An Axisymmetric Cylindrical Resonating Cavity with Hole

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Abstract—The problem of the shift and broadening of the normal modes of electromagnetic oscillations in a cylindrical cavity resonator with axisymmetric interior and ideally conducting walls with a circular hole at the base is solved. It is shown that the existence of the hole perturbs the normal frequencies, and this perturbation is calculated. The method of solution is based on the Rayleigh-Schrödinger perturbation theory. It is found that the frequency shift depends on the value of the perturbed electric field at the hole. This field is calculated using the quasistatic approximation, which involves the solution of a mixed boundary value problem for the potential. An expression for the frequency shift and broadening is obtained.

1. INTRODUCTION

I consider the problem of an axisymmetric cylindrical cavity resonator of radius a that is coupled to the outside via a circular aperture of radius b at its cap. The material outside the cylinder is considered to be the same as the material inside; i.e., an axisymmetric lossless dielectric. I take the hole's axis and the axis of dielectric symmetry to be the same as the z-axis and to coincide with the axis of the cylindrical cavity. The problem that I wish to solve is the calculation of the shift and broadening of the normal mode frequencies due to the existence of the hole.

The interest in this problem is due to the experiments of Strayer et al. [1–3], Iny and Barmatz [4, 5], and Dick and Santiago [6, 7] on superconductor-coated microwave cavity resonators with sapphire interiors. The aim of these experimental researches has been the development of stable microwave frequency standards with very high quality factor. In these researches the experimental setup was more complex than is treated in this paper but they typically consisted of cylindrical cavity resonators that, depending on the experiments, were placed one inside the other, or were coupled to each other, or were used singly. For example, in [1] measurements were performed with a Pb-on-sapphire cavity that was placed inside a Pb-on-copper cavity and also with a Pb-on-sapphire cavity that was mounted in a bare copper cavity. To interpret the results of these experiments one has to understand the underlying boundary-value problems. This paper presents a solution to one of the simplest boundary value problems that arises in one of the simplest building blocks of these experiments.

A different but related area of research has been the computation of the effects of the sample insertion holes in the well-known field of dielectric measurements by means of cavity perturbation method. This area has been treated by many investigators including Estin and Bussey [8], Meyer [9], Thomassen [10], Li and Boisiso [11], and Gauthier et al. [12]. These authors generally consider cylindrical cavities with 2 insertion tubes with homogeneous isotropic interiors. Gautheir considers layered inhomogeneous but isotropic (within each layer) samples that are lossy. All these authors assume that the fields inside the cavity are perturbed due to presence of the insertion tubes, and they fully treat each insertion hole as a coupling between the unperturbed resonating cavity fields and a waveguide. Estin and Bussey and Meyer assume that the field inside the tubes is well approximated by the first evanescent TM mode. Li and Boisiso allow a large number of modes in the tubes. Estin

and Bussey and Li and Bisiso use adiabatic invariance theorem to compute the magnitude of the shift in the lowest mode frequencies. I am, on the other hand, considering a single excitation aperture and with a lossless anisotropic interior. I further differ with these authors in that I model the hole's effect on the cavity modes by computing the quasic-static fields at the hole and then obtaining the frequency shift from a perturbative expansion of the cavity fields directly.

The outline of this paper is as follows: In Section 1, I write the basic equations for an axisymmetric medium and the appropriate boundary conditions. In Section 2, I construct the general solution of the axisymmetric wave equation inside the cylinder and give the equations for the normal mode frequencies. The existence of the hole perturbs the normal modes. The effect of this coupling is discussed in Section 3 where I shall draw heavily on the results of [16] to derive a formula for the shift and broadening of the axisymmetric cylindrical cavity field frequencies due to this hole. And in section Section 4, I discuss the results of numerical calculation of this shift in comparison to the results of other workers in this field.

2. NORMAL MODES OF THE CYLINDRICAL CAVITY

I consider the normal modes of electromagnetic oscillations inside a cylinder of radius a and height d without the hole. I take the origin of the the rectangular coordinate system to coincide with the center of the base circle. The basic equations are the macroscopic Maxwell equations, and I shall write them in Gaussian units. For fields varying harmonically in time $[\vec{E}(\vec{r}, t) = \vec{E}(\vec{r})e^{-i\omega t}]$ they are:

$$\vec{\nabla} \times \vec{E}(\vec{r}) - i\frac{\omega}{c}\vec{B}(\vec{r}) = 0,$$
$$\vec{\nabla} \times \vec{B}(\vec{r}) + i\frac{\omega}{c}\vec{D}(\vec{r}) = 0, \tag{1}$$

where $\vec{B}$ is the magnetic field, $\vec{E}$ the electric field, and $\vec{D}$ the electric displacement field. These equations are completed when the dielectric relation between $\vec{E}$ and $\vec{D}$ field is specified; i.e.,

$$\vec{D} = \overleftrightarrow{\epsilon} \cdot \vec{E} \tag{2}$$

The dielectric tensor appearing in the dielectric relation (2) is assumed to be of the form:

$$\overleftrightarrow{\epsilon} = \begin{pmatrix} \epsilon_{xx} & 0 & 0 \\ 0 & \epsilon_{xx} & 0 \\ 0 & 0 & \epsilon_{zz} \end{pmatrix}. \tag{3}$$

This may, in general, be a complex function of frequency, ω. Since I am interested in a transparent media, I take the dielectric tensor to be real and independent of ω for the frequencies of interest. It is convenient to express the dielectric relation in the axisymmetric case as:

$$\vec{E} = \overleftrightarrow{\epsilon}^{-1} \cdot \vec{D} = (\vec{D} + \widetilde{\gamma}\widehat{z} \cdot \vec{D} \ \widehat{z})/\widetilde{\epsilon}, \tag{4}$$

where

$$\widetilde{\epsilon} = \epsilon_{xx}, \widetilde{\gamma} = \frac{\epsilon_{xx} - \epsilon_{zz}}{\epsilon_{zz}}. \tag{5}$$

Inside the dielectric, $\vec{D}$ must satisfy the equation obtained by eliminating $\vec{B}$ from Equation (1) and invoking Equation (4), i.e.,

$$\vec{\nabla} \times \vec{\nabla} \times (\vec{D} + \widetilde{\gamma}\widehat{z} \cdot \vec{D} \ \widehat{z}) - q_0^2\vec{D} = 0 \tag{6}$$

where $q_0^2 = \widetilde{\epsilon}(\frac{\omega}{c})^2$. I shall call Equation (6) the axisymmetric wave equation.

In this paper, I consider the cavity walls to be ideal conductors. The boundary conditions at the surface of the cavity will then follow from the Maxwell Equation (1) by standard arguments [13]. At the surface the tangential component of $\vec{E}$ must vanish to avoid having infinite surface currents. For the same reason, the normal component of $\vec{B}$ must vanish at the surface. Thus:

$$\widehat{n} \times \vec{E}|_{surface} = 0 \quad \text{or} \quad \widehat{n} \cdot \vec{B}|_{surface} = 0, \tag{7}$$

where $\widehat{n}$ is a unit normal at the surface.

Introducing the usual separation of the fields into components parallel to and transverse to the z-axis: $\vec{D} = \vec{D}_{||} + \vec{D}_{\perp}$, etc. where $\vec{D}_{||} = \hat{z} \cdot \vec{D}$, I can solve the axisymmetric wave equation (Equation (6)) with the boundary conditions in Equation (7) in a straightforward fashion in cylindrical coordinates [13]. The results are the usual TM and TE modes but with modifications due to the presence of the anisotropic dielectric. The TM modes are given by:

$$\vec{D}_{||} = J_m(q\rho) \exp(ikz) \exp(im\phi) \, \hat{z},$$

$$\vec{D}_{\perp} = \frac{ik}{q} \exp(ikz) \left\{ J'_m(q\rho)\hat{\rho} + im\frac{J_m(q\rho)}{q\rho}\hat{\phi} \right\} \exp(im\phi),$$

$$\vec{B}_{\perp} = i\frac{\omega}{cq} \exp(ikz) \, \hat{z} \times \left\{ J'_m(q\rho)\hat{\rho} + im\frac{J_m(q\rho)}{q\rho}\hat{\phi} \right\} \exp(im\phi),$$

(8)

where $q^2 = \frac{q_0^2 - k^2}{1 + \tilde{\gamma}}$ and $k = p\pi/d$, $p = 0, 1, 2, \ldots$, $J_m(q\rho)$ is the cylindrical Bessel function [14], and ρ, ϕ, z are the cylindrical coordinates. For the TM modes the magnetic field has no z component but the electric displacement vector has components both parallel and transverse to the z-axis. Thus the electric displacement field will change as $\tilde{\gamma}$ is varied, and the corresponding TM normal mode frequencies will be dependent on $\tilde{\gamma}$.

The boundary condition require that $J_m(qa) = 0$. The resonance frequencies are given by

$$\sqrt{\tilde{\epsilon}}(\frac{\omega}{c}) = \sqrt{\frac{p^2\pi^2}{d^2} + \frac{(1+\tilde{\gamma})x_{mn}^2}{a^2}} \quad p = 0, 1, 2, , \ldots, \quad m = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots,$$

(9)

where $x_{mn} = qa$ is the n-th root of $J_m(x)$. The TM resonant frequencies are dependent on $\tilde{\gamma}$ and vary as $\tilde{\gamma}$ is varied. This is because the electric field has a component along the z-axis and, therefore, is affected by the existence of the dielectric anisotropy along that axis. For TM modes there exists normal mode frequencies (for $p = 0$) that go to zero as $\tilde{\gamma} \to -1$. In the TM case there are no frequencies that are independent of $\tilde{\gamma}$. Moreover, at the limit of $\tilde{\gamma} = -1$, the modes coalesce into points that are distances π apart. (I have considered this limit because for $\tilde{\gamma} \leq -1$ some of the TM normal mode frequencies become imaginary; i.e., the interior of the cavity behaves as a metal for these wave numbers.) The TM fields may be labeled as TM_{mnp}

The TE modes are given by:

$$\vec{B}_{||} = -iJ_m(q\rho) \exp(ikz) \exp(im\phi) \, \hat{z},$$

$$\vec{D}_{\perp} = \left(-\frac{\omega}{c}q\right) \left(\frac{\tilde{\epsilon}}{1+\tilde{\gamma}}\right) \exp(ikz)\hat{z} \times \left\{ J'_m(q\rho)\hat{\rho} + im\frac{J_m(q\rho)}{q\rho}\hat{\phi} \right\} \exp(im\phi),$$

$$\vec{B}_{\perp} = \frac{k}{q} \left(\frac{\tilde{\epsilon}}{1+\tilde{\gamma}}\right) \exp(ikz) \left\{ J'_m(q\rho)\hat{\rho} + im\frac{J_m(q\rho)}{q\rho}\hat{\phi} \right\} \exp(im\phi),$$

(10)

where $q^2 = q_0^2 - k^2$ and $k = p\pi/d$, $p = 1, 2, \ldots$. The TE modes are those for which the electric displacement field lies in the xy-plane, and therefore it will not change as $\tilde{\gamma}$ is varied. For these modes the electric displacement field experiences an effective dielectric constant $\tilde{\epsilon}$. In this respect, it is as though this field is in an isotropic cavity. The electric field lines would be closed curves lying parallel to the xy-plane. As a consequence, the normal mode frequencies will be constants.

For the TE modes, the boundary condition require that $J'_m(qa) = 0$ and the resonant frequencies are given by

$$\sqrt{\tilde{\epsilon}} \left(\frac{\omega}{c}\right) = \sqrt{\frac{p^2\pi^2}{d^2} + \frac{x\prime_{mn}^2}{a^2}} \quad p = 1, 2, , \ldots, \quad m = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots,$$

(11)

where $x\prime_{mn} = qa$ are the roots of $J'_m(x)$. Note that $p = 0$ does not appear in this equation since that corresponds to no fields inside the cavity. I write TE_{mnp} for a mode with azimuth number m, number of radial nodes $n + 1$, and number of axial nodes $p + 1$.

The cavity fields satisfy the usual orthogonality conditions of Sturm-Liouville boundary value problems. In [16], Appendix B, I demonstrated the orthogonality in normal mode frequency ω for a cavity of arbitrary shape. There I also showed that the normalization constant is:

$$N_\omega^{mp \, 2} = \int d\vec{r} (\vec{E}_\omega^{mp})^* \cdot \vec{D}_\omega^{mp} = \int d\vec{r} (\vec{B}_\omega^{mp})^* \cdot \vec{B}_\omega^{mp}.$$

(12)

The orthogonality in m is a consequence of the orthogonality in $\exp(im\phi)$. The orthogonality in p is a consequence of the orthogonality properties of the trigonometric functions. The orthogonality in n follows from the orthogonality properties of Bessel functions [15]. Inserting the expressions (8) and (9) for the TM and TE magnetic fields in Equation (11) I get:

$$N_\omega^{mp\ 2} = \frac{\pi d a^2 \omega^2}{2 q^2 c^2} J_{m+1}^2 (qa) \left\{ \begin{array}{ll} 1 & \text{where } TM \quad (J_m(qa) = 0) \\ \left(\dfrac{\widetilde{\epsilon}}{1+\widetilde{\gamma}}\right)^2 & \text{where } TE \quad (J_m'(qa) = 0) \end{array} \right\}. \tag{13}$$

I note here that the TM and TE fields doubly degenerate for $m = \pm|m|$ and that $N_\omega^{mp\ 2} = N_\omega^{-mp\ 2}$.

3. EFFECT OF THE HOLE

In order to excite these frequencies, a hole must be made in the surface of the cavity to allow for the coupling with external fields. The existence of this hole perturbs the normal modes and will split the $m = \pm|m|$ modes. In [16], an analogous problem for an axisymmetric spherical cavity resonator was solved. The method of solution was based on the Rayleigh-Schrödinger perturbation theory. It was found that the frequency shift depended on the value of *only* the perturbed electric field at the hole. The field was calculated using the quasistatic approximation, which involved the solution of a mixed-boundary-value problem for the potential.

The hole's radius b is considered small in comparison with the dimensions of the cavity, a and d; see Figure 1. The material outside the cylinder is considered to be the same as the material inside; i.e., an axisymmetric lossless dielectric. I take the hole's axis and the axis of dielectric symmetry to be the same as the z-axis. Naively, one would expect that since the radius of the hole is assumed to be much smaller than that of cavity's dimensions, the effect of the coupling may be treated perturbatively. This assumption is always satisfied when the hole's diameter is much smaller than the resonant wavelength in the dielectric or when the following inequality is satisfied:

$$\frac{\sqrt{\pi^2 p^2 \dfrac{b^2}{d^2} + (1+\widetilde{\gamma}) x_{mn}^2 \dfrac{b^2}{a^2}}}{2\pi\sqrt{\widetilde{\epsilon}}} \ll 1 \text{ with } p = 1, 2, \ldots, \quad m = 0, 1, 2, \ldots, \quad n = 1, 2, \ldots \tag{14}$$
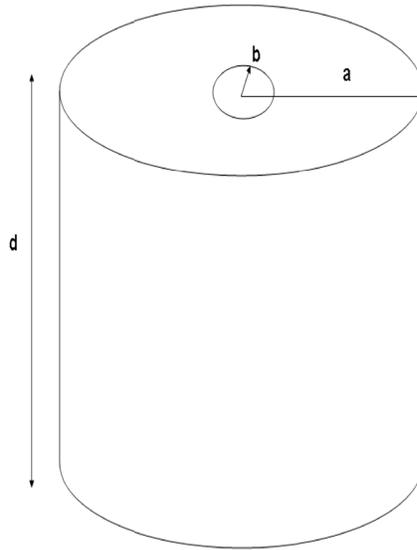
Figure 1. The geometry of the problem with the cylinder of radius a, height d, and excitation hole of radius b.

Clearly, for very thin cavities for which $\frac{d^2}{b^2} \ll 1$ and with large values of parameter p, or large values of the roots of the Bessel function x_{mn} and parameter $\widetilde{\gamma}$, (or a combination thereof) this inequality could be violated and perturbation theory will fail.

Since the hole is small, the fields may be considered to be approximately constant over the hole. I shall use an adaptation of degenerate perturbation theory to the perturbation of the boundary conditions. My treatment is similar to the one given by Goubau [17] and is different than the Galerkin method employed in [11]. The starting point of Goubau is to treat the effect of small apertures in wave guides and resonating cavities by writing the physical quantities of interest as a series expansion of small corrections to their unperturbed quantities. In case of the correction terms, no assumptions are made on the nature of these functions beyond continuity, differentiability, smoothness and so on. In fact, the expansion terms do not even have to satisfy Maxwell's equations. Additionally, within the limit of applicability of the perturbation theory, this treatment is exact. Li and Bosisio, (in [11]) on the other hand, approximate the perturbed fields by a linear combination of the lowest modes of the cavity. This is a trial solution with unknown coefficients with a set of complete functions (in the sense of function spaces) as the basis vectors. Next, they use the orthogonality properties of these basis functions and the boundary conditions to derive an infinite set of linear algebraic equations for the expansion coefficients. They then solve these equations numerically.

I begin by noting that within the cavity both the perturbed and unperturbed fields satisfy the Maxwell Equation (1). The boundary conditions satisfied by the unperturbed fields are: $\widehat{n} \times \vec{E}|_{surface} = 0$, $\widehat{n} \cdot \vec{B}|_{surface} = 0$ where $\widehat{n}$ is a unit normal to the surface. The perturbed fields, on the other hand, satisfy the boundary conditions in Equation (15) everywhere on the surface of the cavity except in the hole; in the hole, the electric field, $\vec{E}$, and the magnetic field, $\vec{B}$, must be continuous. Note that the unperturbed fields vanish outside the cavity. In the subsequent development, the subscript 0 denotes the unperturbed quantities.

The unperturbed cavity modes are labeled by m, p, ω. There is no degeneracy for $m = 0$. The perturbed quantities are assumed to have the form:

$$
\begin{aligned}
\omega &= \omega_0 + \sum_n \omega(n). \\
\vec{D} &= a_m \vec{D}_0^{mp} + a_{-m} \vec{D}_0^{-mp} + \sum_n \vec{D}(n) \\
\vec{E} &= a_m \vec{E}_0^{mp} + a_{-m} \vec{E}_0^{-mp} + \sum_n \vec{E}(n) \\
\vec{B} &= a_m \vec{B}_0^{mp} + a_{-m} \vec{B}_0^{-mp} + \sum_n \vec{B}(n),
\end{aligned}
\tag{15}
$$

where the coefficients a_m and a_{-m} are as yet unknown.

In [16] I used the above expansions to derive the following consistency condition relating the perturbed and unperturbed fields.

$$
\int_{hole} \vec{B}_0^{m'p*} \cdot \widehat{n} \times \vec{E}^{(1)} ds = 2i \frac{\omega^{(1)} N_{\omega_0}^{mp2}}{c} \times (a_m \delta_{mm'} + a_{-m} \delta_{-mm'}).
\tag{16}
$$

(An important condition for the validity of the derivation above has been the fact that I have taken the dielectric tensor to be real.) This is a pair of equations that connects the inhomogeneous terms, $a_m \vec{E}_0^{mp}$, etc., with the frequency $\omega^{(1)}$ and with the solutions at the boundary. To proceed further I need to calculate the electric field perturbation $\vec{E}^{(1)}$ in the neighborhood of the hole.

Consider an ideally conducting screen in the xy-plane, with a hole of radius b centered at the origin. The unperturbed electric field $\vec{E}_0$ is uniform and constant below the hole and vanishes above the hole. The problem is then to determine the electric field in the hole; with the boundary conditions that in the hole it is to be continuous and on the screen its tangential component must vanish. In addition, I require that this solution asymptotically vanish above the screen; and below the screen, to asymptotically approach the unperturbed cavity fields. The analogous problem of the diffraction of

electromagnetic waves by a perfectly absorbing plane screen was treated in detail by Stokes [18] in 1849 and by Lorenz [19] in 1860. An approximate treatment for a perfectly conducting screen was given by Lord Rayleigh [20]. In 1944, Bethe [21] gave a detailed analysis of the diffraction of the electromagnetic radiation by a circular hole small compared with the wavelength. The most accurate treatment of this problem, with extensive criticism of previous work in this area, was given by Bouwkamp [22].

I am aided by the assumption that the hole is small compared to the wavelength, that is $b \ll c/\omega$. This allows me to use the quasistatic approximation [23, 24]. In applying the quasistatic approximation, I am assuming that at the vicinity of the hole, all effects due to the finite velocity of propagation of electromagnetic waves may be neglected. This means that I am neglecting the displacement current in the Ampére's Law. Nevertheless, the approximation is still valid for radiating systems since the fields at short distances from the hole, the near fields, are always the quasistatic fields. In [16] I solved this problem and showed that the electric field in the hole is given by:

$$\vec{E}^{(1)}(\vec{\rho},0) = \left(\frac{E_0}{\pi} \right) \left(\sqrt{\frac{\epsilon_{zz}}{\epsilon_{xx}}} \frac{\vec{\rho}}{\sqrt{b^2 - \rho^2}} + \frac{\pi}{2} \widehat{z} \right). \tag{17}$$

I am now in the position to calculate the shift and broadening of the normal mode frequencies due to the presence of the hole. I go back to the consistency condition, Equation (16). I consider the case in which hole is at the top of the cylinder. My solution consists of several stages; First I replace the incident field in Equation (16) with Equation (17). I then apply Stoke's theorem to the integration over the surface of the hole. I follow that by an application of Ampére's Law to the result and, at the same time, apply the assumption that the unperturbed fields are uniform over the surface of the hole and thus may be removed as integrands. I arrive at:

$$\frac{\omega_0 b^3}{3} \sqrt{\frac{\epsilon_{zz}}{\epsilon_{xx}}} D_0^{m'p'*}(a_m E_0^{mp} + a_{-m} E_0^{-mp}) = \omega^{(1)} N_{\omega_0}^{mp2}(a_m \delta_{mm'} + a_{-m} \delta_{-mm'}). \tag{18}$$

In Equation (18), I first let $m' = m$, and then I put $m' = -m$. The result is a pair of coupled, linear, and homogeneous algebraic equations for the coefficients a_m and a_{-m}. For non-trivial solutions to exist the determinant must vanish. The vanishing of the determinant leads to two solutions for $\omega^{(1)}$:

$$\omega^{(1)} = 0, \quad \omega^{(1)} = \frac{\omega_0 b^3}{3 N_{\omega_0}^{mp2}} \sqrt{\frac{\epsilon_{zz}}{\epsilon_{xx}}} [D_0^{mp*} E_0^{mp} + D_0^{-mp*} E_0^{-mp}]. \tag{19}$$

The frequency perturbation, $\omega^{(1)}$ is complex. The shift in the normal mode is given by the real part of expression (19), and the decay rate is computed from its imaginary part.

Next, I apply these results to the calculation of the shift and broadening of the TM modes of the cylindrical resonator. I approximate the fields at the hole by evaluating them at $\rho = b$, $z = d$, and for arbitrary ϕ. I get:

$$\Delta\omega_{mnp} = \frac{2c(b/a)^3}{3\pi\sqrt{\epsilon_{zz}}} \frac{x_{mn}^2}{\sqrt{\pi^2 a^2 p^2 + (1+\widetilde{\gamma})d^2 x_{mn}^2}} \frac{J_m^2(x_{mn}b/a)}{J_{m+1}^2(x)}, \tag{20}$$

which is completely real. This indicates that, to the first order in perturbation theory, there are no losses associated with the existence of this hole. This is the main result of this paper.

4. NUMERICAL RESULTS

I have evaluated $\Delta\omega$ for the lowest $m = 0$, $n = 1$, $p = 0$ TM mode of the cavity, i.e., TM_{010}. I obtain:

$$\Delta\omega_{010} = \frac{2c(b/a)^3}{3\pi d\sqrt{\epsilon_{zz}}} \frac{x_{01}}{\sqrt{(1+\widetilde{\gamma})}} \frac{J_0^2(x_{01}b/a)}{J_1^2(x_{01})}. \tag{21}$$

I have compared the result in Equation (21) above with the resonant frequency pulling derived in [11], Equations (8) and (9). Due to differences between the physical setup and the mathematical methods employed, an exact comparison of the results is not possible. However, one may note that these expressions are in general qualitative agreement with Equation (21); they display a dependence

on the ratio of Bessel functions of orders zero and one, the cube of ratio of hole radius to the cylinder radius, and are inversely proportional to the height of the cylinder.

I have used Maple Software to compute the formula (21) with $d = 2a$, $b = 0.1a$, and $a = 2.5$ cm. I have chosen the usual values of the dielectric constant for sapphire [25]; $\epsilon_{xx} = 8.6$, $\epsilon_{zz} = 10.55$. With these choices of the parameters, the TM_{010} frequency is $\nu_0 = 1.3772$ GHz, and the corresponding shift in frequency is $\Delta\nu = 6.2174 \times 10^{-4}$ GHz.

I have plotted the frequency shift as the dimensions of the cavity and the amount of dielectric anisotropy are varied for the lowest x_{01} mode in units of $\sqrt{\epsilon_{zz}}a$. In Figure 2, I have plotted the variation of the frequency shift as a function of the ratio of the hole radius to the cylinder radius and the parameter $\widetilde{\gamma}$. As $\widetilde{\gamma}$ is increased from a value of -1, indicating a metallic interior, the amount of the shift decreases. But as the ratio of the hole radius to cavity radius increases, the frequency shift increases until it reaches a maximum. After that, further increase in the b/a ratio will cause the frequency shift to decrease. The appearance of this maximum is not due to the failure of the perturbation theory, but it is a consequence of the Bessel functions in the frequency shift formula. After that maximum is reached, further increase in the ratio b/a tends to decrease the frequency shift until a value of $b/a = 1$ is reached, and the frequency shift becomes zero. However, much earlier than that, I expect that the perturbation theory and this whole approach to fail since the physical situation would then correspond to a rather large hole in a resonating cavity, and the entire program needs to be revisited.
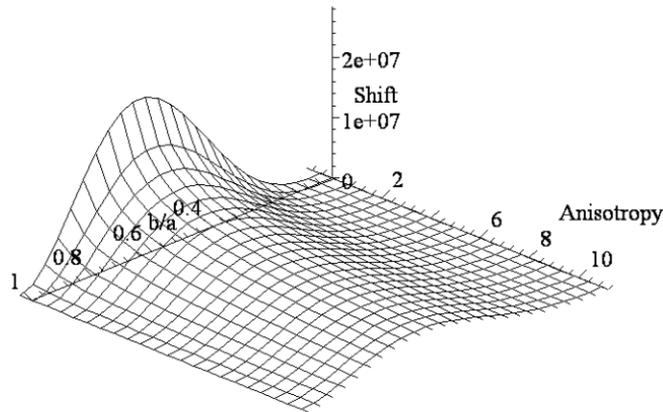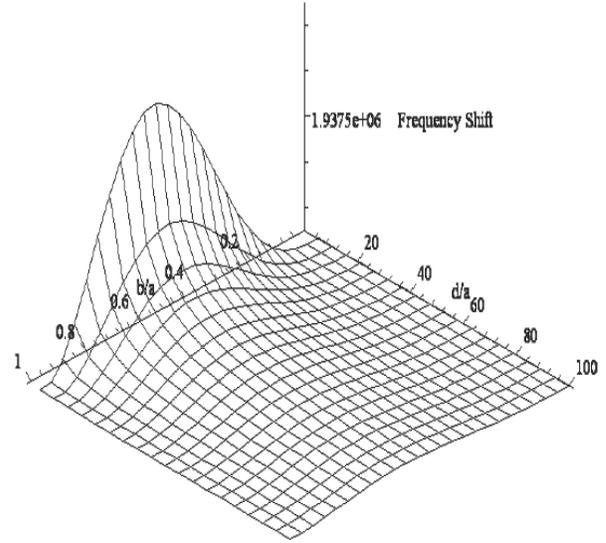

Figure 2. $\Delta\omega$ vs. b/a and $\widetilde{\gamma}$ with $d = 2a$, $x_{01} \equiv 2.40482$ in units of $\sqrt{\epsilon_{zz}}a$.

In Figure 3, I have plotted the same quantities but with a cavity height that is 10 times as large as the one in Figure 2. In this figure, and in Figure 4, where I have plotted the frequency shift against the dimensions of the cavity and the hole for a fixed value of the dielectric constant, I observe the same overall behavior as Figure 2. That is, the amount of frequency shift decreases as the dimensions of the cavity are increased (hole's size being kept fixed). This is to be expected since the shift in frequency is an indication of the ratio of energy loss through the hole to the total electromagnetic energy stored in the cavity. So as the size of the cavity is increased compared to the size of hole, proportionally less and less of the energy is lost. I note here that the peak in the frequency shift is observed in Figures 3 and Figure 4, just as in Figure 2.

There are a number of directions that this work can be extended. The simplest extension will be to include off-diagonal terms in the dielectric tensor. The calculation may also be extended to complex permittivity. And finally, the problem of a hole on the side of the cylindrical cavity may be treated for TE modes. Another direction for further extension of this work is the modeling of the experimental setup in which a cylindrical sapphire core is enclosed by a metallic (niobium or lead) cylinder that is larger than the sapphire core and has two holes at top and bottom.
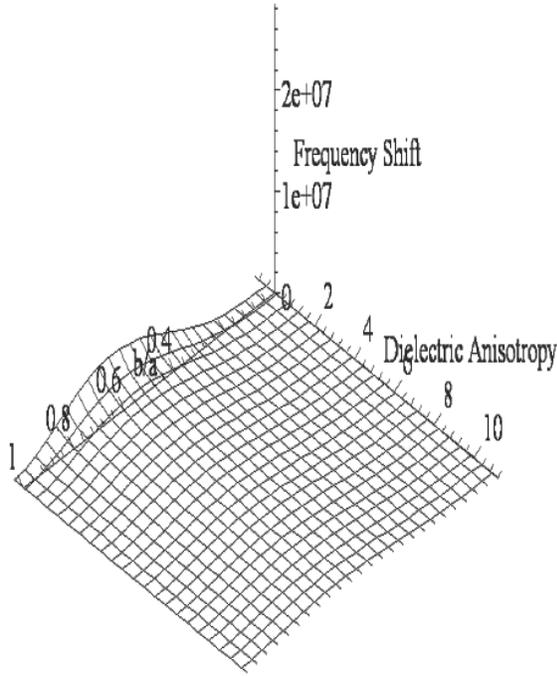
Figure 3. $\Delta\omega$ vs. b/a and $\widetilde{\gamma}$ with $d = 10a$, $x_{01} \equiv 2.40482$ in units of $\sqrt{\epsilon_{zz}}a$.

Figure 4. $\Delta\omega$ vs. b/a and d/a with $\widetilde{\gamma} = -0.1848$, $x_{01} \equiv 2.40482$ in units of $\sqrt{\epsilon_{zz}}a$.

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