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Comparing Different Schemes for Random Arrays

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Abstract—In this work, four types of random arrays are compared. In particular, the mean and variance of the array factor are derived. This provides a partial statistical characterisation that allows pointing out some important aspects of random arrays and link them to the number of elements and array aperture. In the absence of a simple and effective analytical apparatus, here great importance is also given to the experimental aspect, especially as far as the side-lobe level is concerned. To this end, Monte Carlo simulations are run to experimentally build the side-lobe distribution as a function of the number of radiators and the average spacing between two adjacent radiators. The obtained results show that random arrays where one is free to impose constraints on the minimum spacing between adjacent elements can obtain performance analogous to those achievable by other schemes which do not put such constraints. However, the former are preferable because they are able to zero the probability that adjacent radiators are separated with less than a certain minimum distance, which allows the mitigation of mutual coupling effects.

1. INTRODUCTION

In aperiodic antenna arrays, the spacing between the elements is chosen to be *incommensurable*. This distinctive feature offers a number of advantages as compared to equally-spaced arrays. Grating-lobes are in principle avoided according to nonuniform sampling theory [1, 2]. Hence, large scan angles are possible, and/or wide frequency ranges can be covered. This makes somehow the spatial non-uniformity as a synonymous of broadband functionality [3–5]. Again, especially for large aperture, the number of radiators can be reduced to a large extent without causing significant performance degradation, mainly as far as resolution is concerned. Accordingly, the average distance between adjacent radiators is increased, which makes the assumption of negligible mutual couplings more valid [6]. By reversing the perspective, having fixed the number of available radiators, they can be located on average more distant so to have an overall larger aperture. This in turn narrows the main-beam and hence leads to an improvement in resolution. Actually, the bandwidth-steerability product can be made much larger than for conventional equally-spaced arrays [5]. Finally, deploying the elements non-uniformly over the array aperture allows the *control* of side-lobe level without the need to taper someway the excitation currents. Accordingly, all the radiators can be uniformly excited so that the amplifiers can all work at maximum power, and this results in the simplification of the feeding network [7]. Actually, for aperiodic equally excited arrays the parameter that most influences the side-lobe level is the number of radiators [6, 8]. Eventually, it can be stated that while designing aperiodic arrays one has more *spatial* degrees of freedom allowed (i.e., all the positions of radiators instead of only the uniform step), and this in principle offers greater possibility for obtaining high performance arrays. Moreover, in many practical cases, the geometry of the array must necessarily be irregular [7].

Previous discussion justifies why aperiodic arrays have been the subject of a large body of research since long time [3, 8–14], and a number of ways to attack the analysis and synthesis of such a type of arrays have been proposed [15–25].

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In this paper, we focus on a special class of aperiodic array: the random arrays. Random arrays are those for which the positions of radiators are chosen according to some probabilistic law. This naturally leads to a nonuniform element arrangement (i.e., with probability one). The corresponding array factor is thus a stochastic process whose features (if determined) allow for an a priori, though *statistically*, performance analysis of this type of aperiodic arrays [26]. In this regard, the seminal paper of Lo [8] was the first at introducing a systematic theory of random arrays. Nonetheless, this issue is still an open problem. Indeed, while determining the array factor statistics up to the second order can be easily done, characterising the side-lobe level is a much harder problem. Actually, determining the side-lobe distribution requires finding that one of the supremum of the array factor magnitude. To cope with this problem the sampling [8, 27] and the level crossing approaches [28] are among the most used methods. More rigorously, the extreme value theory should be invoked [29]. However, this is not the subject of this paper.

In this paper, the aim is to compare different strategies for randomly generating the positions of radiators. In particular, we compare four types of positions generation rules while the elements are assumed to be equally-excited. Two approaches are borrowed from the array literature according to [8] and [30], respectively. Instead the other two approaches come from the nonuniform sampling theory [1]. For all the methods, the first (mean array factor) and second order (variance) statistics are analytically derived. The side-lobe level distribution is instead studied thanks to Montecarlo simulations.

2. RANDOM ARRAYS UNDER COMPARISON

Consider a linear array of N equally excited isotropic radiators placed at random along the X axis (see Fig. 1). The corresponding (normalised) array factor can be written as [27]

$$F(u) = \frac{1}{N}\sum_{n=1}^{N} e^{j2\pi X_n u} \tag{1}$$

where

- $u = \sin\theta - \sin\alpha$;
- θ is the observation angle from the broad-side;
- α is the scan angle;
- X_n is the location of the nth element normalised to the wavelength.

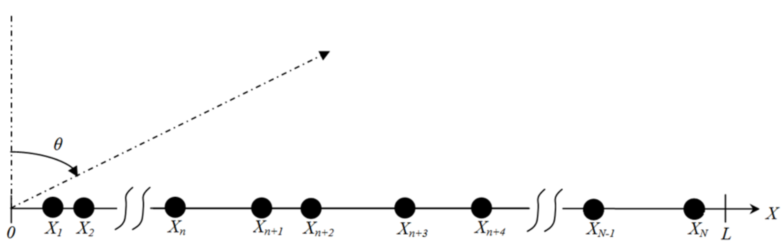

Figure 1. Generic random array where the positions of antenna elements are implicitly ordered.

The positions of radiators are randomly generated. Here, four types of generating rule are analysed.

The first generating rule leads to the so-called *Totally Random Arrays* (*TRA*) [8, 30]. Here, the positions of antenna elements $X_1, X_2, \ldots, X_n$ are *i.i.d.* random variables with probability densities supported over $[0, L]$, L being the length of the array normalised with respect to the wavelength.

The second generation rule is associated with the *Binned Arrays* (*BA*) [30] and can be expressed as follows

$$X_n = (n-1)\frac{L}{N} + Y_n \quad n = 1, 2, \ldots, N \tag{2}$$

wherein $Y_1, Y_2, \ldots, Y_N$ are independent random variables each taking value within the interval $[0, L/N]$. Basically, in this scheme uniform positions are randomly perturbed.

The next rule is similar to the previous one [31, 32]

$$X_n = (n-1)(2\varepsilon + \Delta) + W_n \quad n = 1, 2, \ldots, N \tag{3}$$

where $X_1 = 0$ and $W_1, W_2, \ldots, W_N$ are independent random variables each assuming values in the interval $(-\varepsilon, \varepsilon)$, and Δ is a step parameter. This scheme is similar to some nonuniform sampling method which is called periodic sampling with jitter [1]. For this reason, we address this way to generate random arrays as *Jittered Random Arrays* (*JRA*).

Finally, as a fourth rule the following scheme is considered [32]

$$X_n = X_{n-1} + Z_{n-1} = \sum_{k=1}^{n-1} Z_k \quad n = 2, \ldots, N \tag{4}$$

in which $X_1 = 0$ and $Z_1, Z_2, \ldots, Z_{N-1}$ are independent random variables taking values within the interval $[z_{\min},\ z_{\max}]$. Note that this is another rule used in nonuniform sampling theory which is addressed as additive random sampling [1]. Accordingly, we address this random scheme as *Additive Random Arrays* (*ARA*).

It is noted that these random arrays generally lead to different apertures. For example, the maximum aperture for $TRAs$ and BAs is L, for $JRAs$ is $[(2\varepsilon + \Delta)(N-1) + 2\varepsilon]$ while for $ARAs$ it is $[(N-1)z_{\max}]$. The average distance between adjacent elements can be different as well as shown below.

For BAs, if $Y_1, Y_2, \ldots, Y_N$ are *i.i.d.* random variables, the average spacing between adjacent antenna elements is

$$d_{av} = E[X_n] - E[X_{n-1}] = (n-1)\frac{L}{N} + E[Y_n] - (n-2)\frac{L}{N} - E[Y_{n-1}] = \frac{L}{N} \tag{5}$$

For $JRAs$, if $W_1, W_2, \ldots, W_N$ are *i.i.d.* random variables, the average spacing between adjacent radiators is

$$d_{av} = E[X_n] - E[X_{n-1}] = (n-1)(2\varepsilon + \Delta) + E[W_n] - (n-2)(2\varepsilon + \Delta) - E[W_{n-1}] = (2\varepsilon + \Delta) \tag{6}$$

For $ARAs$, if $Z_1, Z_2, \ldots, Z_N$ are *i.i.d.* random variables, the average spacing between adjacent antenna elements is

$$d_{av} = E[X_n] - E[X_{n-1}] = E[Z] = \int_{z_{\min}}^{z_{\max}} Z\ f(Z)\ dZ \tag{7}$$

In the case of $TRAs$, the element positions are not in general ordered. However, one can resort to the order statistics [33] by putting

$$\begin{aligned}
X_{(1)} &= \min(X_1, X_2, X_3, \ldots, X_N) \\
X_{(2)} &= \min(\{X_1, X_2, X_3, \ldots, X_N\} - \{X_{(1)}\}) \\
&\vdots \\
X_{(n)} &= \min(\{X_1, X_2, X_3, \ldots, X_N\} - \{X_{(1)}, X_{(2)}, \ldots, X_{(n-1)}\}) \\
&\vdots \\
X_{(N)} &= \mathrm{Max}(X_1, X_2, X_3, \ldots, X_N)
\end{aligned} \tag{8}$$

so that $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)} \leq \ldots \leq X_{(N)}$. $X_{(n)}$ is called the nth order statistics whose distribution, since the positions are *i.i.d.*, is given as [33]

$$f_n(x) = \frac{N!}{(n-1)!(N-n)!} F^{n-1}(x)[1 - F(x)]^{N-n} f(x) \tag{9}$$

where $F(x)$ and $f(x)$ are the cumulative distribution function (CDF) and the probability density function (PDF) of the above positions, respectively. If θ_p is the pth quantile of the common CDF of the positions of radiators and $f(\theta_p) \neq 0$, then follows that [34]

$$E[X_{(n)}] \approx \theta_p, \quad p = \frac{n}{N+1} \tag{10}$$

So, in general, $E[X_{(n)}]-E[X_{(n-1)}]\neq E[X_{(m)}]-E[X_{(m-1)}]$, but for uniform distribution

$$E[X_{(n)}]-E[X_{(n-1)}]=\frac{L}{N+1}\quad\forall\, n \tag{11}$$

and then this quantity can be identified as the average spacing, d_{av}, between adjacent radiators†.

Looking at Eqs. (7)–(11), one can also observe that for BAs and $JRAs$ the average spacing between adjacent antenna elements does not depend on the probability distribution, as instead it is for $TRAs$ and $ARAs$. It is also important to remark that while for $TRAs$ and BAs the radiators can be placed at each point within the aperture, no matter how close they are, this does not hold true for $ARAs$ and $JRAs$. Indeed, in BAs two consecutive elements could result in coincidence whereas in $TRAs$ even all the elements could in principle be located at the same position. Instead for $ARAs$, the distance between adjacent radiators cannot be smaller than $z_{\min}$ while in $JRAs$ there are points on the aperture where none antenna element can be placed, i.e., those points that fall within the Δ intervals cannot accommodate any location of radiators. From the point of view of mutual coupling, with $JRAs$ and $ARAs$ one can constraint the minimum distance between two adjacent antenna elements. For example, if d is the minimum acceptable spacing between adjacent radiators, then for $JRAs$ one can set $\Delta\geq d$ whereas for $ARAs$, $z_{\min}=d$.

3. FIRST AND SECOND ORDER CHARACTERISATION

A useful even though partial statistical characterisation of the random arrays (obtained according to the rules reported above) can be easily given. Therefore, here, we derive the mean and variance of the array factors under comparison.

We start with the $TRAs$. In this case, the mean and variance of the array factor are the following

$$\phi_{TRA}(u)=E[F(u)]=\int_0^L f(X)\; e^{j2\pi Xu}dX=\psi_X(u) \tag{12}$$

and

$$\begin{aligned}\sigma^2_{TRA}(u)&=E[|F(u)-\phi_{TRA}(u)|^2]=E[|F(u)|^2]-|\phi_{TRA}(u)|^2\\&=E\left\{\frac{1}{N}\sum_{n=1}^{N}e^{j2\pi X_n u}\cdot\frac{1}{N}\sum_{m=1}^{N}e^{-j2\pi X_m u}\right\}-|\phi_{TRA}(u)|^2\\&=\frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N}E\left\{e^{j2\pi X_n u}\cdot e^{-j2\pi X_m u}\right\}-|\phi_{TRA}(u)|^2\\&=\frac{1}{N}+\sum_{n=1}^{N}\sum_{\substack{m=1\\m\neq n}}^{N}E\left\{e^{j2\pi X_n u}\right\}\cdot E\left\{e^{-j2\pi X_m u}\right\}-|\phi_{TRA}(u)|^2=\frac{1-|\phi_{TRA}(u)|^2}{N}\end{aligned} \tag{13}$$

For BAs, if $Y_1,Y_2,\ldots,Y_N$ are $i.i.d.$ random variables, the mean and variance of the array factor are

$$\begin{aligned}\phi_{BA}(u)&=\frac{1}{N}\sum_{n=1}^{N}\left[e^{j2\pi(n-1)\frac{L}{N}u}\cdot E\left\{e^{j2\pi Y_n u}\right\}\right]=\frac{e^{-j2\pi\frac{L}{N}u}}{N}\cdot\psi_Y(u)\cdot\sum_{n=1}^{N}\left[e^{j2\pi n\frac{L}{N}u}\right]\\&=\frac{e^{j\pi L\frac{N-1}{N}u}}{N}\cdot\psi_Y(u)\cdot\frac{\sin\left(\pi Lu\right)}{\sin\left(\pi\frac{L}{N}u\right)}\end{aligned} \tag{14}$$

† The quantities d_{av} above are all normalised with respect to the wavelength.

whereas

$$\begin{aligned}\sigma_{BA}^2(u) &= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} e^{j2\pi(n-m)\frac{L}{N}u} E\left\{e^{j2\pi Y_n u}\cdot e^{-j2\pi Y_m u}\right\} - |\phi_{BA}(u)|^2 \\ &= \frac{1}{N} + \frac{1}{N^2}|\psi_Y(u)|^2 \sum_{n=1}^{N}\sum_{\substack{m=1\\ m\neq n}}^{N} e^{j2\pi(n-m)\frac{L}{N}u} - |\phi_{BA}(u)|^2 = \frac{1-|\psi_Y(u)|^2}{N} \end{aligned} \tag{15}$$

where $\psi_Y(u) = E\left\{e^{j2\pi Y u}\right\} = \int_0^{L/N} f(Y)\cdot e^{j2\pi Y u}\, dY$ is once again the characteristic function associated to the random variable Y.

Similar results are obtained for *JRAs* after assuming, as done above, that $W_1, W_2, \ldots, W_N$ are *i.i.d.* random variables. Accordingly, the mean and variance of the array factor are

$$\begin{aligned}\phi_{JRA}(u) &= \frac{1}{N}\sum_{n=1}^{N}\left[e^{j2\pi(n-1)(2\varepsilon+\Delta)u}\cdot E\left\{e^{j2\pi W_n u}\right\}\right] = \frac{e^{-j2\pi(2\varepsilon+\Delta)u}}{N}\cdot\psi_W(u)\cdot\sum_{n=1}^{N} e^{j2\pi n(2\varepsilon+\Delta)u} \\ &= \frac{e^{j\pi(N-1)(2\varepsilon+\Delta)u}}{N}\cdot\psi_W(u)\cdot\frac{\sin[\pi N(2\varepsilon+\Delta)u]}{\sin[\pi(2\varepsilon+\Delta)u]}\end{aligned} \tag{16}$$

and

$$\begin{aligned}\sigma_{JRA}^2(u) &= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} e^{j2\pi(n-m)(2\varepsilon+\Delta)u}\cdot E\left\{e^{j2\pi W_n u}\cdot e^{-j2\pi W_m u}\right\} - |\phi_{JRA}(u)|^2 \\ &= \frac{1}{N} + \frac{|\psi_W(u)|^2}{N^2}\cdot\sum_{n=1}^{N}\sum_{\substack{m=1\\ m\neq n}}^{N} e^{j2\pi(n-m)(2\varepsilon+\Delta)u} - |\phi_{JRA}(u)|^2 = \frac{1-|\psi_W(u)|^2}{N}\end{aligned} \tag{17}$$

with $\psi_W(u) = E\left\{e^{j2\pi W u}\right\} = \int_{-\varepsilon}^{\varepsilon} f(W)\cdot e^{j2\pi W u}\, dW$.

Finally, for *ARAs*, if $Z_1, Z_2, \ldots, Z_{N-1}$ are as usual *i.i.d.* random variables, we get

$$\phi_{ARA}(u) = \frac{1}{N}\cdot\sum_{n=1}^{N} E\left\{e^{j2\pi\sum_{k=1}^{n-1} Z_k u}\right\} = \frac{1}{N}\cdot\sum_{n=1}^{N}\left[E\left\{e^{j2\pi Z u}\right\}\right]^{(n-1)} = \frac{1}{N}\cdot\sum_{n=1}^{N}\psi_Z(u)^{(n-1)} \tag{18}$$

and

$$\begin{aligned}\sigma_{ARA}^2(u) &= \frac{1}{N^2}\sum_{n=1}^{N}\sum_{m=1}^{N} E\left\{e^{j2\pi X_n u}\cdot e^{-j2\pi X_m u}\right\} - |\phi_{ARA}(u)|^2 \\ &= \frac{1}{N} + \frac{1}{N^2}\sum_{n=1}^{N}\sum_{\substack{m=1\\ m\neq n}}^{N} E\left\{e^{j2\pi X_n u}\cdot e^{-j2\pi X_m u}\right\} - |\phi_{ARA}(u)|^2 \\ &= \frac{1}{N} + \frac{1}{N^2}\sum_{n=1}^{N}\sum_{\substack{m=1\\ m\neq n}}^{N} E\left\{e^{j2\pi\sum_{k=1}^{n-1} Z_k u}\cdot e^{-j2\pi\sum_{p=1}^{m-1} Z_p u}\right\} - |\phi_{ARA}(u)|^2 \\ &= \frac{1}{N} - |\phi_{ARA}(u)|^2 + \frac{1}{N^2}\sum_{n=1}^{N}\sum_{\substack{m=1\\ m<n}}^{N} E\left\{e^{j2\pi\sum_{k=1}^{n-1} Z_k u}\cdot e^{-j2\pi\sum_{p=1}^{m-1} Z_p u}\right\} \\ &\quad + \frac{1}{N^2}\sum_{n=1}^{N}\sum_{\substack{m=1\\ m>n}}^{N} E\left\{e^{j2\pi\sum_{k=1}^{n-1} Z_k u}\cdot e^{-j2\pi\sum_{p=1}^{m-1} Z_p u}\right\} \\ &= \frac{1}{N} - |\phi_{ARA}(u)|^2 + \frac{1}{N^2}\sum_{n=1}^{N}\sum_{\substack{m=1\\ m<n}}^{N}\psi_Z(u)^{(n-m)} + \frac{1}{N^2}\sum_{n=1}^{N}\sum_{\substack{m=1\\ m>n}}^{N}\psi_Z^*(u)^{(m-n)}\end{aligned} \tag{19}$$

with $\psi_Z(u) = E\{e^{j2\pi Zu}\} = \int_{z_{\min}}^{z_{\max}} f(Z) \cdot e^{j2\pi Zu}\, dZ$.

The central role played by the characteristic functions must be noted.

In order to perform a comparison, we take the same cases as addressed in [30]. The array consists of 100 radiators and has nominal aperture of 400λ. For JRA and ARA, we can set the minimum acceptable distance between adjacent elements. According to common usage for avoiding mutual coupling, this distance is chosen equal to $\lambda/2$. Therefore, $\Delta = 1/2$ and $z_{\min} = 1/2$, respectively. Moreover, since the nominal aperture is 400λ, $\varepsilon = 1.7525$ and $z_{\max} = 4.0404$ for JRA and ARA, respectively. Furthermore, for TRA, the variables $X_1, X_2, \ldots, X_N$ are chosen to be uniformly distributed within $[0, L]$; for BA, the variables $Y_1, Y_2, \ldots, Y_N$ are uniformly distributed within $[0, L/N]$; for JRA, the variables $W_1, W_2, \ldots, W_N$ are uniformly distributed within $[-\varepsilon, \varepsilon]$, and finally for ARA, the variables $Z_1, Z_2, \ldots, Z_{N-1}$ are uniformly distributed within $[z_{\min}, z_{\max}]$. This way, the mean array factors are specialised as

$$\phi_{TRA}(u) = \phi_{BA}(u) = e^{j\pi Lu} \cdot \frac{\sin(\pi Lu)}{\pi Lu} \tag{20}$$

$$\phi_{JRA}(u) = \frac{e^{j\pi(N-1)(2\varepsilon+\Delta)u}}{N} \cdot \frac{\sin(2\pi\varepsilon u)}{2\pi\varepsilon u} \cdot \frac{\sin[\pi N(2\varepsilon+\Delta)u]}{\sin[\pi(2\varepsilon+\Delta)u]} \tag{21}$$

$$\phi_{ARA}(u) = \frac{1}{N} \cdot \sum_{n=1}^{N} \left[\frac{e^{j2\pi z_{\max} u} - e^{j2\pi z_{\min} u}}{j2\pi(z_{\max} - z_{\min})u} \right]^{n-1} \tag{22}$$

In Fig. 2, the magnitude of such mean array factors is reported as a function of the observation angle θ having fixed the steering angle at $\alpha = 25°$. As can be seen, $|E\{F(u)\}|$ for TRA and BA are identical and similar to that of a continuous uniform current distribution. For JRA, the magnitude of the mean array factor is similar to the two previous ones around the main beam region, say for θ approximately within $(10°, 40°)$. However, it presents small peaks in correspondence of $\theta = \sin^{-1}[-(1/(2\varepsilon+\Delta)) + \sin(\theta_0)] \approx 9.96°$, $\theta = \sin^{-1}[(1/(2\varepsilon+\Delta)) + \sin(\theta_0)] \approx 42.24°$ and $\theta = \sin^{-1}[(2/(2\varepsilon+\Delta)) + \sin(\theta_0)] \approx 67.22°$ which are due to the periodic function $\{\sin[\pi N(2\varepsilon+\Delta)u]/\sin[\pi(2\varepsilon+\Delta)u]\}$. Moreover, for θ away from all peaks, owing to the function $\{\sin(2\pi\varepsilon u)/(2\pi\varepsilon u)\}$ the level is lower than $\phi_{TRA}(u)$. It is noted that in order to remedy the presence of the small peaks, one might think to synthesise PDF of the random variables $W_1, W_2, \ldots, W_N$ so as to have zeros just in correspondence of such peak positions. Finally, for $ARAs$ the magnitude of the mean array factor is still similar to the previous ones up to the first side-lobe region, which is close to the main lobe. Away from that region, however, it does not tend to

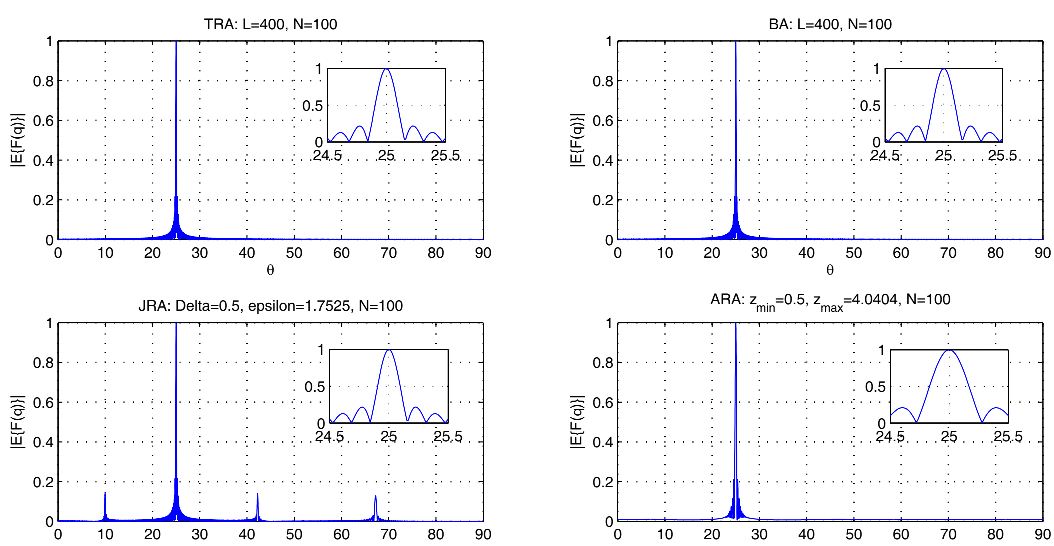

Figure 2. Magnitude of the mean array factor with $N = 100$, maximum aperture equal to 400λ and steering angle equal to $25°$.

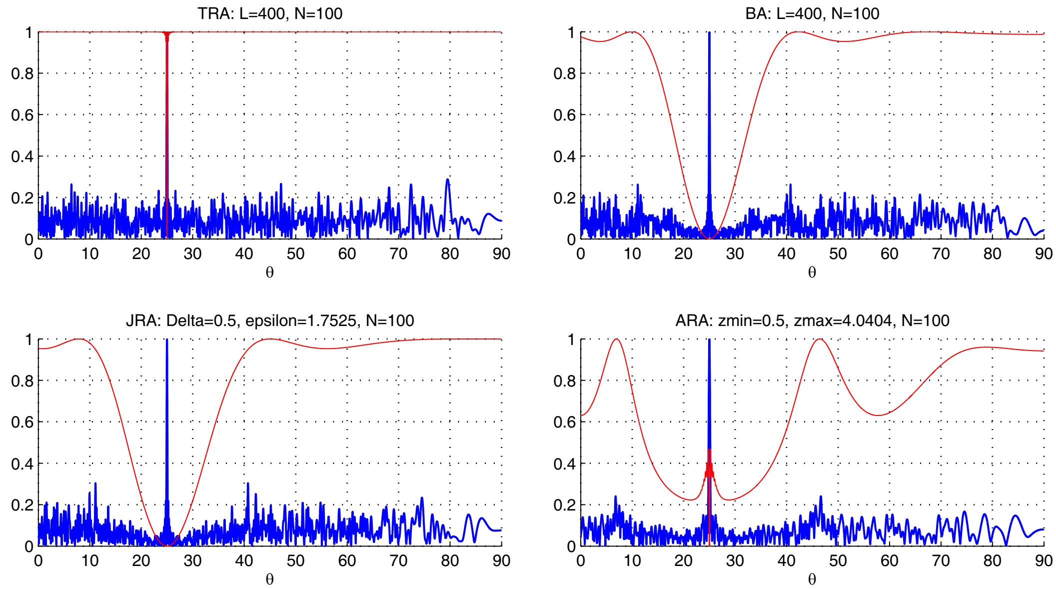

Figure 3. Magnitude of sample array factors (blue lines) with normalised variances (red lines) superimposed. The parameters are the same as in Fig. 2.

zero but assumes a nearly constant value.

In Fig. 3, sample functions of the array factor magnitude for the different cases under consideration are shown (blue lines) along with the corresponding normalised variance (red lines) defined as $\tilde{\sigma}_i^2(u) = \sigma_i^2(u)/\max_u\{\sigma_i^2(u)\}$ with $i = TRA, BA, JRA, ARA$. As already shown in [30], the variance relative to BA is considerably slower at reaching its maximum than TRA. A slightly slower behaviour is observed for JRA while the slowest to achieve the variance maximum, even though with an oscillatory trend, are $ARAs$. However, the latter exhibit higher variance values close to the main beam.

Within the region where the variance assumes low value, the array factor is more probable to be similar to the mean one. Therefore, at least for $TRAs$, BAs and $JRAs$, it can be concluded that the array factor around the main-beam region is practically *certain* to coincide with the mean one. This statement basically coincides with the claim that resolution is mainly affected by the array aperture rather than by the number and the way the elements are deployed. Moreover, in this regard BAs and $JRAs$ have a large *certain* region (as the variance grows up more slowly) and hence should be preferred.

Finally, it is remarked that previous discussion does not dependent on the steering angle.

4. SIDE-LOBE LEVEL

In this section, we turn to consider the comparison in terms of the the side-lobe level (SLL). The SLL is defined as

$$SLL = \max_{u\in[\delta,2]} |F(u)| \tag{23}$$

with $\delta > 0$ being the point where the side-lobe region starts. According to the previous discussion about the almost deterministic nature of the main beam, δ can be directly linked to the reciprocal of the array aperture. In particular, it can be fixed as the first zero of the mean array factor. Only the interval $[\delta, 2]$ is of concern. This is because the array factor magnitude is an even function. Also, limiting u to be ≤ 2 allows covering all the *visible* domain, whatever the steering angle is.

According to Eq. (23), in order to perform the study, the statistical characterisation of $\max_{u\in[\delta,2]} |F(u)|$ is required. This is a hard task which in general cannot be solved in closed form. To cope with this problem, some approximate approaches have been proposed in literature [8, 27–29, 35]. Here, the main focus is on establishing how the different random arrays perform. To this end, it is sufficient to achieve the SLL experimental characterisation. Accordingly, we build up the corresponding distributions by Monte Carlo simulations (20000 trials indeed). More in details, in order to highlight the role played by different array settings, different average spacings $d_{AV} = 1/2, 1, 2.5, 5$ (between adjacent) antenna elements are considered. Also the number of radiators, for each value of the average spacing, is varied from 20 to 200 with a step of 20. For $JRAs$ and $ARAs$, the minimum acceptable spacing between adjacent antenna elements is chosen equal to 0.3 for $d_{AV} = 1/2$, while it is chosen equal to $1/2$ for the other values of d_{AV}. Finally, while performing the simulations, the array factor magnitude is sampled at a step equal to the reciprocal of 20 times the maximum array aperture, which is a much finer step than the one usually used while analysing this kind of arrays [27].

In Fig. 4, the experimental SLL distributions are reported for $d_{av} = 1/2$ and for different numbers N of elements. Note that keeping fixed d_{av} while increasing N entails increasing the array aperture as well. First, it can be observed that as N increases all the distributions tend to be more peaked. This is somehow consistent with the behaviour of the variances as N increases reported in the previous section. Indeed, when N increases, in all the schemes under consideration, the variance decreases making it thus natural to expect that the SLL approaches the one corresponding to the average array factor. It is seen that $JRAs$ have a rather high SLL as compared to the other schemes. This is clearly due to the Dirichlet sine term $\sin[\pi N(2\varepsilon + \Delta)u]/\sin[\pi(2\varepsilon + \Delta)u]$ which is responsible for greeting lobes appearing at $u = 2$. Actually, this holds true as long as d_{av} does not exceed 1. Indeed, when d_{AV} is low and Δ comparable with it, ε is small as well. Hence, the perturbation (on the deterministic parts of the positions of radiators) $(n-1)(2\varepsilon + \Delta)$ with $n = 1, 2, \ldots, N$, is negligible. On the contrary, when the average spacing is high, and hence ε can assume sufficiently great values, the $JRAs$ performance becomes similar to those of the other methods.

That figure also shows that when it is allowed to use an average spacing equal to $\lambda/2$, random arrays generally return a worse SLL than the usual uniform arrays. However, it is interesting to note that as N increases the SLL tends (even though differently for each scheme but the JRA) to the standard -13 dB of uniform arrays, regardless the way the elements are arranged over the aperture.

We now turn to address the case reported in Fig. 5 where d_{av} is increased up to 5. It can be observed that the trend highlighted while discussing the previous figure still persists. Furthermore, as anticipated before, now $JRAs$ are no more affected by high SLL, and their PDF are similar to the ones of BAs.

It must be remarked that now the number of elements is roughly ten times lower than the one that should be used under a uniform setting. This entails that if uniform arrays were used a number of greeting lobes would manifest. Hence, the advantage provided by randomly putting the radiating

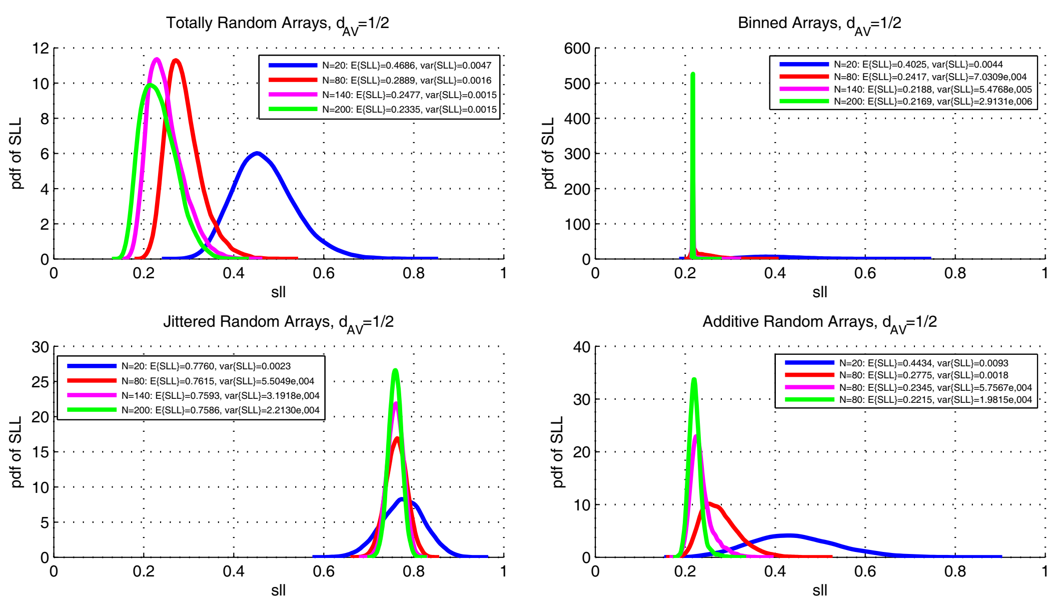

Figure 4. SLL distributions as a function of the number of antenna elements with d_{av} set at $1/2$.

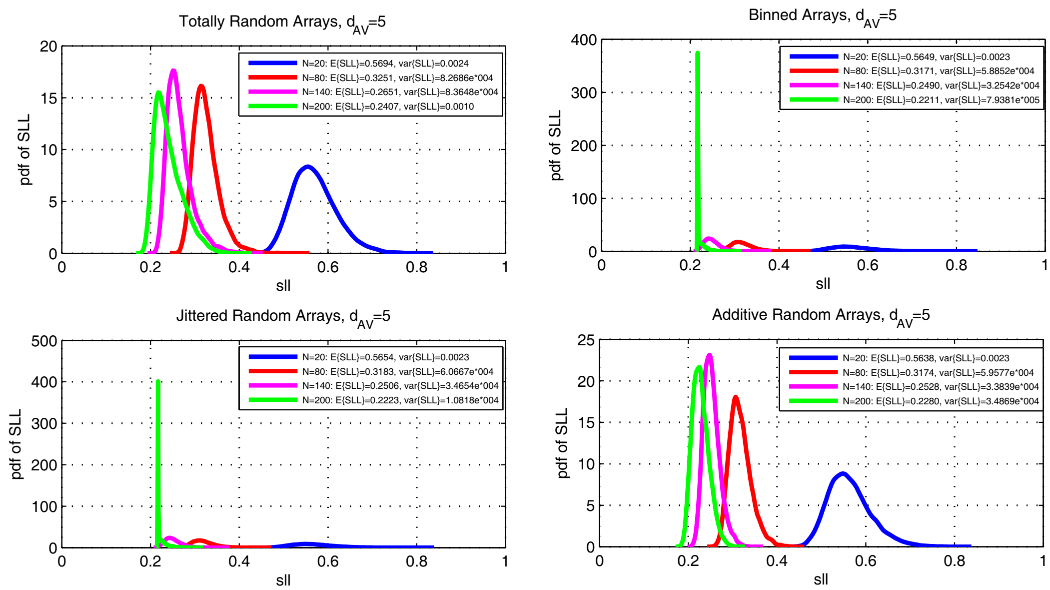

Figure 5. SLL distributions as a function of the number of antenna elements with d_{av} set at 5.

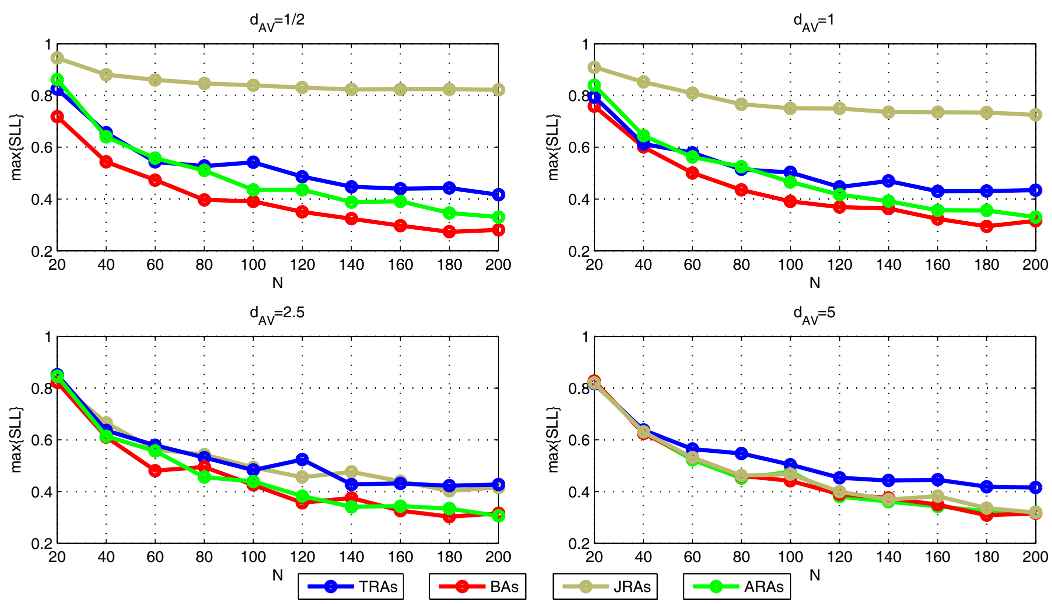

Figure 6. Statistical maximum of the SLL for each type of array as function of the number of antenna elements and for diferent average spacing.

elements is evident, especially when N increases. Moreover, by comparing Figs. 4 and 5 once again it is clear that the most important parameter that governs the SLL is the number of radiators and much less the array aperture. Indeed, the SLL distributions corresponding to the same N but at different d_{av} are very similar (of course, leaving outside the discussion $JRAs$ as explained above) even though the same number of radiators is deployed over two different array apertures (i.e, the second aperture is ten times larger than the first one). Instead, according to the previous discussion, for $JRAs$, the aperture also plays a crucial role. However, when a large array aperture is of concern $JRAs$ should be preferred. This is because the SLL distribution is as peaked as BAs PDF, but $JRAs$ also allow constraint of the distance between adjacent radiators and hence someway mitigation of mutual coupling.

In the next two figures, the experimental statistical maximum and minimum curves of the SLL

(i.e., the lower and upper edges of the support of the SLL distributions) are compared. These curves are basically a measure of the best and worst cases as far as SLL is concerned. Indeed, having fixed N and d_{av}, the probability that the SLL is below the maximum one is 1. Therefore, even by only one trial, it is almost sure that the sample array factor has SLL below the experimental $\max\{SLL\}$. Also, from Fig. 6 it can be seen that BAs are generally the best. Furthermore, as expected, for low values of d_{Av}, $JRAs$ have very high SLL. However, when the average spacing is increased, $JRAs$ behave similarly to the other methods. Also, as remarked above, they should be preferred to BAs for mutual coupling reason, and with respect the other two methods because PDF is more peaked around the mean.

Finally, Fig. 7 compares the experimental statistical minimum curves of the SLL. These curves can be used as a reference for the best achievable case, although to obtain the optimal positions, i.e., the one that provides such minimum SLL, one in general needs to generate many sample positions vectors. As may be observed, $TRAs$ and $ARAs$ are slightly better (this is already evident in Figs. 4 and 5 indeed). However, according to previous discussion, $JRAs$ are still preferable. It is interesting to note that, although for $d_{AV} = 1/2$ random arrays give in general high SLL as compared to uniform arrays, there is at least one random configuration (the one corresponding to the minimum SLL) that can return a SLL slightly below the standard -13.4 dB. In this regard, when radiators are equally excited randomising their locations can lead to lower SLL, confirming what reported in the classical paper [4].

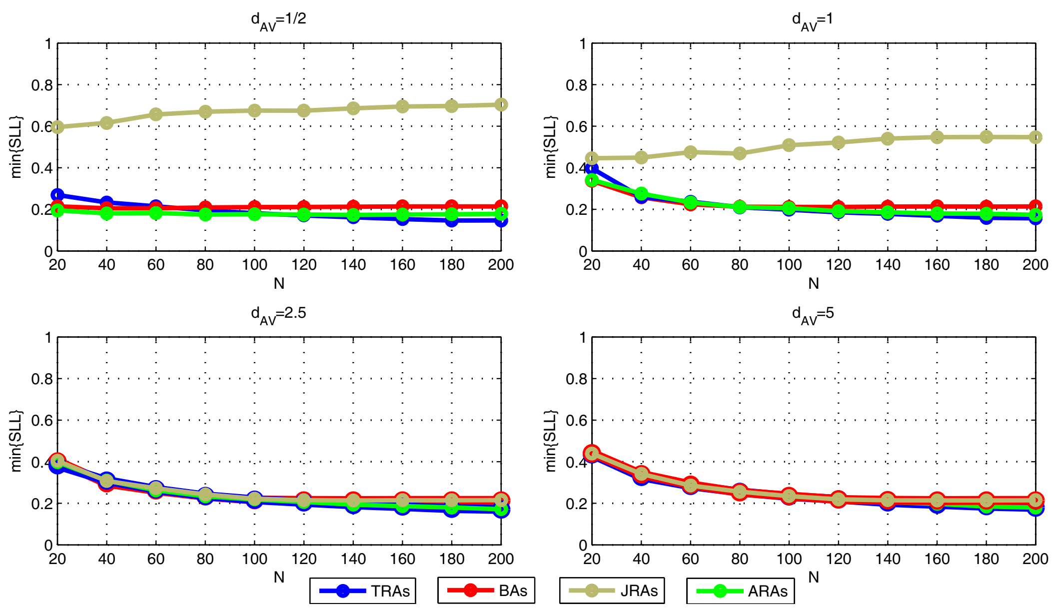

Figure 7. Statistical minimum of the SLL for each type of array as function of the number of antenna elements and for different average spacing.

5. CONCLUSIONS

In this work, four different rules for randomly generating the positions of radiators in random arrays are compared. In particular, we bring together approaches coming from antenna arrays literature (i.e., $TRAs$ and BAs) and nonuniform sampling theory (i.e., $JRAs$ and $ARAs$). The two worlds are naturally linked when the positions of antenna elements are replaced by the sampling points although the applications and specifications to be met could be generally different.

For each generation rule, the mean and variance of the array factor are provided under the assumption (rather common indeed) the random variables describing the element positions are *i.i.d.*. Of course, this statistics characterisation alone is insufficient. However, such formulas already allow one to understand the important role that the number of radiators play not only on the side-lobe level but also in a general synthesis problem.

The side-lobe distribution is worked out via a Monte Carlo analysis. It is shown that for low values of average spacing between antenna elements, BAa and $ARAs$ are the better ones and have similar performance. However, the latter have the advantage of being able to fix a minimum distance allowed between adjacent elements and hence counteract mutual coupling effects. For high values of d_{av}, $JRAs$ become more convenient. In particular, they exhibit performance similar to BAs but more resilient to mutual coupling effects. Furthermore, $JRAs$ are also interesting because they can be optimised by simply controlling the deterministic parameters, ε and Δ [31].

We end this paper by observing that even though $TRAs$ have the worse performance (at least from the SLL point of view that was of main concern herein) they can be analytically studied with less effort also from the synthesis point of view. More in detail, as the larger the number of radiators the smaller the variance (this is actually holds true for all the generating schemes), the array factor resembles more and more the mean array factor. Hence, for a sufficiently high N, the synthesis of the array factor may be re-phrased as the synthesis of the mean array factor, which in turn is linked to the position PDF the same way the far-field pattern is related to a continuous line-source [36]. Therefore, one can set the array factor according to some assigned pattern (not just the main beam width and/or the SLL) and find the corresponding PDF. We plan to show this approach in a future paper.

REFERENCES

1. Bilinskis, I., *Digital Alias-free Signal Processing*, John Wiley & Sons, Inc., 2007.
2. Holm, S., A. Austeng, K. Iranpour, and J. F. Hopperstad, "Nonuniform sampling: Theory and practice," *F. Marvasti Nonuniform Sampling: Theory and Practice*, Chapter 19, 2001.
3. King, D. D., R. F. Packard, and R. K. Thomas, "Unequally-spaced, broad-band antenna arrays," *IRE Trans. Antennas Propagat.*, Vol. 8, 380–384, 1960.
4. Ishimaru, A. and Y. S. Chen, "Thinning and broadbanding antenna arrays by unequal spacings," *IEEE Trans. Antennas Propagat.*, Vol. 13, 34–42, January 1965.
5. Andreasen, M. G., "Linear arrays with variable in-terelement spacing," *IRE Trans. Antennas Propagat.*, Vol. 10, 137–143, 1962.
6. Steinberg, B. D., "The peak sidelobe of the phased array having randomly located elements," *IEEE Trans. Antennas Propagat.*, Vol. 20, 129–136, 1972.
7. Leahy, B. D. and B. D. Jeffs, "On the design of maximally sparse beamforming arrays," *IEEE Trans. Antennas Propagat.*, Vol. 39, No. 8, 1178–1187, 1991.
8. Lo, Y. T., "A mathematical theory of antenna arrays with randomly spaced elements," *IEEE Trans. Antennas Propagat.*, Vol. 12, 257268, 1964.
9. Unz, H., "Linear arrays with arbitrarily distributed elements," *IRE Trans. Antennas Propagat.*, Vol. 8, 222–223, 1960.
10. Ishimaru, A., "Theory of unequally-spaced arrays," *IRE Trans. Antennas Propagat.*, Vol. 10, 691–702, 1962.
11. Willey, R. E., "Space tapering of linear and planar arrays," *IEEE Trans. Antennas Propag.*, Vol. 10, No. 4, 369–377, 1962.
12. Sandler, S., "Some equivalences between equally and unequally spaced arrays," *IRE Trans. Antennas Propagat.*, Vol. 8, 496–500, September 1960.
13. Maffett, A. L., "Array factors with non uniform spacing parameters," *IRE Trans. Antennas Propagat.*, Vol. 10, 131–146, 1962.
14. Skolnik, M., G. Nemhauser, and J. Sherman "Dynamic programming applied to unequally spaced arrays," *IEEE Trans. Antennas Propag.*, 35–43, January 1964.
15. Skolnik, M. I., J. W. Sherman III, and F. C. Ogg, Jr., "Statistically designed density-tapered arrays," *IEEE Trans. Antennas Propag.*, Vol. 12, 408–417, July 1964.
16. Haupt, R. L., "Thinned arrays using genetic algorithms," *IEEE Trans. Antennas Propag.*, Vol. 42, No. 7, 993–999 1994.
17. Haupt, R. L., "Unit circle representation of aperiodic arrays," *IEEE Trans. Antennas Propag.*, Vol. 43, No. 10, 1152–1155, 1995.

18. Kerby, K. C. and J. T. Bernhard, "Sidelobe level and wideband behavior of arrays of random subarrays," *IEEE Trans. Antennas Propag.*, Vol. 54, No. 8, 2253–2262, 2006.
19. Liu, Y., Z.-P. Nie, and Q. H. Liu, "A new method for the synthesis of non-uniform linear arrays with shaped power patterns," *Progress In Electromagnetics Research*, Vol. 107, 349–363, 2010.
20. Zhao, X., Q. Yang, and Y. Zhang, "Compressed sensing approach for pattern synthesis of maximally sparse non-uniform linear array," *IET Microw., Antennas Propag.*, Vol. 8, No. 5, 301–307, 2013.
21. Gregory, M. and D. Werner, "Ultrawideband aperiodic antenna arrays based on optimized raised power series representations," *IEEE Trans. Antennas Propag.*, Vol. 58, No. 3, 756–764 2010.
22. Tokan, F. and F. Gunes, "The multi-objective optimization of non-uniform linear phased arrays using the genetic algorithm," *Progress In Electromagnetics Research B*, Vol. 17, 135–151, 2009.
23. Razavi, A. and K. Forooraghi, "Thinned arrays using pattern search algorithms," *Progress In Electromagnetics Research*, Vol. 78, 61–71, 2008.
24. Kumar, B. P. and G. R. Branner, "Synthesis of unequally spaced arrays using Legendre series expansion," *Proc. IEEE Antennas Propag. Int. Symp.*, Vol. 4, 2236–2239, 1997.
25. Kurup, D. G., M. Himdi, and A. Rydberg, "Synthesis of uniform amplitude unequally spaced antenna arrays using the differential evolution algorithm," *IEEE Trans. Antennas Propag.*, Vol. 51, 2210–2217, 2003.
26. Lo, Y. T. and S. W. Lee, "Aperiod arrays," Chapman and Hall, *Antenna Handbook: Antenna Theory*, Vol. 2, Chapter 14, 1993.
27. Agrawal, V. and Y. Lo, "Distribution of sidelobe level in random arrays," *Proc. IEEE*, Vol. 57, No. 10, 1764–1765, 1969.
28. Donvito, M. B. and S. A. Kassam, "Characterization of the random array peak sidelobe," *IEEE Trans. Antennas Propagat.*, Vol. 27, 379–385, May 1979.
29. Krishnamurthy, S., D. Bliss, C. Richmond, V. Tarokh, "Peak sidelobe level gumbel distribution for arrays of randomly placed antennas," *2015 IEEE National Radar Conference — Proceedings*, 1671–1676, 2015.
30. Hendricks, W. J., "The totally random versus the bin approach for random arrays," *IEEE Trans. Antennas Propag.*, Vol. 39, 1757–1762, 1991.
31. Shinohara, N., B. Shishkov, H. Matsumoto, K. Hashimoto, and A. K. M. Baki, "New stochastic algorithm for optimization of both side lobes and grating lobes in large antenna arrays for MPT," *IEICE Transactions on Communications*, Vol. 91-B, No. 1, 286–296, 2008.
32. Kolghi, H. and S. B. Kesler, "Comparison of two methods of generating sensor locations in random array," *Antennas and Propagation Society International Symposium*, Vol. 24, 1986.
33. Papoulis, A. and S. U. Pillai, *Probability, Random Variables and Stochastic Processes*, 4th Edition, McGraw Hill, 2002.
34. Baglivo, J. A., *Mathematica Laboratories for Mathematical Statistics, Emphasizing Simulation and Computer Intensive Methods, ASA-SIAM*, Chapter 9, 120, 2005.
35. Agrawal, V. D. and Y. T. Lo, "Mutual coupling in phased arrays of randomly spaced antennas," *IEEE Trans. on Antennas and Propagat*, Vol. 20, 288–295, 1972.
36. Balanis, C. A., *Antenna Theory: Analysis and Design*, 2nd Edition, John Wiley and Sons, Inc., New York, 1996.