory, a signal can be captured at a rate significantly below the Nyquist rate if it has a sparse representation in a suitable transform domain, and then it can be exactly reconstructed using recovery algorithms [9]. By means of this theory, some useful schemes are developed to solve partial differential equations (PDEs) problems with the help of sparse approximations [10, 11].

Motivated by these early theoretical frameworks, a novel FETD method improved by CS (CS-FETD) is proposed to accelerate solution for the matrix equations of the mixed FETD method. The implementation of CS for this new scheme can be described by three steps: (1) establish a measurement matrix by randomly extracting some rows from mass matrix; (2) construct a new basis based on redundant dictionary that offered by the prior knowledge included in solutions of previous time steps;

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(3) exactly reconstruct the solutions by recovery algorithms. Differently shaped homogeneous cavity and inhomogeneous cavity are analyzed to show the effectiveness and accuracy of this new fast FETD method.

2. THEORY AND IMPLEMENTATIONS

2.1. Explicit FETD Method

The coupled first-order time dependent Maxwell curl equations in source free region are considered as

$$\varepsilon\frac{\partial}{\partial t}E = \nabla\times\left(\mu^{-1}B\right) \tag{1}$$

$$\frac{\partial}{\partial t}B = -\nabla\times E. \tag{2}$$

To achieve the FETD solution of Equations (1) and (2), the computational domain is assumed to be discretized by a FEM mesh with a triangle faces and d edges. The linear system of ordinary differential equations for TE_z problems are yielded by Galerkin FEM, in which Whitney 1-form vector basis function is used to discretize electrical field intensity and Whitney 2-form vector basis function is used to discretize magnetic flux intensity, such as

$$[T]_{d\times d}\frac{\partial}{\partial t}\{e\}_{d\times 1} = [C]^T_{a\times d}\,[K]_{a\times a}\{b\}_{a\times 1} \tag{3}$$

$$\frac{\partial}{\partial t}\{b\}_{a\times 1} = -[C]_{a\times d}\{e\}_{d\times 1} \tag{4}$$

where $\{e\}$ and $\{b\}$ are the interpolation coefficients of E and B, respectively, and $[C]$ is the Curl operator. $[T]$ is the 1-form mass matrix with the material property function ϵ is used to represent the dielectric properties, and $[K]$ is the 2-form mass matrix with the material property function μ^{-1} is used to represent the magnetic permeability. a and d define the dimension of mass matrix. Applying the leap frog method to (3) and (4) with a stable time step Δt, one can obtain

$$[T]_{d\times d}\{e\}^n_{d\times 1} = [T]_{d\times d}\{e\}^{n-1}_{d\times 1} + \Delta t\,[C]^T_{a\times d}\,[K]_{a\times a}\,\{b\}^{n-1/2}_{a\times 1} \tag{5}$$

$$\{b\}^{n+1/2}_{a\times 1} = \{b\}^{n-1/2}_{a\times 1} - \Delta t[C]_{a\times d}\,\{e\}^n_{d\times 1}\,. \tag{6}$$

2.2. Implementation of CS-FETD Method

The mathematical model of CS can be formulated as

$$[\Phi]_{m\times k}\{X\}_{k\times l} = [\Phi]_{m\times k}[\Psi]_{k\times k}\{\alpha\}_{k\times l} = [S]_{m\times l} \tag{7}$$

in which $\{\alpha\}$ is the sparse representation coefficients of original signal $\{X\}$ with the sparse basis $[\Psi]$, and $[\Phi]$ is the measurement matrix. m, k and l are the dimensions of these matrices, in general, k is much larger than m. The approximation of $\{\alpha\}$ is computed by a L-minimization problem as

$$\{\hat{a}\} = \min\left\|\{a\}_{k\times l}\right\|_L s.t.\left([\Phi]_{m\times k}[\Psi]_{k\times k}\right)\{\hat{a}\}_{k\times l} = [S]_{m\times l}. \tag{8}$$

The process of CS discussed above is a regular application in signal processing and other fields. To implement CS in FETD method, some changes must be made in its procedure and they can be described as follows:

Step 1: At the beginning of the simulation, $\{e\}$ is calculated by the traditional FETD method.

Step 2: CS is implemented into FETD when the electromagnetic wave propagation covers the whole computation area. Based on (5) (the right side of (5) is denoted as $\{V\}$), an improved matrix equation can be obtained as

$$\left[T'\right]_{f\times d}\{e\}_{d\times 1} = \left\{V'\right\}_{f\times 1} \tag{9}$$

in which $[T']$ is formed by extracting f rows from $[T]$, and $\{V'\}$ represents the corresponding elements extracted from $\{V\}$. An interesting discovery is that, as FETD is applied, all the $\{e\}$ calculated at each time step have strong correlation and redundancy. Hence, a ready-made sparse basis named $[e_{pre}]$ can

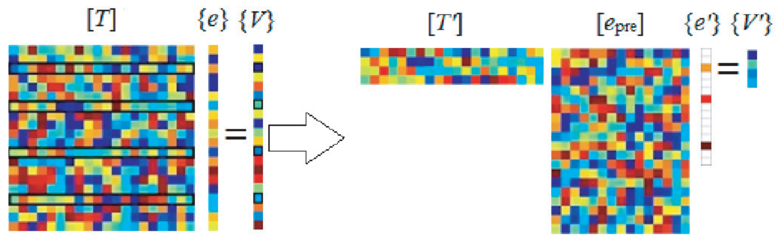

Figure 1. The variation of matrix equation (5) before and after the electromagnetic wave propagation covers the whole computation area.

be obtained as shown in Fig. 1, which is constituted of the column vectors of $\{e\}$ obtained from previous time steps. In other words, the matrix is an existing redundant dictionary for $\{e\}$ at the time step that is currently being calculated. Take $[T']$ as the measurement matrix, Equation (9) at the $i+1$th time step can be transformed as

$$[T']_{f\times d}[e_{pre}]_{d\times i}\{e'\}_{i\times 1} = \{V'\}_{f\times 1} \tag{10}$$

in which $\{e'\}$ represents the sparse projection of $\{e\}$ in $[e_{pre}]$. Especially, sparse basis is required to be updated over time. Fig. 1 shows the performance of Step 2.

Step 3: An appropriate recovery algorithm is applied to compute approximate value of $\{e\}$ by

$$\{\hat{e}'\} = \min \|\{e\}\|_L s.t. \left([T'] [e_{pre}]\right) \{\hat{e}'\} = \{V'\} \tag{11}$$

$$\{\hat{e}\} = [e_{pre}] \{\hat{e}'\}. \tag{12}$$

The complexity of CS-FETD is mainly decided by recovery algorithms after the application of CS. Taking orthogonal matching pursuit (OMP) technique as an example, the complexity is $O(Sdf)$ [12], where S is the number of iteration steps. The complexity of traditional FETD by using the conjugate gradient (CG) algorithm to solve the matrix Equation (5) is $O(Pd^2)$, where P is similar to S. In general, $S \ll P$ and $f \ll d$, so that the proposed method is more efficient. Although the complexity of FETD can be decreased in virtue of the property that mass matrix is sparse and symmetric, CS-FETD also has the obvious advantage under the same conditions.

3. NUMERICAL RESULTS

To illustrate the effectiveness and accuracy by the proposed method, some resonant frequency problems of 2-D cavity with perfectly conducting walls in TE_z case are provided.

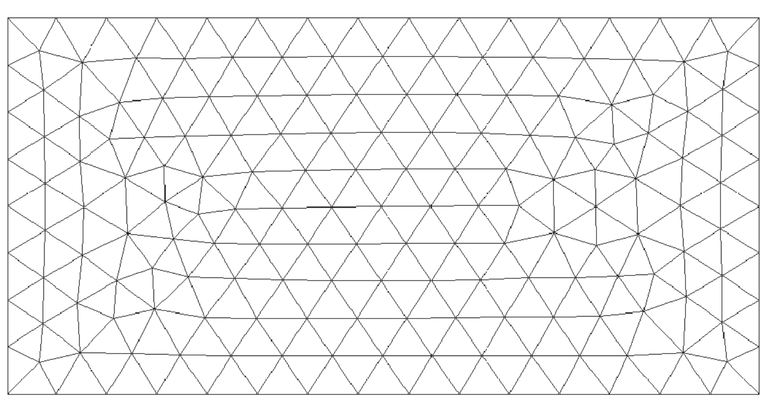

Figure 2. A rectangle cavity ($0.02\,\text{m} \times 0.01\,\text{m}$) discretized with 497 edges and 316 triangle faces.

3.1. Rectangle Cavity

Using the FEM mesh, the rectangle cavity is described in Fig. 2, and the dimension of mass matrix is 497×497 and the time step is set to be $\Delta t = 1\,\text{ps}$. For simplicity we set $\epsilon = \mu = 1$.

For the purpose of comparison, the issue is simulated with both the conventional FETD and CS-FETD methods. The resonant frequencies are acquired from the time domain outcome after certain time steps by fast Fourier transform. Fig. 3 shows that the error between CS-FETD and conventional FETD solutions can be ignored when the number of rows extracted randomly from $[T]$ is greater than 45. Fig. 4 shows values of $\{e\}$ by using two approaches. Clearly, The CS-FETD with 50 extracted rows can obtain an accurate solution. Meanwhile, the resonant frequencies and computing time comparison are also described in Table 1. These comparison results demonstrate that CS-FETD has a high accuracy with relative error less than 0.1%, moreover requires only about half of the physical time of the conventional explicit formulation.

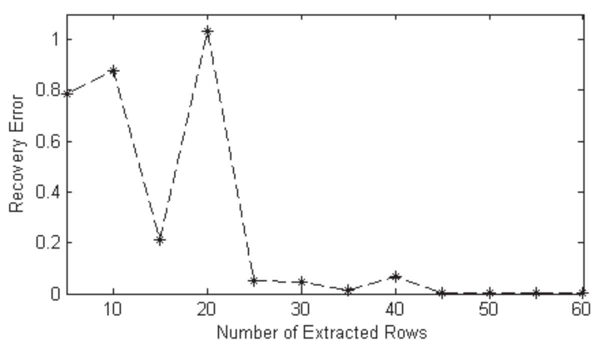

Figure 3. Relationship between recovery error and number of extracted rows at 2000th time step.

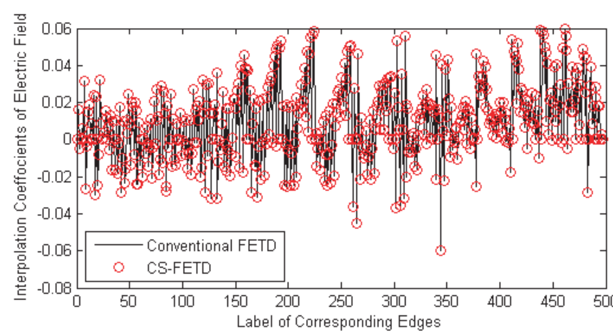

Figure 4. The interpolation coefficients of all edges at 1500th time step.

Table 1. Resonant frequencies of rectangle cavity.

	f/d	NO. steps	Physical Time	TE_{10}	TE_{01}, TE_{20}	TE_{11}
FETD	497/497	20000	3.914e3 s	7.500	15.000	16.667
CS-FETD	50/497	20000	2.146e3 s	7.500	15.000	16.658

3.2. Circular Cavity

Similar design principles are applied in the circular cavity case shown in Fig. 5. The dimension of mass matrix is 843×843 and the time step is set to be $\Delta t = 0.9\,\text{ps}$.

Figure 6 shows the recovery error changes from 1000th to 2400th time step with 70 rows extracted randomly from mass matrix. Fig. 7 depicts interpolation coefficients on certain an edge from 200th to 2400th time step with the same measurement times. Comparisons of computational results and physical time are shown in Table 2.

Table 2. Resonant frequencies of circular cavity.

	f/d	NO. steps	Physical Time	TE_{11}	TE_{21}	TE_{01}
FETD	843/843	20000	5.142e3 s	8.778	14.587	18.267
CS-FETD	70/843	20000	2.964e3 s	8.767	14.565	18.264

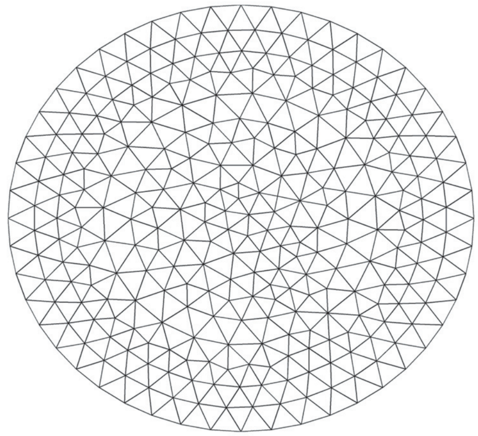

Figure 5. A circular cavity ($r = 0.01\,\text{m}$) discretized with 843 edges and 546 triangle faces.

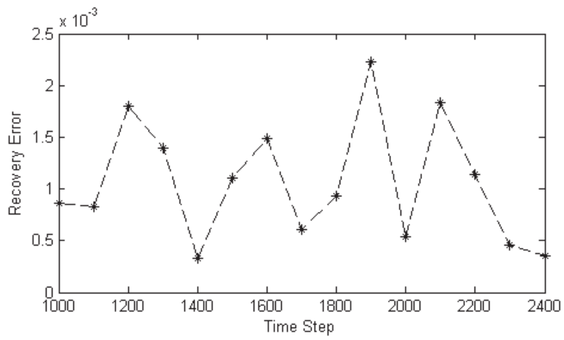

Figure 6. The changes of recovery error.

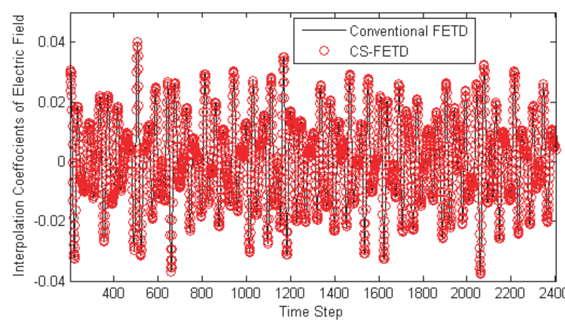

Figure 7. The changes of interpolation coefficients.

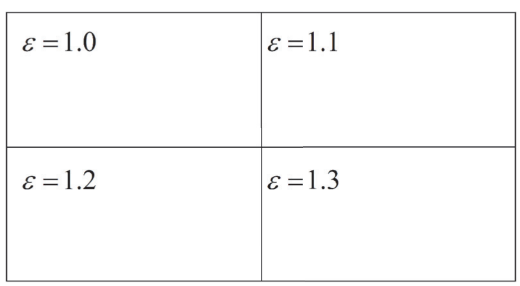

Figure 8. A inhomogeneous cavity with four different dielectric properties areas.

3.3. Inhomogeneous Cavity

In this section, the rectangle cavity offered in the first example is divided into four areas with different dielectric properties. The inhomogeneous cavity depicted in Fig. 8 contains 540 edges and 344 triangles by FEM mesh. The time step is set to be $\Delta t = 1.1\,\text{ps}$.

Figure 9 shows that the accuracy of the computational results by CS-FETD can be satisfied when the number of extracted rows is greater than 40. The variation of recovery error from 1200th to 2600th time step with 50 extracted rows is depicted in Fig. 10. The resonant frequencies of this inhomogeneous cavity are shown in Fig. 11. The effectiveness can be observed by the contrast of computing time listed in Table 3. The advantages of the CS-FETD scheme are further confirmed by these charts.

Table 3. Computing time of inhomogeneous cavity.

	f/d	NO. steps	Physical Time
FETD	540/540	20000	4.179e3 s
CS-FETD	50/843	20000	2.347e3 s

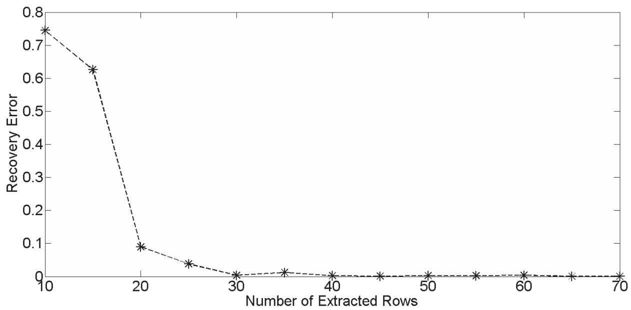

Figure 9. Relationship between recovery error and number of extracted rows at 3000th time step.

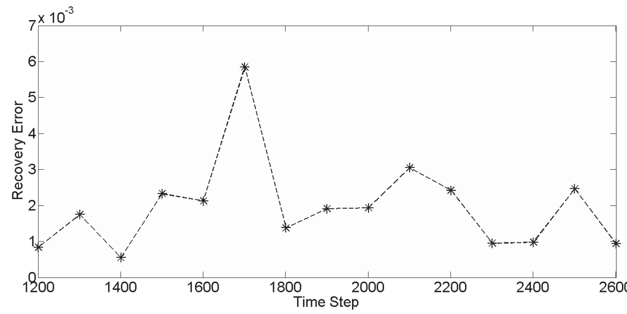

Figure 10. Recovery error changes.

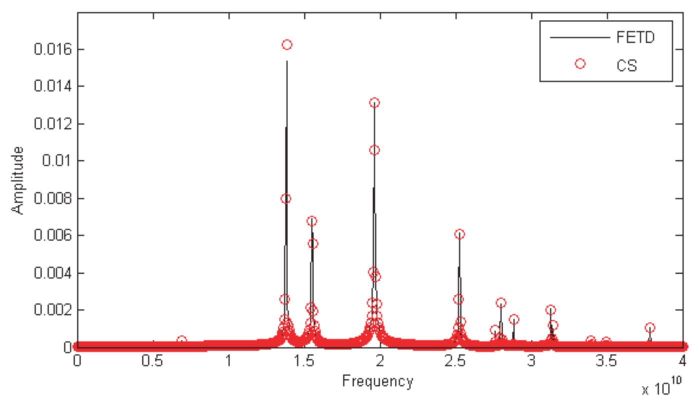

Figure 11. Comparison of resonant frequencies.

4. CONCLUSION

An improved scheme for accelerated FETD method based on CS theory is proposed, in which a low rank measurement matrix is extracted from mass matrix in FETD at each time step, and recovery algorithm is used instead of traditional iterative solution of matrix equations. Meanwhile, with the help of redundant dictionary formed by prior knowledge included in solutions of all previous time steps, the matrix dimension can be reduced drastically.

Numerical results have shown that the proposed method can reduce computing time without accuracy loss. Especially, it is worth mentioning that the proposed scheme can also be used to accelerate other time domain methods as there is a matrix equation to be solved.

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